ANALYTIC CONNECTION BETWEEN THE FIBONACCI SEQUENCE AND DIAGONAL SUMS OF BINOMIAL COEFFICIENTS

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Abstract. Lucas noted in the 19th century that the Fibonacci sequence appears on the diagonals of Pascal's triangle. On the other hand, Binet's formula provides a closed form expression for the Fibonacci sequence using a linear combination of two geometric sequences. Both representations inspire one to produce an analytic function on the complex domain with the Fibonacci recurrence. As the main result, this article proves the equality of the two formulas for the same analytic function obtained by extending those representations. The proof utilizes polynomial sequences of binomial type, which are studied in the umbral calculus.

1. Introduction

Lucas wrote in [8, pp. 5–15] that the Fibonacci sequence [10], defined by $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$, appears on the diagonals of Pascal's triangle. If we lay out the binomial coefficients $\binom{x}{y} = \frac{x(x-1)\cdots(x-y+1)}{y(y-1)\cdots 1}$ ($x \in \mathbb{C},\ y \in \mathbb{N}$) of integral coordinates ($x, y \in \mathbb{N}$) on the xy-plane, adding the diagonals of slope 2 yields the desired sequence: $1, 1, 2, 3, 5, 8, \ldots$. In other words, the Fibonacci sequence can be alternatively written as

$$F_{k+1} = \sum_{n=0}^{k} \binom{\frac{n+k}{2}}{n}_{\mathbb{N}}, \tag{1}$$

defining $\binom{x}{y}_{\mathbb{N}}$ to be $\binom{x}{y}$ on $x, y \in \mathbb{N}$ and 0 everywhere else. On the other hand, Binet's formula [6] describes the Fibonacci sequence in closed form in terms of the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$ and its negative reciprocal $\psi = -\frac{1}{\varphi}$, both of which are solutions to the characteristic equation $x^2 = x + 1$. That is,

$$F_k = \frac{\varphi^k - \psi^k}{\sqrt{5}}. \tag{2}$$

Hence, we have an immediate consequence of equations (1) and (2) that $\sum_{n=0}^{k} \binom{\frac{n+k}{2}}{n}_{\mathbb{N}} = \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}$. Furthermore, a generalization of Binet's formula can describe any sequence U of the recurrence property $U_{k+2} = U_{k+1} + U_k$ by adjusting the coefficients α and β [3, pp. 51–53]:

$$U_k = \alpha\varphi^k + \beta\psi^k.$$

This formula allows us to extend the discrete sequences to analytic functions by replacing the integral argument k with a complex argument z:

$$U(z) = \alpha\varphi^z + \beta\psi^z. \tag{3}$$

Observing that equations (1) and (2) tie together and that equation (2) can be extended to take complex arguments in (3), one may ask the question: does adding up diagonals of the same slope in the complex domain reveal a connection between equations (1) and (3)? Indeed, we will discover that with the complex-valued binomial coefficients, the following identity holds (Theorem 5.4):

$$\sum_{n=0}^{\infty}\binom{\frac{n+z}{2}}{n} = \left(1+\frac{1}{\sqrt{5}}\right)\varphi^{z}. \tag{4}$$

This notable consequence comes from a further generalization that takes place on the diagonals of any rational slope greater than 1 (Theorem 5.2). Using the generalized result, we will determine in reverse that the unique positive solution to equation $x^p = x^q + 1$ ($\gcd(p,q) = 1$, $p > q$) can be obtained by the formula (Theorem 5.3)

$$x = \sqrt[q]{\sum_{n=0}^{\infty}\binom{\frac{n+z+1}{p/q}}{n}\Bigg/\sum_{n=0}^{\infty}\binom{\frac{n+z}{p/q}}{n}}, \tag{5}$$

with any choice of $z \in \mathbb{C}$.

2. Elementary Symmetric Polynomials

The key to proving the introduced equations (4) and (5) is that the diagonal sums of binomial coefficients $f(z) = \sum_{n=0}^{\infty}\binom{\frac{n+z}{\sigma}}{n}$ ($\sigma \in \mathbb{Q}$, $\sigma > 1$) form an exponential function. To show this, we will express the power series of $f(z)$ with a geometric sequence G_n (Theorem 4.3)

$$\sum_{n=0}^{\infty}\binom{\frac{n+z}{\sigma}}{n} = \sum_{n=0}^{\infty}\frac{G_n(\frac{z}{\sigma})^n}{n!}.$$

To expand the left side into the form on the right side, it is convenient to use the elementary symmetric polynomials. We clarify some of the notations involving them.

Definition 2.1 (elementary symmetric polynomials)**.** *The elementary symmetric polynomials (ESPs) are defined below.*

$$\begin{aligned}
E_0(x_1,x_2,\ldots,x_n) &= 1,\\
E_1(x_1,x_2,\ldots,x_n) &= \sum_{i=1}^{n} x_i,\\
E_2(x_1,x_2,\ldots,x_n) &= \sum_{1\le i<j\le n} x_i x_j,\\
&\ \,\vdots\\
E_n(x_1,x_2,\ldots,x_n) &= x_1x_2\cdots x_n.
\end{aligned}$$

