AN ASYMPTOTIC UPPER BOUND FOR COUNTING FIVEN NUMBERS

ABIGAIL HOIT

Abstract. In this paper, we derive an asymptotic upper bound for the number of Fiven (Factorial-Niven) numbers. A Niven number is a number that is a multiple of its digital sum. Dalenberg and Edgar recently defined a Fiven number as a number that is a multiple of its digital sum using the digits from its representation in factorial-base notation. To derive our upper bound, we closely follow the methods first used by Cooper and Kennedy and later by DeKoninck and Doyon and others for their derivations of asymptotic bounds for base-10 and other fixed-base Niven numbers.

1. Introduction

A Niven number is a natural number that is a multiple of its digital sum. Originally introduced by Ivan Niven, there has been extensive research into Niven numbers since then (see, e.g., [3],[10], and [6]). Additionally, the concept was generalized by replacing the base-10 digital sum with the sum of digits in other digital representations (see, e.g., [1], [9], and [8]). Recently, Dalenberg and Edgar [5] generalized Niven numbers to the factorial-base system, calling such numbers Fiven numbers, and went on to prove that there are an infinite number of sets of four consecutive Fiven numbers, but no set of five consecutive Fiven numbers. Formally, the definition of Fiven numbers is the following.

Definition 1.1. *If $n = \sum_{i\geq 1} d_i i!$, where for all i, $0 \leq d_i \leq i$, then n is **Fiven** (or **Factorial-Niven**) if $\sum_{i\geq 1} d_i$ divides n.*

With regard to asymptotic frequency, Cooper and Kennedy [2] demonstrated an upper bound for the counting of ordinary Niven numbers and their method was expanded to derive improved bounds for ordinary Niven numbers and for Niven numbers generalized to other fixed bases (see [4], [6], and [7]). In this paper, we closely follow their method of treating the sum of digits as a random variable and calculating its mean and variance on select subsets of the natural numbers. However, their choice of subsets worked because the ratios of the base elements in their representations were fixed, or at least bounded. This is not the case for the factorial representation, so we will use a slightly different choice of subsets.

2. Main Theorem

Theorem 2.1. *The number $N(x)$ of Fiven numbers up to x is bounded above by*

$$N(x) \ll \frac{(\ln(\ln x))^{5/4}}{(\ln x)^{1/4}} x.$$

To prove this theorem, we will show that for all positive integers $r \geq 8$, $1 \leq m \leq r$, we have

$$N((m+1)r!) \leq \frac{32}{3r^{1/2}}(m+1)r! + \frac{160 \ln r}{r^{1/4}}(m+1)\,r!, \qquad (*)$$

which together with the following lemma implies the result in the theorem.

Lemma 2.2. *If $r \geq 8$, $1 \leq m \leq r$, and $mr! \leq x < (m+1)r!$, then $\frac{\ln x}{2\ln(\ln x)} < r \leq \frac{4\ln x}{\ln(\ln x)}$, $\ln r \leq \ln\ln x$, and $(m+1)r! \leq 2x$.*

Proof. We use the following bounds on $\ln(r!)$ derived in [11],

$$r\ln r - r \leq \ln(r!) \leq r\ln r, \qquad (**)$$

and then bound $\ln x$ above by

$$\ln x < \ln((m+1)\,r!) \leq \ln(r+1) + \ln(r!) < 2r\ln r,$$

and below by

$$\ln x \geq \ln(mr!) \geq \ln r! \geq r\ln r - r \geq r(\ln r - 1) > r.$$

It follows that

$$\frac{\ln x}{\ln\ln x} < \frac{\ln x}{\ln r} < 2r$$

and

$$\frac{\ln x}{\ln\ln x} > \frac{r(\ln r - 1)}{\ln(2r\ln r)} = r\frac{\ln r - 1}{\ln 2 + \ln r + \ln\ln r} > \frac{r}{4},$$

which is the first part of the claim in the lemma. The inequality $\ln r \leq \ln\ln x$ now follows easily. Finally, the last inequality claimed in the lemma is demonstrated by $(m+1)r! = \frac{m+1}{m}mr! \leq 2mr! \leq 2x$. □

3. Proof of the Main Theorem

We let $s(n)$ be the sum of digits in the factorial-base representation of n. That is, if $n = \sum_{i\geq 1} d_i i!$, where for all i, $0 \leq d_i \leq i$, then $s(n) = \sum_{i\geq 1} d_i$. It will be convenient to consider $s(n)$ as a random variable and derive its mean and variance over bounded subsets.

