PRODUCTS INVOLVING RECIPROCALS OF JACOBSTHAL POLYNOMIALS

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Abstract. We explore the Jacobsthal versions of six products involving reciprocals of gibonacci polynomials.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \ge 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial* [1, 6].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [3, 6].

Suppose $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, then $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, then $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 6].

Finally, suppose $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, then $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, then $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [4, 6].

1.1. Relationships in the Extended Gibonacci Family.

The subfamilies of the extended gibonacci family are linked, as Table 1 shows [4, 5], where $i = \sqrt{-1}$. We will employ them in our discourse.

Table 1: Relationships Among the Extended Gibonacci Subfamilies

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(x) = U_{n-1}(x/2)$	$v_n(x) = 2T_n(x/2),$

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $c_n = J_n(x)$ or $j_n(x)$, $\Delta = \sqrt{x^2 + 4}$, $D = \sqrt{4x + 1}, 2\alpha(x) = x + \Delta$, and $2u(x) = 1 + D$, and omit a lot of basic algebra. It follows by the *Binet-like* formulas [6] that $\lim_{m\to\infty} \frac{c_{m+k}}{c_m} = u^k(x)$.

1.2. **Products Involving Reciprocals of Gibonacci Polynomials.** We explored the following products involving reciprocals of gibonacci polynomials in [7]:

$$\prod_{n=2}^{m}\left(1+\frac{x^2}{f_{2n-1}^2}\right) = \frac{1}{x^2+1}\cdot\frac{f_{2m+1}}{f_{2m-1}}. \tag{1.1}$$

$$\prod_{n=2}^{m}\left(1-\frac{x^2}{f_{2n}^2}\right) = \frac{1}{x^2+2}\cdot\frac{f_{2m+2}}{f_{2m}}. \tag{1.2}$$

$$\prod_{n=2}^{m}\left(1-\frac{1}{f_{2n-1}^2}\right)\left(1+\frac{1}{f_{2n}^2}\right) = \frac{x}{x^2+1}\cdot\frac{f_{2m+1}}{f_{2m}}. \tag{1.3}$$

$$\prod_{n=2}^{m}\left(1-\frac{\Delta^2 x^2}{l_{2n-1}^2}\right) = \frac{x}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m-1}}. \tag{1.4}$$

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2 x^2}{l_{2n}^2}\right) = \frac{x^2+2}{x^4+4x^2+2}\cdot\frac{l_{2m+2}}{l_{2m}}. \tag{1.5}$$

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2}{l_{2n-1}^2}\right)\left(1-\frac{\Delta^2}{l_{2n}^2}\right) = \frac{x^2+2}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m}}. \tag{1.6}$$

We will now find their Jacobsthal versions using the Jacobsthal-gibonacci links in Table 1.

2. Products Involving Reciprocals of Jacobsthal Polynomials

We begin our explorations with products containing reciprocals of squares of odd-numbered Fibonacci polynomials.

2.1. **Jacobsthal Version of Formula** (1.1)**.** Let $A = 1+\dfrac{x^2}{f_{2n-1}^2}$. Replacing x with $1/\sqrt{x}$ and then multiplying the numerator and denominator of the resulting expression with x^{2n-2}, we get

$$\begin{aligned} A &= 1+\frac{1}{xf_{2n-1}^2} \\ &= 1+\frac{x^{2n-2}}{x\left[x^{(2n-2)/2}f_{2n-1}\right]^2} \\ &= 1+\frac{x^{2n-3}}{J_{2n-1}^2} \\ \text{LHS} &= \prod_{n=2}^{m}\left(1+\frac{x^{2n-3}}{J_{2n-1}^2}\right), \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now let $B = \dfrac{f_{2m+1}}{(x^2+1)f_{2m-1}}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with $x^{2m/2}$, we get

$$\begin{aligned} B &= \frac{xf_{2m+1}}{(x+1)f_{2m-1}} \\ &= \frac{x\left(x^{2m/2}f_{2m+1}\right)}{(x+1)x\left[x^{(2m-2)/2}f_{2m-1}\right]} \\ \text{RHS} &= \frac{J_{2m+1}}{(x+1)J_{2m-1}}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_m = J_m(x)$.

Equating the two sides, we get

$$\prod_{n=2}^{m}\left(1+\frac{x^{2n-3}}{J_{2n-1}^2}\right) = \frac{1}{x+1}\cdot\frac{J_{2m+1}}{J_{2m-1}}, \tag{2.1}$$

where $J_m = J_m(x)$.