Most notably, they appear in the expansion of $\prod_{i=1}^{n}(X+x_i)$. That is,

$$\prod_{i=1}^{n}(X+x_i) = \sum_{k=0}^{n} E_k(x_1,x_2,\ldots,x_n)X^{n-k}.$$

Notation 2.2. *We adopt an indexing convention for the function arguments, i.e.,*

$$f(x_n,x_{n+1},\ldots,x_m) = f(x_i)_{i=n}^{m} \quad (n \le m).$$

As an example, we can express the ESPs more concisely:

$$E_k(x_1,x_2,\ldots,x_n) = E_k(x_i)_{i=1}^{n}.$$

Definition 2.3 (multinomial coefficient)**.** *We will also use multinomial (binomial if $n = 1$) coefficients,*

$$\binom{x}{k_1, k_2, \dots, k_n} = \frac{x(x-1)\cdots(x-\sum_{i=1}^n k_i + 1)}{k_1! \cdots k_n!},$$

where $x \in \mathbb{C}$, $k \in \mathbb{N}$, and $k! = k(k-1)\cdots 1$.

Definition 2.4 (falling factorial)**.** *Pochhammer notation* [13] *is used for the falling factorials ($x \in \mathbb{C}$, $k \in \mathbb{N}$),*

$$(x)_k = x(x-1)\cdots(x-k+1),$$

which means $(k)_k = k!$ and

$$(x)_k = E_k(x-\nu)_{\nu=0}^{k-1}.$$

Then, the binomial coefficients can be written using the ESPs, i.e.,

$$\binom{x}{k} = \frac{(x)_k}{k!} = \frac{E_k(x-\nu)_{\nu=0}^{k-1}}{k!}.$$

Apart from these, we will assume basic knowledge on combinatorics, infinite series, and analysis, which can be found in [1, 7, 12].

3. Polynomial Sequences of Binomial Type

A sequence of polynomials whose entries have degrees equal to their indices is called a *polynomial sequence*, and a polynomial sequence (P_k) that satisfies the binomial identity

$$P_k(x+y) = \sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}(x) P_{k_2}(y)$$

is said to be of *binomial type*. Polynomial sequences of binomial type are often studied in the umbral calculus [11]. The definition can be extended to the sequences of multivariate polynomials.

Definition 3.1. *A sequence of multivariate polynomials whose entries have degrees equal to their indices is called a* multivariate polynomial sequence*, and a t-variate polynomial sequence (P_k) is of* binomial type *if and only if it satisfies the following binomial identity:*

$$P_k(x_i+y_i)_{i=1}^t = \sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}(x_i)_{i=1}^t P_{k_2}(y_i)_{i=1}^t.$$

This section is devoted to building a useful example of a (multivariate) binomial type sequence and recognizing important properties about such sequences.

Typical examples of binomial type sequences include the sequence of powers $P_k(x) = x^k$ (binomial theorem) and the sequence of falling factorials $Q_k(x) = (x)_k$ (Vandermonde's identity[1]). Now, we introduce a bivariate extension of the falling factorial sequence.

Theorem 3.2. *The sequence $\Theta_k(x,n) = \frac{E_k(x-\nu)_{\nu=0}^{n+k-1}}{\binom{n+k}{k}}$ is of binomial type.*

[1]See Theorem 5.1

Proof. Consider the series $f(\xi) = (1+y)^{\xi+x} \in \mathbb{C}[[x,y]][[\xi]]$.

$$\begin{aligned}
(1+y)^{\xi+x} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{y^n \xi^{n-k} E_k(x-\nu)_{\nu=0}^{n-1}}{n!} \\
&= \sum_{0\le k\le n} \frac{y^n \xi^{n-k} E_k(x-\nu)_{\nu=0}^{n-1}}{n!} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{y^n \xi^{n-k} E_k(x-\nu)_{\nu=0}^{n-1}}{n!} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{y^{n+k} \xi^{n} E_k(x-\nu)_{\nu=0}^{n+k-1}}{(n+k)!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{y^{n+k} E_k(x-\nu)_{\nu=0}^{n+k-1}}{(n+k)_k}\right)\frac{\xi^n}{n!}.
\end{aligned}$$

Its nth derivative is then computed below.

$$\left.\frac{d^n}{d\xi^n} f(\xi)\right|_{\xi=0} = (\ln(1+y))^n(1+y)^x = \sum_{k=0}^{\infty} \frac{y^{n+k} E_k(x-\nu)_{\nu=0}^{n+k-1}}{(n+k)_k}.$$

This implies the equality of the two power series of y in different forms.

$$(\ln(1+y))^{n_1+n_2}(1+y)^{x_1+x_2} = (\ln(1+y))^{n_1}(1+y)^{x_1}\cdot(\ln(1+y))^{n_2}(1+y)^{x_2}.$$

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{E_k(x_1+x_2-\nu)_{\nu=0}^{n_1+n_2+k-1}}{(n_1+n_2+k)_k}\right) y^{n_1+n_2+k} \\
&= \sum_{k=0}^{\infty} \frac{y^{n_1+k} E_k(x_1-\nu)_{\nu=0}^{n_1+k-1}}{(n_1+k)_k} \sum_{k=0}^{\infty} \frac{y^{n_2+k} E_k(x_2-\nu)_{\nu=0}^{n_2+k-1}}{(n_2+k)_k} \\
&= \sum_{k=0}^{\infty}\sum_{k_1+k_2=k} \frac{y^{n_1+k_1} y^{n_2+k_2} E_{k_1}(x_1-\nu)_{\nu=0}^{n_1+k_1-1} E_{k_2}(x_2-\nu)_{\nu=0}^{n_2+k_2-1}}{(n_1+k_1)_{k_1}(n_2+k_2)_{k_2}} \\
&= \sum_{k=0}^{\infty}\left(\sum_{k_1+k_2=k} \frac{E_{k_1}(x_1-\nu)_{\nu=0}^{n_1+k_1-1} E_{k_2}(x_2-\nu)_{\nu=0}^{n_2+k_2-1}}{(n_1+k_1)_{k_1}(n_2+k_2)_{k_2}}\right) y^{n_1+n_2+k}.
\end{aligned}$$