Lemma 3.1. *Let $r \geq 2$, $1 \leq m \leq r$, and let $\mu_{m,r}$ and $\sigma^2_{m,r}$ be the mean and variance, respectively, of $s(n)$ for $1 \leq n < (m+1)r!$. Then, we have*

$$\mu_{m,r} = \frac{1}{4}r^2 - \frac{1}{4}r + \frac{m}{2} \text{ and } \sigma^2_{m,r} \leq \frac{1}{6}r^3.$$

Proof. Because the digits of n are independent, the mean and variance over all of the possible digital sums is, respectively, the sum of the means and variances of all the possible r digits. Thus, we have

$$\mu_{m,r} = \left(\sum_{j=1}^{r-1}\frac{1}{j+1}\left(\sum_{i=0}^{j} i\right)\right) + \frac{m}{2} = \frac{1}{4}r^2 - \frac{1}{4}r + \frac{m}{2} \geq \frac{1}{4}r^2 - \frac{1}{4}r$$

and

$$\sigma^2_{m,r} = \left(\sum_{j=1}^{r-1}\left(\frac{1}{j+1}\left(\sum_{i=0}^{j} i^2\right) - \left(\frac{j}{2}\right)^2\right)\right) + \frac{1}{m+1}\left(\sum_{i=0}^{m} i^2\right) - \left(\frac{m}{2}\right)^2$$
$$= \frac{1}{36}r^3 + \frac{1}{24}r^2 - \frac{5}{72}r + \frac{1}{6}m + \frac{1}{12}m^2 \leq \frac{7}{72}r + \frac{1}{8}r^2 + \frac{1}{36}r^3.$$

The desired inequality follows (see Remark on Inequalities). □

For $r \geq 2$, $0 \leq m \leq r$, the Fiven numbers up to $(m+1)r!$ can be separated into two sets, based on how close $s(n)$ is to the mean $\mu_{m,r}$.

We let
$$A(m,r) = \#\left\{1 \leq n < (m+1)r! : s(n) \mid n \text{ and } |s(n) - \mu_{m,r}| > \frac{1}{8}r^{7/4}\right\}$$
and
$$B(m,r) = \#\left\{1 \leq n < (m+1)r! : s(n) \mid n \text{ and } |s(n) - \mu_{m,r}| \leq \frac{1}{8}r^{7/4}\right\}.$$

Then, we have $N((m+1)r!) = A(m,r) + B(m,r) + 1$. To bound $A(m,r)$, we will be generous and count natural numbers n even if they are not Fiven. For $B(m,r)$, we will be more precise.

By Chebyshev's Inequality, for every $k > 0$, a random variable X with mean μ and standard deviation σ satisfies
$$P(|X - \mu| > k) < \frac{\sigma^2}{k^2}.$$

Thus, by Lemma 3.1, we have
$$A(m,r) < \frac{\sigma^2_{m,r}}{\left(\frac{1}{8}r^{7/4}\right)^2}(m+1)r! \leq \frac{\frac{1}{6}r^3}{\frac{1}{64}r^{7/2}}(m+1)r! = \frac{32}{3r^{1/2}}(m+1)r!.$$

This is our desired upper bound for $A(m,r)$ as indicated in $(*)$. It remains to bound $B(m,r)$.

For n to be counted in $B(m,r)$, it must satisfy the three requirements (1) $1 \leq n < (m+1)r!$, (2) $s(n)|n$, and (3) $|s(n) - \mu_{m,r}| \leq \frac{1}{8}r^{7/4}$. First, let n be a natural number such that condition (1) is satisfied. Then, n has at most r digits in its factorial representation. Let α be the number of digits in the factorial representation of $s(n)$, so that $\alpha! \leq s(n) < (\alpha+1)!$. Note that, because $s(n) \leq n$, we have $\alpha \leq r$. Write n in its factorial expansion form as follows.
$$n = d_r r! + d_{r-1}(r-1)! + \cdots + d_\alpha \alpha! + d_{\alpha-1}(\alpha-1)! + \cdots + d_1$$
with $0 \leq d_i \leq i$ for $i < r$, and $0 \leq d_r \leq m$. Let $c = d_{r-1}(r-1)! + \cdots + d_{\alpha+1}(\alpha+1)!$. There are $\frac{r!}{(\alpha+1)!}$ possible values of the set of digits of c.

For convenience, we use the sequence of triangular numbers by defining $T(t) = \sum_{i=0}^{\lfloor t \rfloor} i$. Then, we have $s(n) \leq T(r)$, $\sum_{i=1}^{\alpha} d_i \leq T(\alpha)$, and $s(n) - T(\alpha) - d_r \leq s(c) \leq s(n) - d_r$.

Now, if requirement (2) is also satisfied, then $s(n)$ divides n. Because $d_r r! + c \leq n \leq d_r r! + c + \alpha\alpha!$, we find an upper bound for the number of multiples of $s(n) = \sum_{i=1}^{r} d_i$ in this interval. That upper bound is one more than the ratio of the length of the interval to the minimum possible value of $s(n)$. So, we can bound it by $1 + \frac{(\alpha+1)!}{\alpha!} = \alpha + 2$.

Finally, we consider requirement (3). If n satisfies $|s(n) - \mu_{m,r}| \leq \dfrac{1}{8}r^{7/4}$, then $\dfrac{1}{4}r^2 - \dfrac{1}{8}r^{7/4} - \dfrac{1}{4}r + \dfrac{1}{2}m \leq s(n) \leq \dfrac{1}{4}r^2 + \dfrac{1}{8}r^{7/4} - \dfrac{1}{4}r + \dfrac{1}{2}m$, by Lemma 3.1.

