INFINITE SUMS INVOLVING JACOBSTHAL POLYNOMIAL PRODUCTS REVISITED

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ABSTRACT. Using graph-theoretic tools, we confirm five finite sums of Jacobsthal polynomial products and a Jacobsthal-Lucas version.

1. INTRODUCTION

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. They can also be defined by the *Binet-like* formulas

$$J_n(x) = \frac{u^n(x) - v^n(x)}{D} \text{ and } j_n(x) = u^n(x) + v^n(x),$$

where $D = \sqrt{4x+1}$, $2u(x) = 1 + D$, and $2v(x) = 1 - D$. It then follows that $\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = u(x)$ and $\lim_{n\to\infty} \frac{J_n}{j_n} = \frac{1}{D}$. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$ [2, 3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $\Delta = \sqrt{x^2+4}$, $2\alpha = 1 + \sqrt{5}$, and $2\beta = 1 - \sqrt{5}$, and omit a lot of basic algebra.

Table 1 showcases some fundamental Jacobsthal Identities [3]; we will use them in our discourse.

$J_{n+1} + xJ_{n-1}$	$=$	j_n	J_{2n}	$=$	$J_n j_n$
$J_{n+1}^2 + xJ_n^2$	$=$	J_{2n+1}	$J_{n+2} + x^2 J_{n-2}$	$=$	$(2x+1)J_n$
$j_{n+2} + x^2 j_{n-2}$	$=$	$(2x+1)j_n$	$(-x)^n J_{m-n}$	$=$	$J_m J_{n+1} - J_{m+1} J_n$
$J_{n+k} J_{n-k} - J_n^2$	$=$	$-(-x)^{n-k} J_k^2$	$j_{n+k} j_{n-k} - j_n^2$	$=$	$(-x)^{n-k} D^2 J_k^2$

Table 1: Some Fundamental Jacobsthal Identities

1.1. **Finite Sums of Jacobsthal Polynomial Products.** In [6], we established the following sums of Jacobsthal polynomial products:

$$\sum_{n=0}^{m} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} = \frac{J_{2m+2}}{J_{2m+1}}; \tag{1.1}$$

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}} = \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}}; \tag{1.2}$$

$$\frac{2x+1}{J_n^4 + (x-1)(-x)^{n-2}J_n^2 - x^{2n-3}} = \frac{1}{J_{n-2}J_{n-1}J_nJ_{n+1}} + \frac{x^2}{J_{n-1}J_nJ_{n+1}J_{n+2}}; \tag{1.3}$$

$$\sum_{n=0}^{m} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}} = \frac{J_{2m+2}}{(4x+1)J_{2m+1}}; \tag{1.4}$$

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{j_{2n+1}^2 + (2x+1)^2x^{2n-1}} = \frac{J_{4m+4}}{(4x+1)J_{2m+3}J_{2m+1}}. \tag{1.5}$$

1.2. **A Jacobsthal-Lucas Version.** In the proof of Theorem 4.1 in [5], we established that

$$\frac{x^2+2}{l_n^4 + (-1)^n(x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2} = \frac{1}{l_{n-2}l_{n-1}l_nl_{n+1}} + \frac{1}{l_{n-1}l_nl_{n+1}l_{n+2}}.$$

This, coupled with the relationship $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$ [3, 6], can be used to find the Jacobsthal-Lucas version of equation (1.3).

To this end, we let $A = \text{LHS}$ and $B = \text{RHS}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with x^{2n-3}, we get

$$\begin{aligned} A &= \frac{(2x+1)x^2}{x^3 l_n^4 - (-1)^n x(x-1)D^2 l_n^2 - D^4} \\ &= \frac{(2x+1)x^{2n-1}}{(x^{n/2}l_n)^4 - (-1)^n(x-1)x^{n-2}D^2(x^{n/2}l_n)^2 - D^4x^{2n-3}}; \\ \text{LHS} &= \frac{(2x+1)x^{2n-1}}{j_n^4 - (x-1)(-x)^{n-2}D^2 j_n^2 - D^4x^{2n-3}}, \end{aligned}$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Now, replace x with $1/\sqrt{x}$ in B, and then multiply each numerator and denominator with x^{2n+1}. This yields

$$\text{RHS} = \frac{x^{2n-1}}{j_{n-2}j_{n-1}j_nj_{n+1}} + \frac{x^{2n+1}}{j_{n-1}j_nj_{n+1}j_{n+2}},$$

where $j_n = j_n(x)$.