Hence, the corresponding coefficients equal

$$\begin{aligned}
&\frac{E_k(x_1+x_2-\nu)_{\nu=0}^{n_1+n_2+k-1}}{(n_1+n_2+k)_k} \\
&= \sum_{k_1+k_2=k} \frac{E_{k_1}(x_1-\nu)_{\nu=0}^{n_1+k_1-1} E_{k_2}(x_2-\nu)_{\nu=0}^{n_2+k_2-1}}{(n_1+k_1)_{k_1}(n_2+k_2)_{k_2}}
\end{aligned}$$

and this gives us the equation we desired.

$$\frac{\Theta_k(x_1+x_2,n_1+n_2)}{k!} = \sum_{k_1+k_2=k} \frac{\Theta_{k_1}(x_1,n_1)\Theta_{k_2}(x_2,n_2)}{k_1!k_2!},$$

$$\Theta_k(x_1+x_2,n_1+n_2) = \sum_{k_1+k_2=k} \binom{k}{k_1}\Theta_{k_1}(x_1,n_1)\Theta_{k_2}(x_2,n_2). \qquad \square$$

Vertical sums of binomial type sequences form geometric sequences as we observe in the familiar examples[2] $\sum_{k=0}^{\infty} \frac{n^k}{k!} = e^n$, $\sum_{k=0}^{\infty} \binom{n}{k} = 2^n$. The motivation behind Lemma 3.4 is to show in Theorem 3.5 that the diagonal sums of binomial type sequences make geometric sequences as well. Lemma 3.4 and Theorem 3.5 are multivariate extensions of the original results in [14].

Lemma 3.3. *Given any binomial type sequence* (P_k)*, we have* $P_0 = 1$.

Proof. Because $P_0 = (P_0)^2$ by the binomial identity, $P_0 = 0$ or $P_0 = 1$. Definition 3.1 requires that P_0 be a degree 0 polynomial, but the degree of the constant 0 is left unassigned (or a value other than a nonnegative integer). Therefore, $P_0 = 1$. $\square$

Lemma 3.4 (Shifting)**.** *Let* (P_k) *be a* t*-variate binomial type sequence, and let* $(L_k^{(1)})$, $(L_k^{(2)})$, …, $(L_k^{(t)})$ *be arithmetic sequences. Suppose each sequence* $(L_k^{(i)})$ *is assigned another arithmetic sequence* $(\overline{L}_k^{(i)})$ *that shares the common difference (i.e.,* $L_{k+1}^{(i)} - L_k^{(i)} = \overline{L}_{k+1}^{(i)} - \overline{L}_k^{(i)}$*). Then, the following identity holds for constants* $a_1, a_2, \ldots, a_t$*.*

$$\sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}\left(L_{k_1}^{(i)} + a_i\right)_{i=1}^t P_{k_2}\left(\overline{L}_{k_2}^{(i)}\right)_{i=1}^t$$
$$= \sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}\left(L_{k_1}^{(i)}\right)_{i=1}^t P_{k_2}\left(\overline{L}_{k_2}^{(i)} + a_i\right)_{i=1}^t.$$

Proof. By induction on k. For the case $k = 0$, both sides of the equation are 1 (Lemma 3.3). Now, assume the equality holds up to $k-1$. Then,

$$\sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}\left(L_{k_1}^{(i)} + a_i\right)_{i=1}^t P_{k_2}\left(\overline{L}_{k_2}^{(i)}\right)_{i=1}^t$$
$$= \sum_{k_1+k_2=k} \binom{k}{k_1} \left(\sum_{k_1'+j=k_1} \binom{k_1}{k_1'} P_{k_1'}\left(L_{k_1}^{(i)}\right)_{i=1}^t P_j(a_i)_{i=1}^t \right) P_{k_2}\left(\overline{L}_{k_2}^{(i)}\right)_{i=1}^t$$
$$= \sum_{k_1'+j+k_2=k} \binom{k}{k_1', j} P_{k_1'}\left(L_{k_1'+j}^{(i)}\right)_{i=1}^t P_j(a_i)_{i=1}^t P_{k_2}\left(\overline{L}_{k_2}^{(i)}\right)_{i=1}^t$$
$$= \sum_{j=0}^{k} \binom{k}{j} P_j(a_i)_{i=1}^t \left(\sum_{k_1'+k_2=k-j} \binom{k-j}{k_1'} P_{k_1'}\left(L_{k_1'+j}^{(i)}\right)_{i=1}^t P_{k_2}\left(\overline{L}_{k_2}^{(i)}\right)_{i=1}^t \right).$$

[2]The constant $e = 2.71828\ldots$ in the example is Euler's number. This will appear again in Theorem 4.3.