This yields

$$\begin{aligned} \prod_{n=2}^{m}\left(1+\frac{1}{F_{2n-1}^2}\right) &= \frac{F_{2m+1}}{2F_{2m-1}}; \\ \prod_{n=2}^{m}\left(1+\frac{2^{2n-3}}{J_{2n-1}^2}\right) &= \frac{J_{2m+1}}{3J_{2m-1}}; \\ \prod_{n=2}^{\infty}\left(1+\frac{x^{2n-3}}{J_{2n-1}^2}\right) &= \frac{u^2(x)}{x+1}; \\ \prod_{n=2}^{\infty}\left(1+\frac{1}{F_{2n-1}^2}\right) &= \frac{\alpha^2}{2}; \\ \prod_{n=2}^{\infty}\left(1+\frac{2^{2n-3}}{J_{2n-1}^2}\right) &= \frac{4}{3}. \end{aligned} \tag{2.2}$$

2.2. **Jacobsthal Version of Formula** (1.2)**.** With $A = 1 - \dfrac{x^2}{f_{2n}^2}$, replace x with $1/\sqrt{x}$ and then multiply the numerator and denominator in the resulting expression with x^{2n-1}. We then get

$$\begin{aligned} A &= 1-\frac{1}{xf_{2n}^2} \\ &= 1-\frac{x^{2n-2}}{\left[x^{(2n-1)/2}f_{2n}\right]^2} \\ &= 1-\frac{x^{2n-2}}{J_{2n}^2} \\ \text{LHS} &= \prod_{n=2}^{m}\left(1-\frac{x^{2n-2}}{J_{2n}^2}\right); \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{x}{2x+1} \cdot \frac{f_{2m+2}}{f_{2m}} \\ &= \frac{x}{2x+1} \cdot \frac{x^{(2m+1)/2} f_{2m+2}}{x\left[x^{(2m-1)/2} f_{2m}\right]} \\ &= \frac{1}{2x+1} \cdot \frac{J_{2m+2}}{J_{2m}}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides, we get

$$\prod_{n=2}^{m} \left(1 - \frac{x^{2n-2}}{J_{2n}^2}\right) = \frac{1}{2x+1} \cdot \frac{J_{2m+2}}{J_{2m}}. \tag{2.3}$$

This implies,

$$\begin{aligned} \prod_{n=2}^{m} \left(1 - \frac{1}{F_{2n}^2}\right) &= \frac{F_{2m+2}}{3F_{2m}}; \\ \prod_{n=2}^{m} \left(1 - \frac{2^{2n-2}}{J_{2n}^2}\right) &= \frac{J_{2m+2}}{5J_{2m}}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{x^{2n-2}}{J_{2n}^2}\right) &= \frac{u^2(x)}{2x+1}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{1}{F_{2n}^2}\right) &= \frac{\alpha^2}{3}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{2^{2n-2}}{J_{2n}^2}\right) &= \frac{4}{5}. \end{aligned} \tag{2.4}$$

2.3. **Jacobsthal Version of Formula** (1.3)**.** Let $A = \left(1 - \frac{1}{f_{2n-1}^2}\right)\left(1 + \frac{1}{f_{2n}^2}\right)$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator in the first factor with x^{2n-2} and that of those in the second factor with x^{2n-1}, we get

$$\begin{aligned} A &= \left[1 - \frac{x^{2n-2}}{x^{2n-2} f_{2n-1}^2}\right]\left[1 + \frac{x^{2n-1}}{x^{2n-1} f_{2n}^2}\right] \\ &= \left(1 - \frac{x^{2n-2}}{J_{2n-1}^2}\right)\left(1 + \frac{x^{2n-1}}{J_{2n}^2}\right) \\ \text{LHS} &= \prod_{n=2}^{m} \left(1 - \frac{x^{2n-2}}{J_{2n-1}^2}\right)\left(1 + \frac{x^{2n-1}}{J_{2n}^2}\right), \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

With $B = \frac{x}{x^2+1} \cdot \frac{f_{2m+1}}{f_{2m}}$, similarly we get

$$\begin{aligned} B &= \frac{\sqrt{x}}{x+1} \cdot \frac{f_{2m+1}}{f_{2m}} \\ \text{RHS} &= \frac{\sqrt{x}}{x+1} \cdot \frac{\left[x^{(2m/2)} f_{2m+1}\right]}{\sqrt{x}\left[x^{(2m-1)/2} f_{2m}\right]} \\ &= \frac{1}{x+1} \cdot \frac{J_{2m+1}}{J_{2m}}. \end{aligned}$$