Equating the two sides, we get the Jacobsthal-Lucas version:

$$\frac{2x+1}{j_n^4 - (x-1)(-x)^{n-2}D^2 j_n^2 - D^4x^{2n-3}} = \frac{1}{j_{n-2}j_{n-1}j_nj_{n+1}} + \frac{x^2}{j_{n-1}j_nj_{n+1}j_{n+2}}. \tag{1.6}$$

Our objective is to confirm the six formulas using graph-theoretic techniques. To this end, we first summarize the needed tools.

2. Graph-theoretic Tools

Consider the *Jacobthal digraph* D in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [3, 4].

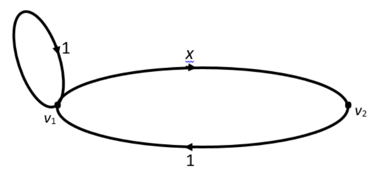

Figure 1. Weighted Fibonacci Digraph D

It follows by induction from its *weighted adjacency matrix* $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ that

$$M^n = \begin{bmatrix} J_{n+1} & xJ_n \\ J_n & xJ_{n-1} \end{bmatrix},$$

where $J_n = J_n(x)$ and $n \geq 1$ [3, 4].

The sum of the weights of closed walks of length n originating at v_1 in the digraph is J_{n+1} and that of those originating at v_2 is xJ_{n-1} [3, 4]. Consequently, the sum of the weights of all closed walks of length n in the digraph is $J_{n+1} + xJ_{n-1} = j_n$. These facts play a major role in the graph-theoretic proofs.

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then, the sum of the weights of the elements (a, b) in the product set $A \times B$ is *defined* as the product of the sums of weights from each component. This definition can be extended to any finite number of components [4].

With these tools at our disposal, we are ready for the graph-theoretic proofs. They hinge on the identities in Table 1.

3. Graph-theoretic Confirmations

3.1. **Confirmation of Identity** (1.1).

Proof. First, we will establish that

$$\sum_{n=1}^{m} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} = \frac{xJ_{2m}}{J_{2m+1}}.$$

Let A_n denote the sum of the weights of elements in the set A of closed walks of length $2n-1$ in the digraph originating at v_1, where $1 \leq n \leq m$. Then, the sum of the weights of the elements in the product set $A \times A$ is given by A_n^2. Let $S_n^* = A_n^2 + w^{2n-1}$, where $w =$ weight of edge v_1v_2. Let

$$S_m = \sum_{n=1}^{m} \frac{w^{2n-1}}{S_n^*} = \sum_{n=1}^{m} \frac{x^{2n-1}}{A_n^2 + x^{2n-1}}.$$

We will now compute S_m in a different way. Let w be an arbitrary walk in A. It can land at v_1 or v_2 at the $(n-1)$st step: $w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n-1} \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}$, where $v = v_1$ or v_2.

Table 2 shows the possible cases and the corresponding sums of the weights, where $J_n = J_n(x)$.

w lands at v_1 at the $(n-1)$st step?	w lands at v_1 at the $(2n-1)$st step?	sum of the weights of walks w
yes	yes	$J_n \cdot J_{n+1}$
no	yes	$xJ_{n-1} \cdot J_n$

Table 2: Sums of the Weights of Closed Walks Originating at v_1

It follows from the table that the sum A_n of the weights of walks in A is given by $A_n = J_n(J_{n+1} + xJ_{n-1}) = J_n j_n = J_{2n}$. So, $S_n^* = J_{2n}^2 + x^{2n-1}$, and hence,

$$S_m = \sum_{n=1}^{m} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}}.$$

Using the initial values

$$\begin{aligned} S_1 &= \frac{x}{x+1} = \frac{xJ_2}{J_3}; \\ S_2 &= \frac{x(2x+1)}{x^2+3x+1} = \frac{xJ_4}{J_5}; \text{ and} \\ S_3 &= \frac{x(3x^2+4x+1)}{x^3+6x^2+5x+1} = \frac{xJ_6}{J_7}, \end{aligned}$$

we conjecture that

$$\sum_{n=1}^{m} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} = \frac{xJ_{2m}}{J_{2m+1}}.$$