For each j satisfying the condition $1 \le j \le k$, we invoke the induction hypothesis. Recall that $L_{n+1}^{(i)} - L_n^{(i)} = \overline{L}_{n+1}^{(i)} - \overline{L}_n^{(i)}$. Continuing our equalities, we have

$$= \sum_{j=0}^{k} \binom{k}{j} P_j(a_i)_{i=1}^t \left(\sum_{k_1'+k_2=k-j} \binom{k-j}{k_1'} P_{k_1'} \left(L_{k_1'}^{(i)}\right)_{i=1}^t P_{k_2} \left(\overline{L}_{k_2+j}^{(i)}\right)_{i=1}^t \right)$$

$$= \sum_{k_1'+j+k_2=k} \binom{k}{k_1', j} P_{k_1'} \left(L_{k_1'}^{(i)}\right)_{i=1}^t P_j(a_i)_{i=1}^t P_{k_2} \left(\overline{L}_{k_2+j}^{(i)}\right)_{i=1}^t$$

$$= \sum_{k_1'+k_2'=k} \binom{k}{k_1'} P_{k_1'} \left(L_{k_1'}^{(i)}\right)_{i=1}^t \left(\sum_{j+k_2=k_2'} \binom{k_2'}{j} P_j(a_i)_{i=1}^t P_{k_2} \left(\overline{L}_{k_2'}^{(i)}\right)_{i=1}^t \right)$$

$$= \sum_{k_1'+k_2'=k} \binom{k}{k_1'} P_{k_1'} \left(L_{k_1'}^{(i)}\right)_{i=1}^t P_{k_2'} \left(\overline{L}_{k_2'}^{(i)} + a_1\right)_{i=1}^t . \qquad \square$$

Theorem 3.5 (Diagonalization)**.** *Let (P_k) be a t-variate binomial type sequence, and let $(A_n^{(1)})$, …, $(A_n^{(t)})$, $(L_k^{(1)})$, …, $(L_k^{(t)})$ be arithmetic sequences. Then, the sequence (G_n) defined by*

$$G_n = \sum_{k=0}^{\infty} \frac{P_k\left(A_n^{(i)} + L_k^{(i)}\right)_{i=1}^t}{k!}$$

is a geometric sequence, provided that the sum is absolutely convergent for all n.

Proof. We will prove this by verifying that $G_{n+d}G_m = G_n G_{m+d}$. First,

$$\begin{aligned}
&G_{n+d}G_m \\
&= \sum_{k=0}^{\infty} \frac{P_k\left(A_{n+d}^{(i)} + L_k^{(i)}\right)_{i=1}^t}{k!} \sum_{k=0}^{\infty} \frac{P_k\left(A_m^{(i)} + L_k^{(i)}\right)_{i=1}^t}{k!} \\
&= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \frac{P_{k_1}\left(A_{n+d}^{(i)} + L_{k_1}^{(i)}\right)_{i=1}^t P_{k_2}\left(A_m^{(i)} + L_{k_2}^{(i)}\right)_{i=1}^t}{k_1!k_2!} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}\left(A_{n+d}^{(i)} + L_{k_1}^{(i)}\right)_{i=1}^t P_{k_2}\left(A_m^{(i)} + L_{k_2}^{(i)}\right)_{i=1}^t .
\end{aligned}$$

To continue, we apply Lemma 3.4 to obtain

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1+k_2=k} \binom{k}{k_1} P_{k_1}\left(A_n^{(i)} + L_{k_1}^{(i)}\right)_{i=1}^t P_{k_2}\left(A_{m+d}^{(i)} + L_{k_2}^{(i)}\right)_{i=1}^t \\
&= \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \frac{P_{k_1}\left(A_n^{(i)} + L_{k_1}^{(i)}\right)_{i=1}^t P_{k_2}\left(A_{m+d}^{(i)} + L_{k_2}^{(i)}\right)_{i=1}^t}{k_1!k_2!} \\
&= \sum_{k=0}^{\infty} \frac{P_k\left(A_n^{(i)} + L_k^{(i)}\right)_{i=1}^t}{k!} \sum_{k=0}^{\infty} \frac{P_k\left(A_{m+d}^{(i)} + L_k^{(i)}\right)_{i=1}^t}{k!} \\
&= G_n G_{m+d}. \qquad \square
\end{aligned}$$

4. Diagonal Sums of Binomial Coefficients

We will introduce functions β_σ ($\sigma \in \mathbb{Q}$, $\sigma > 1$) formed by summing the diagonals of binomial coefficients and show three basic facts about them.

- They have infinite radii of convergence; hence, $\beta_\sigma : \mathbb{C} \to \mathbb{C}$.
- They are exponential. There exists $\gamma, \rho \in \mathbb{C}$ such that for all $z \in \mathbb{C}$, $\beta_\sigma(z) = \gamma e^{\rho z}$.
- They are positive on the real line. $\lim_{x\to\infty} \beta_\sigma(x) = \infty$ for $x \in \mathbb{R}$.