Combining the two sides yields

$$\prod_{n=2}^{m} \left(1 - \frac{x^{2n-2}}{J_{2n-1}^2}\right)\left(1 + \frac{x^{2n-1}}{J_{2n}^2}\right) = \frac{1}{x+1} \cdot \frac{J_{2m+1}}{J_{2m}}. \tag{2.5}$$

It follows from this formula that

$$\begin{aligned} \prod_{n=2}^{m} \left(1 - \frac{1}{F_{2n-1}^2}\right)\left(1 + \frac{1}{F_{2n}^2}\right) &= \frac{1}{2} \cdot \frac{F_{2m+1}}{F_{2m}}; \qquad (2.6) \\ \prod_{n=2}^{m} \left(1 - \frac{2^{2n-2}}{J_{2n-1}^2}\right)\left(1 + \frac{2^{2n-1}}{J_{2n}^2}\right) &= \frac{1}{3} \cdot \frac{J_{2m+1}}{J_{2m}}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{x^{2n-2}}{J_{2n-1}^2}\right)\left(1 + \frac{x^{2n-1}}{J_{2n}^2}\right) &= \frac{1}{x+1} u(x); \\ \prod_{n=2}^{\infty} \left(1 - \frac{2^{2n-2}}{J_{2n-1}^2}\right)\left(1 + \frac{2^{2n-1}}{J_{2n}^2}\right) &= \frac{2}{3}. \end{aligned}$$

Combining formulas (2.2), (2.4), and (2.6), we get

$$\begin{aligned} \prod_{n=3}^{2m} \left(1 - \frac{1}{F_n^4}\right) &= \frac{F_{2m+2} F_{2m+1}^2}{12 F_{2m}^2 F_{2m-1}} \\ \prod_{n=3}^{\infty} \left(1 - \frac{1}{F_n^4}\right) &= \frac{\alpha^5}{12}, \end{aligned}$$

as in [2, 8].

2.4. **Alternate Versions.** Using the Jacobsthal identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$ [6], we can rewrite formulas (2.1), (2.3), and (2.5) in a different way:

$$\begin{aligned} \prod_{n=2}^{m} \left(1 + \frac{D^2 x^{2n-3}}{j_{2n-1}^2 + 4x^{2n-1}}\right) &= \frac{1}{x+1} \cdot \frac{J_{2m+1}}{J_{2m-1}}; \\ \prod_{n=2}^{\infty} \left(1 + \frac{D^2 x^{2n-3}}{j_{2n-1}^2 + 4x^{2n-1}}\right) &= \frac{u^2(x)}{x+1}; \\ \prod_{n=2}^{\infty} \left(1 + \frac{5}{L_{2n-1}^2 + 4}\right) &= \frac{\alpha^2}{2}; \qquad (2.7) \end{aligned}$$

$$\begin{aligned}
\prod_{n=2}^{\infty}\left(1+\frac{9\cdot 2^{2n-3}}{j_{2n-1}^2+2^{2n+1}}\right) &= \frac{4}{3}.\\
\prod_{n=2}^{m}\left(1-\frac{D^2x^{2n-2}}{j_{2n}^2-4x^{2n}}\right) &= \frac{1}{2x+1}\cdot\frac{J_{2m+2}}{J_{2m}};\\
\prod_{n=2}^{\infty}\left(1-\frac{D^2x^{2n-2}}{j_{2n}^2-4x^{2n}}\right) &= \frac{u^2(x)}{2x+1};\\
\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n}^2-4}\right) &= \frac{\alpha^2}{3}; \qquad (2.8)\\
\prod_{n=2}^{\infty}\left(1-\frac{9\cdot 2^{2n-2}}{j_{2n}^2-2^{2n+2}}\right) &= \frac{4}{5};\\
\prod_{n=2}^{m}\left(1-\frac{D^2x^{2n-2}}{j_{2n-1}^2+4x^{2n-1}}\right)\left(1+\frac{D^2x^{2n-1}}{j_{2n}^2-4x^{2n}}\right) &= \frac{1}{x+1}\cdot\frac{J_{2m+1}}{J_{2m}};\\
\prod_{n=2}^{\infty}\left(1-\frac{D^2x^{2n-2}}{j_{2n-1}^2+4x^{2n-1}}\right)\left(1+\frac{D^2x^{2n-1}}{j_{2n}^2-4x^{2n}}\right) &= \frac{u(x)}{x+1};\\
\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n-1}^2+4}\right)\left(1+\frac{5}{L_{2n}^2-4}\right) &= \frac{\alpha}{2}; \qquad (2.9)\\
\prod_{n=2}^{\infty}\left(1-\frac{9\cdot 2^{2n-2}}{j_{2n-1}^2+2^{2n+1}}\right)\left(1+\frac{9\cdot 2^{2n-1}}{j_{2n}^2-2^{2n+2}}\right) &= \frac{2}{3}.
\end{aligned}$$