We will now confirm this using recursion [3, 5]. Let C_m = LHS and D_m = RHS. Then,

$$\begin{aligned} D_m - D_{m-1} &= \frac{xJ_{2m}}{J_{2m+1}} - \frac{xJ_{2m-2}}{J_{2m-1}} \\ &= \frac{x(J_{2m}J_{2m-1} - J_{2m+1}J_{2m-2})}{J_{2m+1}J_{2m-1}} \\ &= \frac{x(-x)^{2m-2}J_{2m-(2m-2)}}{J_{2m}^2 - (-x)^{2m-1}} \\ &= \frac{x^{2m-1}}{J_{2m}^2 + x^{2m-1}} \\ &= C_m - C_{m-1}. \end{aligned}$$

So, $C_m - D_m = C_{m-1} - D_{m-1} = \cdots = C_1 - D_1 = \frac{x}{x+1} - \frac{x}{x+1} = 0$, and hence, $C_m = D_m$, as expected. Thus, the conjecture is true for $m \geq 1$.

Letting $n = 0$, this yields

$$\begin{aligned} \sum_{n=0}^{m} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} &= \frac{xJ_{2m}}{J_{2m+1}} + 1 \\ &= \frac{J_{2m+2}}{J_{2m+1}}, \end{aligned}$$

as desired. □

It then follows that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n}^2 + 1} = \frac{1+\sqrt{5}}{2},$$

as in [6, 8]. It also yields

$$\sum_{n=0}^{\infty} \frac{2^{2n-1}}{J_{2n}^2 + 2^{2n-1}} = 2.$$

3.2. **Confirmation of Identity** (1.2).

Proof. Let B_n denote the sum of the weights of elements in the set B of closed walks of length $2n$ originating at v_1, where $n \geq 1$. Then, the sum of the weights of the elements in the product set $B \times B$ is given by B_n^2. Let w = weight of the edge v_1v_2, $z = 2w+1$, $S_n^* = B_n^2 + w^{2n-1}$, and

$$S_m^* = \sum_{n=1}^{m} \frac{zw^{2n-1}}{S_n^*} = \sum_{n=1}^{m} \frac{(2x+1)x^{2n-1}}{B_n^2 + x^{2n-1}}.$$

We will now compute B_n and hence S_m^* in a different way. Let w be an arbitrary walk in B. It can land at v_1 or v_2 at the nth step: $w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n} \; \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}$, where $v = v_1$ or v_2.

Table 3 implies that the sum B_n of the weights of walks w in B is given by $B_n = J_{n+1}^2 + xJ_n^2 = J_{2n+1}$.

w lands at v_1 at the nth step?	w lands at v_1 at the $(2n)$th step?	sum of the weights of walks w
yes	yes	$J_{n+1} \cdot J_{n+1}$
no	yes	$xJ_n \cdot J_n$

Table 3: Sums of the Weights of Closed Walks Originating at v_1

So, $S_n^* = J_{2n+1}^2 + x^{2n-1}$. Then,

$$S_m^* = \sum_{n=1}^{m} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}}.$$

Next, we let

$$\begin{aligned} S'_m &= S_m^* + \frac{2x+1}{x+1} \\ &= \sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}}. \end{aligned}$$

With the initial values

$$\begin{aligned} S'_0 &= \frac{2x+1}{x+1} = \frac{J_4}{J_3J_1}; \\ S'_1 &= \frac{(2x+1)(2x^2+4x+1)}{(x^2+3x+1)(x+1)} = \frac{J_8}{J_5J_3}; \text{ and} \\ S'_2 &= \frac{(3x^2+4x+1)(2x^3+9x^2+6x+1)}{(x^3+6x^2+5x+1)(x^2+3x+1)} = \frac{J_{12}}{J_7J_5}, \end{aligned}$$

we conjecture that

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2k-1}}{J_{2n+1}^2 + x^{2n-1}} = \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}}.$$