Definition 4.1. *To each rational number $\sigma > 1$, define a complex-valued function β_σ by the infinite series:*

$$\beta_\sigma(z) = \sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}}{n}.$$

Theorem 4.2. *Let $\sigma > 1$ be a rational number. The radius of convergence for $\beta_\sigma(z) = \sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}}{n}$ is infinite.*

Proof. First, we expand the series $\beta_\sigma(z) = \sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}}{n}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}}{n} &= \sum_{n=0}^{\infty} \frac{E_n\left(\frac{n+z}{\sigma} - \nu\right)_{\nu=0}^{n-1}}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\left(\frac{z}{\sigma}\right)^{n-k} E_k\left(\frac{n}{\sigma} - \nu\right)_{\nu=0}^{n-1}}{n!}.\end{aligned}$$

Now, we will show that the double sum $\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\left(\frac{z}{\sigma}\right)^{n-k} E_k\left(\frac{n}{\sigma} - \nu\right)_{\nu=0}^{n-1}}{n!}$ is absolutely convergent, which leads to its rearrangeability and hence, the convergence of $\beta_\sigma(z) = \sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}}{n}$ for all $z \in \mathbb{C}$. Let us obtain a bounding series above.

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \left| \frac{\left(\frac{z}{\sigma}\right)^{n-k} E_k\left(\frac{n}{\sigma} - \nu\right)_{\nu=0}^{n-1}}{n!} \right| &\le \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\left|\frac{z}{\sigma}\right|^{n-k} E_k\left(\left|\frac{n}{\sigma} - \nu\right|\right)_{\nu=0}^{n-1}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{\nu=0}^{n-1}\left(\left|\frac{z}{\sigma}\right| + \left|\frac{n}{\sigma} - \nu\right|\right)}{n!}.\end{aligned}$$

Because $\sigma > 1$ is rational, it can be expressed in a reduced fraction form $\sigma = \frac{p}{q}$, where $\gcd(p, q) = 1$ and $p > q \ge 1$. We partition the terms in this bounding series by their indices n modulo p.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\prod_{\nu=0}^{n-1}\left(\left|\frac{z}{\sigma}\right| + \left|\frac{n}{\sigma} - \nu\right|\right)}{n!} &= \sum_{r=0}^{p-1}\sum_{m=0}^{\infty} \frac{\prod_{\nu=0}^{pm+r-1}\left(\left|\frac{z}{\sigma}\right| + \left|\frac{q}{p}(pm+r) - \nu\right|\right)}{(pm+r)!} \\ &= \sum_{r=0}^{p-1}\sum_{m=0}^{\infty} \frac{\prod_{\nu=0}^{pm+r-1}\left(\left|\frac{z}{\sigma}\right| + \left|qm - \nu + \frac{qr}{p}\right|\right)}{(pm+r)!}.\end{aligned}$$

Notice in the case $z = 0$, $r = 0$, the sum $\sum_{m=0}^{\infty} \frac{\prod_{\nu=0}^{pm-1}(|qm-\nu|)}{(pm)!}$ has the only nonzero term at $m = 0$ (which evaluates to 1), so it is absolutely convergent. Apart from the special case, we apply the ratio test on each subseries of residue r. Recall that $\sigma > 1$, so $p > q$.

$$\lim_{m\to\infty} \frac{(pm+r)!\prod_{\nu=0}^{pm+p+r-1}\left(\left|\frac{z}{\sigma}\right| + \left|qm+q-\nu+\frac{qr}{p}\right|\right)}{(pm+p+r)!\prod_{\nu=0}^{pm+r-1}\left(\left|\frac{z}{\sigma}\right| + \left|qm-\nu+\frac{qr}{p}\right|\right)}$$
$$= \lim_{m\to\infty} \frac{\prod_{\nu=1}^{q}\left(\left|\frac{z}{\sigma}\right| + \left|qm+\nu+\frac{qr}{p}\right|\right)\prod_{\nu=pm+r}^{pm+p-q+r-1}\left(\left|\frac{z}{\sigma}\right| + \left|qm-\nu+\frac{qr}{p}\right|\right)}{(pm+p+r)_p}$$
$$= \lim_{m\to\infty} \frac{\prod_{\nu=1}^{q}\left(\left|\frac{z}{\sigma}\right| + \left|qm+\nu+\frac{qr}{p}\right|\right)\prod_{\nu=0}^{p-q-1}\left(\left|\frac{z}{\sigma}\right| + \left|(q-p)m-\nu+\frac{(q-p)r}{p}\right|\right)}{(pm+p+r)_p}$$
$$= \left|\frac{q}{p}\right|^{q}\left|\frac{q-p}{p}\right|^{p-q} = \left(\frac{1}{\sigma}\right)^{q}\left(1-\frac{1}{\sigma}\right)^{p-q} < 1.$$

Therefore, each subseries converges absolutely. This implies that the double sum $\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\frac{z}{\sigma})^{n-k}E_k(\frac{n}{\sigma}-\nu)_{\nu=0}^{n-1}}{n!}$ is absolutely convergent as well. □

Theorem 4.3. *Let $\sigma > 1$ be a rational number. The function β_σ is exponential:*

$$\beta_\sigma(z) = \gamma e^{\rho z} \text{ (for all } z \in \mathbb{C})$$

for some $\gamma, \rho \in \mathbb{C}$.

