ON THE FAMILY OF DIOPHANTINE PAIRS $\{P_{2k}, 2P_{2k+2}\}$

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Abstract. Let $k \geq 1$ be an integer and let P_k and Q_k be the kth Pell number and kth Pell-Lucas number, respectively. In this paper, we prove that if d is a positive integer such that

$$\{P_{2k}, P_{2k+2}, 2P_{2k+2}, d\}$$

is a Diophantine quadruple, then $d = P_{2k+1}Q_{2k+1}Q_{2k+2}$. We deduce that the pair $\{P_{2k}, 2P_{2k+2}\}$ cannot be extended to an irregular Diophantine quadruple.

1. Introduction

For a nonzero integer n, a set of m distinct positive integers $\{a_1, \dots, a_m\}$ such that $a_ia_j + n$ is a perfect square for all $1 \leq i < j \leq m$, is called a Diophantine m-tuple with property $D(n)$ or a $D(n)$-m-tuple (or a P_n-set of size m). Diophantus was the first who considered the problem of finding such sets in the case $n = 1$. Particularly, he found the set of four positive rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with property $D(1)$. However, the first integral $D(1)$-quadruple was found by Fermat and it was the set $\{1, 3, 8, 120\}$. Moreover, Baker and Davenport [1] proved that the set $\{1, 3, 8, 120\}$ cannot be extended to a $D(1)$-quintuple. Several generalizations of the result of Baker and Davenport were obtained. In 1997, Dujella [5] proved that $D(1)$-triples of the form $\{k - 1, k + 1, 4k\}$ for $k \geq 2$ cannot be extended to a $D(1)$-quintuple. In 1998, Dujella and Pethő [8] proved that the $D(1)$-pair $\{1, 3\}$ cannot be extended to a $D(1)$-quintuple. In 2008, Fujita obtained a more general result by proving that the $D(1)$-pairs of the form $\{k-1, k+1\}$ for $k \geq 2$ cannot be extended to a $D(1)$-quintuple. A folklore conjecture is that there does not exist a Diophantine quintuple. In 2004, Dujella [7] proved that there are only finitely many $D(1)$-quintuples. This conjecture was proved by He, the third author, and Ziegler [11]. For

$$d = d_{\pm} = a + b + c + 2abc \pm 2\sqrt{(ab + 1)(ac + 1)(bc + 1)},$$

both sets $\{a, b, c, d_+\}$ and $\{a, b, c, d_-\}$ are Diophantine quadruples provided $d_- \neq 0$. Such quadruples are said to be regular. We state a stronger version and still open conjecture.

Conjecture 1.1. *If $\{a, b, c, d\}$ is a Diophantine quadruple such that $a < b < c < d$, then $d = d_+$.*

To prove this conjecture, Fujita and Miyazaki [9] recently proved that any fixed Diophantine triple can only be extended to a Diophantine quadruple in at most 11 ways by joining a fourth element exceeding the maximal element in the triple, while Cipu, Fujita, and Miyazaki [3] improved this result by replacing 11 by 8. For other results concerning Diophantine m-tuples and their generalizations, we refer the interested reader to the homepage of Dujella [4].

The next two propositions are important results, giving us an idea on the extendibility of Diophantine pairs under some precise conditions. For a fixed Diophantine triple $\{a, b, c\}$ with $ab+1 = r^2$, where r is positive integer, we denote by N the number of positive integers $d > d_+$ such that $\{a, b, c, d\}$ is a Diophantine quadruple. We have the following propositions.

Proposition 1. (See [3, Proposition 1.5]) *Let $\{a, b\}$ be a Diophantine pair with $a < b$. Define an integer c_ν^τ by*

$$c_\nu^\tau = \frac{1}{4ab}\left[(\sqrt{b} + \tau\sqrt{a})^2(r + \sqrt{ab})^{2\nu} + (\sqrt{b} - \tau\sqrt{a})^2(r - \sqrt{ab})^{2\nu} - 2(a + b)\right] \quad (1)$$

with ν a positive integer and $\tau \in \{-, +\}$.