We will now establish this using recursion [3, 5]. Let K_m = LHS and R_m = RHS. Then,

$$\begin{aligned}
R_m - R_{m-1} &= \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}} - \frac{J_{4m}}{J_{2m+1}J_{2m-1}} \\
&= \frac{J_{2m+2}(J_{2m+3} + xJ_{2m+1})}{J_{2m+3}J_{2m+1}} - \frac{J_{2m}(J_{2m+1} + xJ_{2m-1})}{J_{2m+1}J_{2m-1}} \\
&= \frac{J_{2m+3}(J_{2m+2}J_{2m-1} - J_{2m+1}J_{2m}) - xJ_{2m-1}(J_{2m+3}J_{2m} - J_{2m+2}J_{2m+1})}{J_{2m+3}J_{2m+1}J_{2m-1}} \\
&= \frac{x^{2m-1}J_{2m+3}J_2 - x(-x^{2m})J_{2m-1}J_2}{J_{2m+3}J_{2m+1}J_{2m-1}} \\
&= \frac{x^{2m-1}(J_{2m+3} + x^2J_{2m-1})}{J_{2m+3}J_{2m+1}J_{2m-1}} \\
&= \frac{(2x+1)x^{2m-1}}{J_{2m+3}J_{2m-1}} \\
&= \frac{(2x+1)x^{2m-1}}{J_{2m+1}^2 + x^{2m-1}} \\
&= K_m - K_{m-1}.
\end{aligned}$$

Consequently, $K_m - R_m = K_{m-1} - R_{m-1} = \cdots = K_0 - R_0 = \frac{2x+1}{x+1} - \frac{2x+1}{x+1} = 0$. So, $K_m = R_m$.

Thus

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}} = \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}},$$

as expected. □

It then follows that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2 + 1} = \frac{\sqrt{5}}{3},$$

as in [6, 8]. Additionally, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}} &= D; \\
\sum_{n=0}^{\infty} \frac{2^{2n-1}}{J_{2n+1}^2 + 2^{2n-1}} &= \frac{3}{5}.
\end{aligned}$$

Next, we pursue the sum in equation (1.3).

3.3. **Confirmation of Identity** (1.3).

Proof. Let A_n denote the sum of the weights of closed walks of length n originating at v_1. Let $S_1 = A_{n-3}A_{n-2}A_{n-1}A_n$, $S_2 = A_{n-2}A_{n-1}A_nA_{n+1}$, and $S = \frac{1}{S_1} + \frac{w^2}{S_2}$, where w = weight of edge v_1v_2. Because $A_n = J_{n+1}$, we then have

$$\begin{aligned}
S &= \frac{1}{A_{n-3}A_{n-2}A_{n-1}A_n} + \frac{x^2}{A_{n-2}A_{n-1}A_nA_{n+1}} \\
&= \frac{1}{J_{n-2}J_{n-1}J_nJ_{n+1}} + \frac{x^2}{J_{n-1}J_nJ_{n+1}J_{n+2}}.
\end{aligned}$$

Now, let $T_n = A_{n-3}A_{n-2}A_{n-1}A_nA_{n+1}$. Using the identities $J_{n+2} + x^2J_{n-2} = (2x+1)J_n$ and $J_{n+k}J_{n-k} - J_n^2 = -(-x)^{n-k}J_k^2$, we then get

$$\begin{aligned} S &= \frac{A_{n+1}}{A_{n-3}A_{n-2}A_{n-1}A_nA_{n+1}} + \frac{x^2A_{n-3}}{A_{n-3}A_{n-2}A_{n-1}A_nA_{n+1}} \\ &= \frac{A_{n+1}}{T_n} + \frac{x^2A_{n-3}}{T_n} \\ &= \frac{J_{n+2} + x^2J_{n-2}}{J_{n-2}J_{n-1}J_nJ_{n+1}J_{n+2}} \\ &= \frac{(2x+1)J_n}{J_{n-2}J_{n-1}J_nJ_{n+1}J_{n+2}} \\ &= \frac{2x+1}{(J_{n+2}J_{n-2})(J_{n+1}J_{n-1})} \\ &= \frac{2x+1}{[J_n^2 - (-x)^{n-2}][J_n^2 - (-x)^{n-1}]} \\ &= \frac{2x+1}{J_n^4 + (x-1)(-x)^{n-2}J_n^2 - x^{2n-3}}. \end{aligned}$$