Proof. In the proof of Theorem 4.2, we showed the absolute convergence of the double sum $\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\frac{z}{\sigma})^{n-k}E_k(\frac{n}{\sigma}-\nu)_{\nu=0}^{n-1}}{n!}$. This allows the rearrangement of the series below.

$$\sum_{n=0}^{\infty}\binom{\frac{n+z}{\sigma}}{n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\frac{z}{\sigma})^{n-k}E_k(\frac{n}{\sigma}-\nu)_{\nu=0}^{n-1}}{n!}$$
$$= \sum_{0\le k\le n}\frac{(\frac{z}{\sigma})^{n-k}E_k(\frac{n}{\sigma}-\nu)_{\nu=0}^{n-1}}{n!}$$
$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(\frac{z}{\sigma})^{n-k}E_k(\frac{n}{\sigma}-\nu)_{\nu=0}^{n-1}}{n!}$$
$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{z}{\sigma})^{n}E_k(\frac{n+k}{\sigma}-\nu)_{\nu=0}^{n+k-1}}{(n+k)!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\Theta_k(\frac{n+k}{\sigma},n)}{k!}\right)\frac{(\frac{z}{\sigma})^n}{n!}.$$

From Theorem 3.2 and Theorem 3.5, we conclude that

$$\sum_{n=0}^{\infty}\binom{\frac{n+z}{\sigma}}{n} = \sum_{n=0}^{\infty}\frac{G_n(\frac{z}{\sigma})^n}{n!}$$

for some geometric sequence (G_n). Hence, β_σ satisfies the equation

$$\begin{aligned}\frac{d}{dz}\beta_\sigma(z) &= \sum_{n=0}^{\infty} \frac{G_{n+1}(\frac{z}{\sigma})^n}{\sigma \cdot n!} \\ &= \rho \sum_{n=0}^{\infty} \frac{G_n(\frac{z}{\sigma})^n}{n!} \\ &= \rho\beta_\sigma(z),\end{aligned}$$

where ρ is the common ratio of the geometric sequence divided by σ. Solutions to the differential equation $\frac{d}{dz}f(z) = \rho f(z)$ have the form $f(z) = \gamma e^{\rho z}$ for some complex constant γ. $\square$

Theorem 4.4. *Let $\sigma > 1$ be a rational number. Then,*

$$\lim_{x\to\infty} \beta_\sigma(x) = \infty$$

on the real line $x \in \mathbb{R}$. Consequently, for all $z \in \mathbb{C}$, $\beta_\sigma(z) = \gamma e^{\rho z}$ for some positive real numbers γ, ρ, which means $\beta_\sigma(x) > 0$ for all $x \in \mathbb{R}$.

Proof. Recall that by Theorem 4.2, β_σ converges everywhere on $\mathbb{C}$. In particular, $\beta_\sigma(x) = \sum_{n=0}^{\infty} \binom{\frac{n+x}{\sigma}}{n}$ converges to some real value for all $x \in \mathbb{R}$ because all of the terms in the series will be in $\mathbb{R}$. Together with Theorem 4.3, we can further deduce that for all $z \in \mathbb{C}$, $\beta_\sigma(z) = \gamma e^{\rho z}$ for some $\gamma, \rho \in \mathbb{R}$. Upon showing that $\lim_{x\to\infty} \beta_\sigma(x) = \infty$, we obtain $\gamma, \rho > 0$.

Write in reduced fraction form $\sigma = \frac{p}{q}$ ($\gcd(p, q) = 1$, $p > q \geq 1$) and assume $x \geq p - q > 0$. To construct a lower bound for $\beta_\sigma(x)$, which tends to infinity as $x \to \infty$, we observe two facts about the sum $\sum_{n=0}^{\infty} \binom{\frac{n+x}{\sigma}}{n}$.

(1) The first nontrivially indexed term ($n = 1$) is positive and increasing without bound.
(2) The sum of the first negative term ($n = n_0$) and the terms after ($n > n_0$) is bounded by a constant.

The first claim is true: $\binom{\frac{1+x}{\sigma}}{1} = \frac{1+x}{\sigma} > 0$ is increasing without bound. To prove the second claim, we analyze the behavior of $\left(\frac{n+x}{\sigma}\right)_n$. Because $x > 0$, all factors in $\left(\frac{n+x}{\sigma}\right)_n = \left(\frac{n+x}{\sigma}\right)\left(\frac{n+x}{\sigma} - 1\right)\cdots\left(\frac{n+x}{\sigma} - n + 1\right)$ remain positive for small n, but $\sigma > 1$ causes $\left(\frac{n+x}{\sigma}\right)_n$ to accumulate negative factors eventually: that is, $\frac{n+x}{\sigma} - n + 1 < 0$ for all sufficiently large n. Let n_0 be the least index for which $\frac{n_0+x}{\sigma} - n_0 + 1 < 0$.

Provided that $n \geq n_0$, we can build a permutation π on $\{1, 2, \ldots, n\}$ so that the factors $a_1 = \frac{n+x}{\sigma}$, $a_2 = \frac{n+x}{\sigma} - 1$, $\ldots$, $a_n = \frac{n+x}{\sigma} - n + 1$ satisfy $|a_\nu| \leq \pi(\nu)$. Pick the least ν_0 such that a_{ν_0} is negative. The permutation π is defined as follows.

$$\pi(\nu) = \begin{cases} \nu_0 - \nu, & 1 \leq \nu < \nu_0; \\ \nu, & \nu_0 \leq \nu \leq n. \end{cases}$$

Because the sequence (a_ν) is decreasing by 1, and a_{ν_0} is the first negative term, $0 \leq a_{\nu_0-1} < 1$. Inductively, it follows that $1 \leq a_{\nu_0-2} < 2 \leq a_{\nu_0-3} < 3 \leq \cdots < \nu_0 - 2 \leq a_1 < \nu_0 - 1$. Furthermore, $|a_n| < n$ because the sequence (a_ν), although beginning with a positive number and falling by 1, is not long enough to reach the magnitude n on the negative side. Again, we observe inductively that $|a_{n-1}| < n - 1$, $|a_{n-2}| < n - 2$, $\ldots$, $|a_{\nu_0}| < \nu_0$. Therefore,

$$\prod_{\nu=1}^{n} |a_\nu| < n! \;\Rightarrow\; \frac{\left|\left(\frac{n+x}{\sigma}\right)_n\right|}{n!} < 1 \;\Rightarrow\; \left|\binom{\frac{n+x}{\sigma}}{n}\right| < 1,$$

where $n \geq n_0$.