3. Products Involving Reciprocals of Jacobsthal-Lucas Polynomials

Using the relationship $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$, we now find the Jacobsthal-Lucas versions of formulas (1.4), (1.5), and (1.6).

3.1. **Jacobsthal-Lucas Version of Formula** (1.4)**.** Let $A = 1 - \frac{\Delta^2x^2}{l_{2n-1}^2}$. Replace x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the fractional expression with x^{2n-3}. We then get

$$\begin{aligned}
A &= 1-\frac{D^2}{x^2l_{2n-1}^2}\\
&= 1-\frac{D^2x^{2n-3}}{\left[x^{(2n-1)/2}l_{2n-1}\right]^2}\\
&= 1-\frac{D^2x^{2n-3}}{j_{2n-1}^2}\\
\text{LHS} &= \prod_{n=2}^{m}\left(1-\frac{D^2x^{2n-3}}{j_{2n-1}^2}\right),
\end{aligned}$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

With $B = \dfrac{x}{x^3+3x} \cdot \dfrac{l_{2m+1}}{l_{2m-1}}$, similarly we get

$$\begin{aligned} B &= \frac{x}{3x+1} \cdot \frac{l_{2m+1}}{l_{2m-1}} \\ &= \frac{x}{3x+1} \cdot \frac{x^{(2m+1)/2} l_{2m+1}}{x\left[x^{(2m-1)/2} l_{2m-1}\right]} \\ \text{RHS} &= \frac{1}{3x+1} \cdot \frac{j_{2m+1}}{j_{2m-1}}. \end{aligned}$$

Equating the two sides, we get

$$\prod_{n=2}^{m} \left(1 - \frac{D^2 x^{2n-3}}{j_{2n-1}^2}\right) = \frac{1}{3x+1} \cdot \frac{j_{2m+1}}{j_{2m-1}}. \tag{3.1}$$

This yields

$$\begin{aligned} \prod_{n=2}^{m} \left(1 - \frac{5}{L_{2n-1}^2}\right) &= \frac{L_{2m+1}}{4L_{2m-1}}; \qquad (3.2) \\ \prod_{n=2}^{m} \left(1 - \frac{9 \cdot 2^{2n-3}}{j_{2n-1}^2}\right) &= \frac{1}{7} \cdot \frac{j_{2m+1}}{j_{2m-1}}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{D^2 x^{2n-3}}{j_{2n-1}^2}\right) &= \frac{u^2(x)}{3x+1}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{9 \cdot 2^{2n-3}}{j_{2n-1}^2}\right) &= \frac{4}{7}. \end{aligned}$$

3.2. **Jacobsthal-Lucas Version of Formula** (1.5)**.** Using similar steps as before with $A = 1 + \dfrac{\Delta^2 x^2}{l_{2n}^2}$ and $B = \dfrac{x^2+2}{x^4+4x^2+2} \cdot \dfrac{l_{2m+2}}{l_{2m}}$, we get

$$\begin{aligned} A &= 1 + \frac{D^2}{x^2 l_{2n}^2} \\ &= 1 + \frac{D^2 x^{2n-2}}{\left[x^{(2n)/2} l_{2n}\right]^2} \\ &= 1 + \frac{D^2 x^{2n-2}}{j_{2n}^2} \\ \text{LHS} &= \prod_{n=2}^{m} \left(1 + \frac{D^2 x^{2n-2}}{j_{2n}^2}\right); \\ B &= \frac{(2x+1)x}{2x^2+4x+1} \cdot \frac{l_{2m+2}}{l_{2m}} \\ &= \frac{(2x+1)x}{2x^2+4x+1} \cdot \frac{x^{(2m+2)/2} l_{2m+2}}{x\left[x^{(2m)/2} l_{2m}\right]} \\ \text{RHS} &= \frac{2x+1}{2x^2+4x+1} \cdot \frac{j_{2m+2}}{j_{2m}}, \end{aligned}$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Combining the two sides, we get

$$\prod_{n=2}^{m}\left(1+\frac{D^2x^{2n-2}}{j_{2n}^2}\right)=\frac{2x+1}{2x^2+4x+1}\cdot\frac{j_{2m+2}}{j_{2m}}. \tag{3.3}$$