Equating the two values of S yields the desired result. □

It follows from this result that

$$\begin{aligned} \frac{3}{F_n^4 - 1} &= \frac{1}{F_{n-2}F_{n-1}F_nF_{n+1}} + \frac{1}{F_{n-1}F_nF_{n+1}F_{n+2}}; \qquad (3.1) \\ \frac{5}{J_n^4 + (-2)^{n-2}J_n^2 - 2^{2n-3}} &= \frac{1}{J_{n-3}J_{n-2}J_{n-1}J_n} + \frac{4}{J_{n-2}J_{n-1}J_nJ_{n+1}}. \end{aligned}$$

Equation (3.1) has an interesting byproduct. Using equations (2.7) and (2.8) in [5], we get

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{2}{F_{n-2}F_{n-1}F_nF_{n+1}} &= \frac{7}{2} + 5\beta; \\ \sum_{n=3}^{\infty} \frac{2}{F_{n-1}F_nF_{n+1}F_{n+2}} &= \frac{7}{2} + 5\beta - \frac{1}{3}; \\ \sum_{n=3}^{\infty} \frac{1}{F_n^4 - 1} &= \frac{35}{18} - \frac{5\sqrt{5}}{6}, \end{aligned}$$

as in [5, 8].

Next, we explore the Jacobsthal-Lucas sums in equations (1.4) through (1.6).

3.4. **Confirmation of Identity** (1.4).

Proof. Let A_n denote the sum of the weights of the elements in the set A of all closed walks of length $2n$ and w the weight of the edge v_1v_2, where $n \geq 1$. Then, the sum of the weights of the elements in the product set $A \times A$ is A_n^2. Let $S_n = A_n^2 + w^{2n-1}$, and

$$S_m = \sum_{n=1}^{m} \frac{w^{2n-1}}{S_n} = \sum_{n=1}^{m} \frac{x^{2n-1}}{A_n^2 + x^{2n-1}}.$$

We will now compute A_n and hence S_m in a different way. Let w be an arbitrary element in A.

Case 1. Suppose w originates at v_1. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 4 that the sum of the weights of such walks w is $J_{n+1}^2 + xJ_n^2 = J_{2n+1}$.

w lands at v_1 at the nth step?	w lands at v_1 at the $(2n)$th step?	sum of the weights of walks w
yes	yes	$J_{n+1}J_{n+1}$
no	yes	$xJ_n \cdot J_n$

Table 4: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. Then, also it can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

Table 5 implies that the sum of the weights of such walks w is $xJ_n^2 + x^2J_{n-1}^2 = xJ_{2n-1}$.

w lands at v_1 at the nth step?	w lands at v_2 at the $(2n)$th step?	sum of the weights of walks w
yes	yes	$J_n \cdot xJ_n$
no	yes	$xJ_{n-1} \cdot xJ_{n-1}$

Table 5: Sums of the Weights of Closed Walks Originating at v_2

Thus, the sum B_n of the weights of all closed walks w is given by $A_n = J_{2n+1} + xJ_{2n-1} = j_{2n}$. So,

$$S_m = \sum_{n=1}^{m} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}}.$$

For convenience, we let

$$\begin{aligned} S_m^* &= S_m + \frac{1}{4x+1} \\ &= \sum_{n=0}^{m} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}}. \end{aligned}$$

Then,

$$\begin{aligned} S_0^* &= \frac{1}{D^2} = \frac{J_2}{D^2J_1}; \\ S_1^* &= \frac{2x+1}{D^2(x+1)} = \frac{J_4}{D^2J_3}; \text{ and} \\ S_2^* &= \frac{3x^2+4x+1}{D^2(x^2+3x+1)} = \frac{J_6}{D^2J_5}. \end{aligned}$$

Using these initial values of S_m^*, we conjecture that

$$\sum_{n=0}^{m} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}} = \frac{J_{2m+2}}{D^2J_{2m+1}}.$$