We have seen that the terms in $\sum_{n=n_0}^{\infty}\binom{\frac{n+x}{\sigma}}{n}$ are absolutely bounded by 1. Let us enhance this to obtain a bound on the series. Partition the terms in $\sum_{n=n_0}^{\infty}\binom{\frac{n+x}{\sigma}}{n}$ by their indices modulo p. That is,

$$\begin{aligned}\sum_{n=n_0}^{\infty}\binom{\frac{n+x}{\sigma}}{n} &= \sum_{r=0}^{p-1}\sum_{m=0}^{\infty}\binom{\frac{q}{p}(pm+r+n_0+x)}{pm+r+n_0}\\ &= \sum_{r=0}^{p-1}\sum_{m=0}^{\infty}\binom{qm+\frac{q(r+n_0+x)}{p}}{pm+r+n_0}.\end{aligned}$$

Notice that if a zero-term ever occurs in this double sum, it occurs only in one class and for all terms in that class. This allows a bound on the ratio between two consecutive terms in each class without a zero-term.

$$\begin{aligned}&R_{m,r,x}\\ &=\left|\frac{\binom{q(m+1)+\frac{q(r+n_0+x)}{p}}{p(m+1)+r+n_0}}{\binom{qm+\frac{q(r+n_0+x)}{p}}{pm+r+n_0}}\right| = \left|\frac{(pm+r+n_0)!\left(qm+q+\frac{q(r+n_0+x)}{p}\right)_{pm+p+r+n_0}}{(pm+p+r+n_0)!\left(qm+\frac{q(r+n_0+x)}{p}\right)_{pm+r+n_0}}\right|\\ &=\frac{\left(qm+q+\frac{q(r+n_0+x)}{p}\right)_q\left((pm+p+r+n_0-1)-(qm+q+\frac{q(r+n_0+x)}{p})\right)_{p-q}}{(pm+p+r+n_0)_p}\\ &=\frac{\left(qm+q+\frac{q(r+n_0+x)}{p}\right)_q}{(pm+p+r+n_0)_q}\cdot\frac{\left((p-q)m+(p-q)+\frac{p-q}{p}(r+n_0)-1-\frac{qx}{p}\right)_{p-q}}{(pm+(p-q)+r+n_0)_{p-q}}.\end{aligned}$$

By our choice of n_0, we have $qm+q+\frac{q(r+n_0+x)}{p} < pm+p+r+n_0$.

$$\begin{aligned}R_{m,r,x} &< \frac{\left((p-q)m+(p-q)+\frac{p-q}{p}(r+n_0)-1-\frac{qx}{p}\right)_{p-q}}{(pm+(p-q)+r+n_0)_{p-q}}\\ &= \prod_{\nu=0}^{p-q-1}\frac{(p-q)m+(p-q)+\frac{p-q}{p}(r+n_0)-1-\frac{qx}{p}-\nu}{pm+(p-q)+r+n_0-\nu}\\ &= \prod_{\nu=0}^{p-q-1}\left(\frac{p-q}{p}+\frac{\frac{q}{p}(p-q)-1-\frac{qx}{p}-\frac{q}{p}\nu}{pm+(p-q)+r+n_0-\nu}\right).\end{aligned}$$

Because $x \geq p-q$, we further obtain $\frac{q}{p}(p-q)-1-\frac{qx}{p}-\frac{q}{p}\nu < 0$. Hence,

$$R_{m,r,x} < \prod_{\nu=0}^{p-q-1}\frac{p-q}{p} = (1-\tfrac{1}{\sigma})^{p-q} \leq (1-\tfrac{1}{\sigma}).$$

This gives us the desired upper bound:

$$\left|\sum_{n=n_0}^{\infty}\binom{\frac{n+x}{\sigma}}{n}\right| < \sum_{r=0}^{p-1}\sum_{m=0}^{\infty}(R_{m,r,x})^m < \sum_{r=0}^{p-1}\sum_{m=0}^{\infty}(1-\tfrac{1}{\sigma})^m = p\sigma.$$

Now, a lower bound for β_σ can be obtained. We should keep in mind that our choice of $n_0 \geq 2$ depended on x (i.e., $n_0 = n_0(x)$). For $x \geq p - q > 0$,

$$\begin{aligned}\beta_\sigma(x) &= \sum_{n=0}^{\infty} \binom{\frac{n+x}{\sigma}}{n} \\ &= 1 + \frac{1+x}{\sigma} + \sum_{n=2}^{n_0(x)-1} \binom{\frac{n+x}{\sigma}}{n} + \sum_{n=n_0(x)}^{\infty} \binom{\frac{n+x}{\sigma}}{n} \\ &\geq 1 + \frac{1+x}{\sigma} + \sum_{n=n_0(x)}^{\infty} \binom{\frac{n+x}{\sigma}}{n} = b(x),\end{aligned}$$

where $\left|\sum_{n=n_0(x)}^{\infty} \binom{\frac{n+x}{\sigma}}{n}\right| < p\sigma$. That is,

$$\lim_{x\to\infty} \beta_\sigma(x) \geq \lim_{x\to\infty} b(x) \geq 1 - p\sigma + \lim_{x\to\infty} \frac{1+x}{\sigma} = \infty. \qquad \square$$