It then follows that

$$\begin{aligned}\prod_{n=2}^{m}\left(1+\frac{5}{L_{2n}^2}\right) &= \frac{3}{7}\cdot\frac{L_{2m+1}}{L_{2m-1}}; \\ \prod_{n=2}^{m}\left(1+\frac{9\cdot 2^{2n-2}}{j_{2n}^2}\right) &= \frac{5}{17}\cdot\frac{j_{2m+2}}{j_{2m}}; \\ \prod_{n=2}^{\infty}\left(1+\frac{D^2x^{2n-2}}{j_{2n}^2}\right) &= \frac{2x+1}{2x^2+4x+1}u^2(x); \\ \prod_{n=2}^{\infty}\left(1+\frac{9\cdot 2^{2n-2}}{j_{2n}^2}\right) &= \frac{20}{17}.\end{aligned} \tag{3.4}$$

Finally, we investigate the Jacobsthal-Lucas counterpart of formula (1.6).

3.3. **Jacobsthal-Lucas Version of Formula** (1.6)**.** With $A=\left(1+\frac{\Delta^2}{l_{2n-1}^2}\right)\left(1-\frac{\Delta^2}{l_{2n}^2}\right)$ and $B=\frac{x^2+2}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m}}$, using similar steps as before yields

$$\begin{aligned} A &= \left(1+\frac{D^2}{xl_{2n-1}^2}\right)\left(1-\frac{D^2}{xl_{2n}^2}\right) \\ &= \left(1+\frac{D^2x^{2n-2}}{\left[x^{(2n-1)/2}l_{2n-1}\right]^2}\right)\left(1-\frac{D^2x^{2n-1}}{\left[x^{(2n)/2}l_{2n}\right]^2}\right) \\ &= \left(1+\frac{D^2x^{2n-2}}{j_{2n-1}^2}\right)\left(1-\frac{D^2x^{2n-1}}{j_{2n}^2}\right) \\ \text{LHS} &= \prod_{n=2}^{m}\left(1+\frac{D^2x^{2n-2}}{j_{2n-1}^2}\right)\left(1-\frac{D^2x^{2n-1}}{j_{2n}^2}\right); \\ B &= \frac{(2x+1)\sqrt{x}}{3x+1}\cdot\frac{l_{2m+1}}{l_{2m}} \\ &= \frac{(2x+1)\sqrt{x}}{3x+1}\cdot\frac{x^{(2m+1)/2}l_{2m+1}}{\sqrt{x}\left[x^{(2m)/2}l_{2m}\right]} \\ \text{RHS} &= \frac{2x+1}{3x+1}\cdot\frac{j_{2m+1}}{j_{2m}}, \end{aligned}$$

where $l_n=l_n(1/\sqrt{x})$ and $j_n=j_n(x)$.

Equating the two sides yields

$$\prod_{n=2}^{m}\left(1+\frac{D^2x^{2n-2}}{j_{2n-1}^2}\right)\left(1-\frac{D^2x^{2n-1}}{j_{2n}^2}\right)=\frac{2x+1}{3x+1}\cdot\frac{j_{2m+1}}{j_{2m}}. \tag{3.5}$$

It then follows that

$$\prod_{n=2}^{m}\left(1+\frac{5}{L_{2n-1}^2}\right)\left(1-\frac{5}{L_{2n}^2}\right) = \frac{3}{4}\cdot\frac{L_{2m+1}}{L_{2m}}; \tag{3.6}$$

$$\prod_{n=2}^{m}\left(1+\frac{9\cdot 2^{2n-2}}{j_{2n-1}^2}\right)\left(1-\frac{9\cdot 2^{2n-1}}{j_{2n}^2}\right) = \frac{5}{7}\cdot\frac{j_{2m+1}}{j_{2m}};$$

$$\prod_{n=2}^{\infty}\left(1+\frac{D^2x^{2n-2}}{j_{2n-1}^2}\right)\left(1-\frac{D^2x^{2n-1}}{j_{2n}^2}\right) = \frac{2x+1}{3x+1}u(x);$$

$$\prod_{n=2}^{\infty}\left(1+\frac{9\cdot 2^{2n-2}}{j_{2n-1}^2}\right)\left(1-\frac{9\cdot 2^{2n-1}}{j_{2n}^2}\right) = \frac{10}{7}.$$