We will now establish its validity by recursion [3, 5]. Let K_m and R_m denote the LHS and RHS of this equation, respectively. Using the identities $-(-x)^{n-1}J_{m-n} = J_mJ_{n-1} - J_{m-1}J_n$

and $j_n^2 - D^2J_n^2 = 4(-x)^n$, we then have

$$\begin{aligned} R_m - R_{m-1} &= \frac{J_{2m+2}}{D^2J_{2m+1}} - \frac{J_{2m}}{D^2J_{2m-1}} \\ &= \frac{J_{2m+2}J_{2m-1} - J_{2m+1}J_{2m}}{D^2J_{2m+1}J_{2m-1}} \\ &= \frac{x^{2m-1}J_{(2m+1)-(2m-1)}}{D^2\left(J_{2m}^2 + x^{2m-1}\right)} \\ &= \frac{x^{2m-1}}{\left(j_{2m}^2 - 4x^{2m}\right) + (4x+1)x^{2m-1}} \\ &= \frac{x^{2m-1}}{j_{2m}^2 + x^{2m-1}} \\ &= K_m - K_{m-1}. \end{aligned}$$

Then, $K_m - R_m = K_{m-1} - R_{m-1} = \cdots = K_0 - R_0 = \dfrac{1}{D^2} - \dfrac{1}{D^2} = 0$. So, $K_m = R_m$.

Thus, the conjecture is true and hence, the desired result holds. □

It then follows that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}^2 + 1} = \frac{1+\sqrt{5}}{10},$$

as in [5]. It also yields

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}} &= \frac{u(x)}{D^2}; \\ \sum_{n=0}^{\infty} \frac{2^{2n-1}}{j_{2n}^2 + 2^{2n-1}} &= \frac{2}{9}. \end{aligned}$$

We will now confirm equation (1.5).

3.5. **Confirmation of Identity** (1.5).

Proof. Let B_n denote the sum of the weights of elements in the set B of all closed walks of length $2n+1$ in the digraph, where $0 \le n \le m$. So, the sum of the weights of the elements in the product set $B \times B$ is B_n^2. Let w = weight of edge v_1v_2, $z = 2w+1$, $S_n^* = B_n^2 + z^2w^{2n-1}$, and

$$S_m^* = \sum_{n=0}^{m} \frac{zw^{2n-1}}{S_n^*} = \sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{B_n^2 + (2x+1)^2x^{2n-1}}.$$

We will now compute S_m^* in a different way. To this end, we let w be an arbitrary walk in B.

Case 1. Suppose w originates at v_1. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n}\ \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n+1}, \text{ where } v = v_1 \text{ or } v_2.$$

Table 6 implies that the sum of the weights of such walks w is given by $J_{n+1}(J_{n+2} + xJ_n) = J_{n+1}j_{n+1} = J_{2n+2}$.

w lands at v_1 at the nth step?	w lands at v_1 at the $(2n+1)$st step?	sum of the weights of walks w
yes	yes	$J_{n+1}J_{n+2}$
no	yes	$xJ_n \cdot J_{n+1}$

Table 6: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. Then, also w can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n} \; \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n+1}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 7 that the sum of the weights of such walks is $xJ_n(J_{n+1} + xJ_{n-1}) = xJ_n j_n = xJ_{2n}$.

w lands at v_1 at the nth step?	w lands at v_2 at the $(2n+1)$st step?	sum of the weights of walks w
yes	yes	$J_n \cdot xJ_{n+1}$
no	yes	$xJ_{n-1} \cdot xJ_n$

Table 7: Sums of the Weights of Closed Walks Originating at v_1

Combining the two cases, we get $B_n = J_{2n+2} + xJ_{2n} = j_{2n+1}$. Consequently,

$$S_m^* = \sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{j_{2n+1}^2 + (2x+1)^2 x^{2n-1}}.$$

This yields

$$\begin{aligned} S_0^* &= \frac{2x+1}{D^2(x+1)} = \frac{J_4}{D^2 J_3 J_1}; \\ S_1^* &= \frac{(2x+1)(2x^2+4x+1)}{D^2(x^2+3x+1)(x+1)} = \frac{J_8}{D^2 J_5 J_3}; \text{ and} \\ S_2^* &= \frac{(3x^2+4x+1)(2x^3+9x^2+6x+1)}{D^2(x^3+6x^2+5x+1)(x^2+3x+1)} = \frac{J_{12}}{D^2 J_7 J_5}. \end{aligned}$$

Based on these initial values of S_m^*, we conjecture that

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{j_{2n+1}^2 + (2x+1)^2 x^{2n-1}} = \frac{J_{4m+4}}{D^2 J_{2m+3} J_{2m+1}}.$$

We can confirm this using recursion, as in [5].