5. Recurrence Properties of β_σ

The main results, Theorem 5.3 and Theorem 5.4, justify equations (4) and (5) in the introduction. These theorems exploit the recurrence properties of the binomial coefficients. In particular, Vandermonde's identity is used to establish Theorem 5.2 and Theorem 5.3. The derivation of a closed form formula in Theorem 5.4 is assisted by Theorem 5.3.

Theorem 5.1 (Vandermonde's identity [4])**.**

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}.$$

Theorem 5.2. *Let $\sigma > 1$ be a rational number. Then, β_σ satisfies the recurrence equation*

$$\beta_\sigma(z+\sigma) = \beta_\sigma(z+1) + \beta_\sigma(z)$$

for all $z \in \mathbb{C}$.

Proof. We use Vandermonde's identity (Theorem 5.1).

$$\begin{aligned}\beta_\sigma(z+\sigma) &= \sum_{n=0}^{\infty} \binom{\frac{n+z+\sigma}{\sigma}}{n} \\ &= \sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}+1}{n} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{\frac{n+z}{\sigma}}{k}\binom{1}{n-k} \\ &= \sum_{n=0}^{\infty} \binom{\frac{n+1+z}{\sigma}}{n} + \sum_{n=0}^{\infty} \binom{\frac{n+z}{\sigma}}{n} \\ &= \beta_\sigma(z+1) + \beta_\sigma(z). \qquad \square\end{aligned}$$

Theorem 5.3. *Let $\gcd(p,q) = 1$ and $p > q \geq 1$. For all $z \in \mathbb{C}$,*

$$\sqrt[q]{\frac{\beta_{p/q}(z+1)}{\beta_{p/q}(z)}}$$

is equal to the unique positive root of the polynomial $x^p - x^q - 1$.

Proof. Descartes' Rule of Signs states that the number of positive roots of a polynomial is at most the number of sign changes in its coefficients [2, 5]. Therefore, $x^p - x^q - 1$ has at most one positive root.

Fix an arbitrary $z \in \mathbb{C}$, and let $\alpha = \frac{\beta_{p/q}(z+1)}{\beta_{p/q}(z)}$. By Theorem 5.2 and Theorem 4.3,

$$\begin{aligned}\beta_{p/q}(z + \tfrac{p}{q}) &= \beta_{p/q}(z+1) + \beta_{p/q}(z),\\ \beta_{p/q}(z) \cdot \alpha^{\frac{p}{q}} &= \beta_{p/q}(z) \cdot \alpha + \beta_{p/q}(z),\\ \alpha^{\frac{p}{q}} &= \alpha + 1.\end{aligned}$$

Hence, $(\sqrt[q]{\alpha})^p = (\sqrt[q]{\alpha})^q + 1$. □

Theorem 5.4. *For all* $z \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \binom{\frac{n+z}{2}}{n} = \left(1 + \frac{1}{\sqrt{5}}\right) \varphi^z.$$

Proof. By Theorem 4.3, for all $z \in \mathbb{C}$, $\beta_2(z) = \gamma e^{\rho z}$ for some $\gamma, \rho \in \mathbb{C}$. With $p = 2$ and $q = 1$, Theorem 5.3 shows that e^ρ is the positive root of $x^2 - x - 1$:

$$e^\rho = \frac{1+\sqrt{5}}{2} = \varphi.$$

Furthermore, $\gamma = \beta_2(0)$ can be evaluated.

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{\frac{n}{2}}{n} &= 1 + \sum_{n=0}^{\infty} \binom{\frac{2n+1}{2}}{2n+1}\\ &= 1 + \sum_{n=0}^{\infty} \frac{\left(\frac{2n+1}{2}\right)_n \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right)_n}{(2n+1)!}\\ &= 1 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n+1) \cdot (2n-1) \cdots 5 \cdot 3 \left(-\frac{1}{2}\right)_n}{(2n+1)!} \left(\frac{1}{2}\right)^n\\ &= 1 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n}{2n \cdot (2n-2) \cdots 4 \cdot 2} \left(\frac{1}{2}\right)^n\\ &= 1 + \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{1}{4}\right)^n\\ &= 1 + \frac{1}{2} \left(1 + \frac{1}{4}\right)^{-\frac{1}{2}}\\ &= 1 + \frac{1}{\sqrt{5}}.\end{aligned}$$

□

Remark 5.5. *The Lucas sequence*[3] (L_k) *appears in harmony with the Fibonacci sequence* (F_k) *in* β_2.

$$\sum_{n=0}^{\infty} \binom{\frac{n+k}{2}}{n} = F_{k+1} + \frac{L_{k+1}}{\sqrt{5}}.$$

[3] It appears in the Online Encyclopedia of Integer Sequences (OEIS) [9]

6. Future Work

It remains an open problem to provide nice identities for other values of σ and z in $\beta_\sigma(z)$ (other than the ones in Theorem 5.4 and Remark 5.5).

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