It follows by equations (3.2), (3.4), and (3.6) that [7]

$$\prod_{n=3}^{2m}\left(1-\frac{25}{L_n^4}\right) = \frac{9}{112}\cdot\frac{L_{2m+2}L_{2m+1}^2}{L_{2m}^2L_{2m-1}};$$

$$\prod_{n=3}^{\infty}\left(1-\frac{25}{L_n^4}\right) = \frac{9}{112}\alpha^5.$$

3.4. **Alternate Forms.** Using the identity $j_n^2 - D^2J_n^2 = 4(-x)^n$ [6], we can rewrite formulas (3.1), (3.3), and (3.5) also in a different way:

$$\prod_{n=2}^{m}\left(1-\frac{D^2x^{2n-3}}{D^2J_{2n-1}^2-4x^{2n-1}}\right) = \frac{1}{3x+1}\cdot\frac{j_{2m+1}}{j_{2m-1}};$$

$$\prod_{n=2}^{\infty}\left(1-\frac{D^2x^{2n-3}}{D^2J_{2n-1}^2-4x^{2n-1}}\right) = \frac{u^2(x)}{3x+1};$$

$$\prod_{n=2}^{\infty}\left(1-\frac{5}{5F_{2n-1}^2-4}\right) = \frac{\alpha^2}{4}; \tag{3.7}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{9\cdot 2^{2n-3}}{9J_{2n-1}^2-2^{2n+1}}\right) = \frac{4}{7};$$

$$\prod_{n=2}^{m}\left(1+\frac{D^2x^{2n-2}}{D^2J_{2n}^2+4x^{2n}}\right) = \frac{2x+1}{2x^2+4x+1}\cdot\frac{j_{2m+2}}{j_{2m}};$$

$$\prod_{n=2}^{\infty}\left(1+\frac{D^2x^{2n-2}}{D^2J_{2n}^2+4x^{2n}}\right) = \frac{2x+1}{2x^2+4x+1}u^2(x);$$

$$\prod_{n=2}^{\infty}\left(1+\frac{5}{5F_{2n}^2+4}\right) = \frac{3\alpha^2}{7}; \tag{3.8}$$

$$\prod_{n=2}^{\infty}\left(1+\frac{9\cdot 2^{2n-2}}{9J_{2n}^2+2^{2n+2}}\right) = \frac{20}{17}.$$

$$\begin{aligned}
\prod_{n=2}^{m}\left(1+\frac{D^2x^{2n-2}}{D^2J_{2n-1}^2-4x^{2n-1}}\right)\left(1-\frac{D^2x^{2n-1}}{D^2J_{2n}^2+4x^{2n}}\right) &= \frac{2x+1}{3x+1}\cdot\frac{j_{2m+1}}{j_{2m}};\\
\prod_{n=2}^{\infty}\left(1+\frac{D^2x^{2n-2}}{D^2J_{2n-1}^2-4x^{2n-1}}\right)\left(1-\frac{D^2x^{2n-1}}{D^2J_{2n}^2+4x^{2n}}\right) &= \frac{2x+1}{3x+1}u(x);\\
\prod_{n=2}^{\infty}\left(1+\frac{5}{5F_{2n-1}^2-4}\right)\left(1-\frac{5}{5F_{2n}^2+4}\right) &= \frac{3\alpha}{4}; \qquad (3.9)\\
\prod_{n=2}^{\infty}\left(1+\frac{9\cdot 2^{2n-2}}{9J_{2n-1}^2-2^{2n+1}}\right)\left(1-\frac{9\cdot 2^{2n-1}}{9J_{2n}^2+2^{2n+2}}\right) &= \frac{10}{7}.
\end{aligned}$$

An Interesting Consequence: It follows by equations (3.7), (3.8), and (3.9) that

$$\prod_{n=2}^{\infty}\left[1-\frac{25}{(5F_{2n-1}^2-4)^2}\right]\left[1-\frac{25}{(5F_{2n}^2+4)^2}\right]=\frac{9}{112}\alpha^5.$$

4. Conclusion

We can extract the Vieta and Chebyshev versions of formulas (1.1) through (1.6) using the gibonacci-Vieta and Vieta-Chebyshev relationships in Table 1, respectively. In the interest of brevity, we omit the details.

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