Equating now the two values of S_m^* yields the desired result. □

This result implies that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n+1}^2 + 9} = \frac{\sqrt{5}}{15},$$

as in [5]. In addition, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(2x+1)x^{2n-1}}{j_{2n+1}^2 + (2x+1)^2 x^{2n-1}} &= \frac{1}{D}; \\ \sum_{n=0}^{\infty} \frac{2^{2n-1}}{j_{2n+1}^2 + 25 \cdot 2^{2n-1}} &= \frac{1}{15}. \end{aligned}$$

Finally, we confirm equation (1.6).

3.6. **Confirmation of Identity** (1.6)**.**
Proof. Let C_n denote the sum of the weights of all closed walks of length n in the digraph. We also let $S_1 = C_{n-2}C_{n-1}C_nC_{n+1}$, $S_2 = C_{n-1}C_nC_{n+1}C_{n+2}$, and $S = \frac{1}{S_1} + \frac{w^2}{S_2}$, where $w =$ weight of edge v_1v_2. Because $C_n = J_{n+1} + xJ_{n-1} = j_n$, we get

$$\begin{aligned} S &= \frac{1}{C_{n-2}C_{n-1}C_nC_{n+1}} + \frac{x^2}{C_{n-1}C_nC_{n+1}C_{n+2}} \\ &= \frac{1}{j_{n-2}j_{n-1}j_nj_{n+1}} + \frac{x^2}{j_{n-1}j_nj_{n+1}j_{n+2}}. \end{aligned}$$

We will now compute S in a different way. Let $T_n = C_{n-2}C_{n-1}C_nC_{n+1}C_{n+2}$. Using the identities $j_{n+2} + x^2 j_{n-2} = (2x+1)j_n$ and $j_{n+k}j_{n-k} - j_n^2 = (-x)^{n-k}D^2J_k^2$, we then have

$$\begin{aligned} S &= \frac{C_{n+2}}{C_{n-2}C_{n-1}C_nC_{n+1}C_{n+2}} + \frac{x^2C_{n-2}}{C_{n-2}C_{n-1}C_nC_{n+1}C_{n+2}} \\ &= \frac{C_{n+2} + x^2C_{n-2}}{T_n} \\ &= \frac{j_{n+2} + x^2 j_{n-2}}{j_{n-2}j_{n-1}j_nj_{n+1}j_{n+2}} \\ &= \frac{(2x+1)j_n}{j_{n-2}j_{n-1}j_nj_{n+1}j_{n+2}} \\ &= \frac{2x+1}{(j_{n+2}j_{n-2})(j_{n+1}j_{n-1})} \\ &= \frac{2x+1}{[j_n^2 + (-x)^{n-2}D^2][j_n^2 + (-x)^{n-1}D^2]} \\ &= \frac{2x+1}{j_n^4 - (x-1)(-x)^{n-2}D^2j_n^2 - D^4x^{2n-3}}. \end{aligned}$$

This value of S, coupled with the earlier one, gives the desired result. □

In particular, we then get

$$\frac{3}{L_n^4 - 25} = \frac{1}{L_{n-2}L_{n-1}L_nL_{n+1}} + \frac{1}{L_{n-1}L_nL_{n+1}L_{n+2}}; \tag{3.2}$$

$$\frac{5}{j_n^4 - 9(-2)^{n-2}j_n^2 - 81 \cdot 2^{2n-3}} = \frac{1}{j_{n-2}j_{n-1}j_nj_{n+1}} + \frac{4}{j_{n-1}j_nj_{n+1}j_{n+2}}.$$

Using equation (4.6) in [5], equation (3.2) yields

$$\sum_{n=3}^{\infty} \frac{1}{L_n^4 - 25} = \frac{5}{63} - \frac{\sqrt{5}}{30},$$

as in [6, 7, 9].

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