Assuming requirements (1) and (3) together, we have the next lemma.

Lemma 3.2. *If $r \geq 8$, and α is defined as above for n such that $1 \leq n < (m+1)r!$ and $|s(n) - \mu_{m,r}| \leq \frac{1}{8}r^{7/4}$, then $(\alpha+1)! > \frac{r^2}{8}$ and $\alpha < 4\ln r$.*

Proof. The first inequality in the claim follows from
$$\frac{1}{8}r^2 \leq \frac{1}{4}r^2 - \frac{1}{8}r^{7/4} - \frac{1}{4}r \leq s(n) < (\alpha+1)!.$$

The second inequality is clear for $\alpha \le 8$. For $\alpha > 8$, by $(**)$ we have

$$\alpha \le \alpha \ln \alpha - \alpha + \ln 2 \le \ln(\alpha!) + \ln 2 \le \ln(s(n)) + \ln 2 \le \ln(T(r)) + \ln 2 < 4 \ln r.$$

$\square$

Now, we bound $B(m,r) = \#\left\{1 \le n < (m+1)r! : s(n) \mid n \text{ and } |s(n) - \mu_{m,r}| \le \frac{1}{8}r^{7/4}\right\}$, using what we have shown. That is, for n to be counted in $B(m,r)$, d_r can be one of $m+1$ possible values from the set $\{0, \ldots, m\}$, the set of digits in c can be any one of the $\frac{r!}{(\alpha+1)!}$ lists, where the ith element is at most i, with $\alpha + 1 \le i \le r - 1$ and $s(n) - T(\alpha) - d_r \le s(c) \le s(n) - d_r$. Also, the remaining digits that determine n must make n one of the fewer than $\alpha + 2$ multiples of $s(n)$ that are in the permissible range for n, and $\frac{1}{4}r^2 - \frac{1}{8}r^{7/4} - \frac{1}{4}r + \frac{1}{2}m \le s(n) \le \frac{1}{4}r^2 + \frac{1}{8}r^{7/4} - \frac{1}{4}r + \frac{1}{2}m$. Putting this all together with Lemma 3.2, and noting that $T(t) \le \frac{1}{2}t^2 + \frac{1}{2}t$, we have

$$\begin{aligned}
B(m,r) &\le (m+1) \sum_{s(n)=\frac{1}{4}r^2-\frac{1}{8}r^{7/4}-\frac{1}{4}r+\frac{1}{2}m}^{\frac{1}{4}r^2+\frac{1}{8}r^{7/4}-\frac{1}{4}r+\frac{1}{2}m} (\alpha+2) \sum_{s(c)=s(n)-T(4\ln r)-d_r}^{s(n)-d_r} \frac{r!}{(\alpha+1)!} \\
&\le (m+1)(4\ln r + 2)\frac{8r!}{r^2} \sum_{s(c)=\frac{1}{4}r^2-\frac{1}{8}r^{7/4}-\frac{1}{4}r+\frac{1}{2}m-T(4\ln r)-d_r}^{\frac{1}{4}r^2+\frac{1}{8}r^{7/4}-\frac{1}{4}r+\frac{1}{2}m-d_r} 1 \\
&\le (m+1)(4\ln r + 2)\frac{8r!}{r^2}\left(\frac{1}{4}r^{7/4} + \frac{1}{2}(4\ln r)^2 + \frac{1}{2}4\ln r\right) \\
&\le (m+1)(4\ln r + 2)\frac{8r!}{r^2}\left(2r^{7/4}\right) \\
&\le \frac{160\ln r}{r^{1/4}}(m+1)\,r!,
\end{aligned}$$

which is the desired upper bound.

4. Remark on Inequalities

Throughout the paper, some basic inequalities were used without proof, but can be easily verified. We list them here for completeness. In some cases, they are more generous than necessary.

$$\begin{aligned}
\ln r &> 2 \text{ for } r \ge 8 \text{ (Lemma 2.1 and bound on } B(m,r)\text{)} \\
\frac{\ln r - 1}{\ln r + \ln 2 + \ln\ln r} &> \frac{1}{4} \text{ for } r \ge 8 \text{ (Lemma 2.1)} \\
\frac{7}{72}r + \frac{1}{8}r^2 + \frac{1}{36}r^3 &\le \frac{1}{6}r^3 \text{ for } r \ge 2, \text{ (Lemma 3.1)} \\
\frac{1}{4}r^2 - \frac{1}{8}r^{7/4} - \frac{1}{4}r &\ge \frac{1}{8}r^2 \text{ for } r \ge 8, \text{ (Lemma 3.2)} \\
\alpha \ln \alpha - \alpha + \ln 2 &\ge \alpha \text{ for } \alpha \ge 9\text{ , (Lemma 3.2)} \\
\frac{1}{8}r^{7/4} + 8(\ln r)^2 + 2\ln r &\le 2r^{7/4} \text{ for } r \ge 8 \text{ (bound on } B(m,r)\text{)}
\end{aligned}$$

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Department of Mathematics, Elmhurst University, Elmhurst, IL
Email address: abigailh@elmhurst.edu