1. *If $c = c_1^\tau$ for some τ, then $N \le 3$.*
2. *If $c_2^- \le c \le c_3^+$, then $N \le 6$.*
3. *If $c \ge c_4^-$, then $N = 0$.*
4. *If $c = c_3^\tau$ for some τ and $b < a^2$, then $N = 0$.*

Proposition 2. (See [3, Corollary 1.6]) *Let $\{a, b, c\}$ be a Diophantine triple with $a < b \le 13a$. Then, $N \le 6$.*

Recall that the Pell and Pell-Lucas numbers are respectively given by

$$P_0 = 0, \quad P_1 = 1, \quad P_{k+2} = 2P_{k+1} + P_k \quad \text{for} \quad k \ge 0,$$

and

$$Q_0 = 2, \quad Q_1 = 2, \quad Q_{k+2} = 2Q_{k+1} + Q_k \quad \text{for} \quad k \ge 0.$$

Let $a = P_{2k}$ and $b = 2P_{2k+2}$. We have $a < b \le 13a$. For this Diophantine pair, using Proposition 1 and Proposition 2, it is enough to consider the extensions of $D(1)$-triples of the forms $\{P_{2k}, 2P_{2k+2}, c\}$, where c_1^-, c_1^+, c_2^-, c_2^+, c_3^-, c_3^+, as all possible c's are given by equation (1). Also, note that from equation (1), we see that $c_1^- = P_{2k+2} < b$. But, in the other cases we have $b < c$. Thus, we have to study the extensibility of the Diophantine triples

$$\{P_{2k}, P_{2k+2}, 2P_{2k+2}\},\ \{P_{2k}, 2P_{2k+2}, c\},\ \text{with } c \in \{c_1^+, c_2^-, c_2^+, c_3^-, c_3^+\}.$$

In this paper, we prove the following results.

Theorem 1. *Let k be a positive integer. If d is a positive integer such that*

$$\{P_{2k}, P_{2k+2}, 2P_{2k+2}, d\}$$

is a Diophantine quadruple with $d > 2P_{2k+2}$, then

$$d = P_{2k+1}Q_{2k+1}Q_{2k+2},$$

where P_k and Q_k are respectively the kth Pell and Pell-Lucas numbers.

Theorem 2. *Let $c \in \{c_1^+, c_2^-, c_2^+, c_3^-, c_3^+\}$ and d be a positive integer. If $\{P_{2k}, 2P_{2k+2}, c, d\}$ is a Diophantine quadruple with $d > c$, then $d = d_+$.*

Taking into account the observations mentioned above, Theorem 1 and Theorem 2 allow us to deduce the following statement.

Corollary 1. *Let $k \ge 1$ be an integer. Any Diophantine quadruple that contains the pair $\{P_{2k}, 2P_{2k+2}\}$ is regular.*

The purpose of this paper is to prove the uniqueness of the extensions of $D(1)$-triples $\{P_{2k}, 2P_{2k+2}, c\}$, where $c \in \{c_1^-, c_1^+, c_2^-, c_2^+, c_3^-, c_3^+\}$. To do this, we will use the standard method for solving finitely many Diophantine equations $z = v_m = w_n$. To get a lower bound for the indices, we will consider the congruence method or obtain an upper bound using linear forms in logarithms.

The organization of this paper is as follows. In Section 2, we will recall or prove some useful results. In Section 3, we will use some results of linear forms in three logarithms. In Section 4, we use the Baker-Davenport reduction method to prove Theorem 1. Section 5 of the paper will be devoted to the proof of Theorem 2 in the case where $c = c_1^+$ and we will finish in Subsection 5.2, where we will discuss the cases $c = c_2^{\pm}$ and $c = c_3^{\pm}$.

2. Some Useful Lemmas

In this section, we recall or prove some useful lemmas that will be used to prove Theorem 1. Here, we will keep the following notation: $a = P_{2k}$, $b = P_{2k+2}$, $c = 2P_{2k+2}$, and let r, s, t be positive integers defined by

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2. \tag{2}$$

To extend the Diophantine triple $\{a, b, c\}$ to a Diophantine quadruple $\{a, b, c, d\}$, we have to solve the system

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2. \tag{3}$$

If we eliminate d, we obtain the following system of Pellian equations:

$$az^2 - cx^2 = a - c, \tag{4}$$

$$bz^2 - cy^2 = b - c. \tag{5}$$

Moreover, by [7, Lemma 1] or the arguments of Nagell [13, Theorem 108a], the positive solutions of Diophantine equations (4) and (5) are respectively given by

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m, \tag{6}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n, \tag{7}$$

where m, n are nonnegative integers, and (z_0, x_0), (z_1, y_1) are solutions of (4) and (5), respectively, satisfying

$$1 \le x_0 < \sqrt{\frac{s+1}{2}}, \quad 1 \le |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}, \tag{8}$$

$$1 \le y_1 < \sqrt{\frac{t+1}{2}}, \quad 1 \le |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}. \tag{9}$$

In any case, we have $z = v_m = w_n$, where

$$v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m, \tag{10}$$

$$w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n. \tag{11}$$

The initial terms z_0 and z_1 are almost completely determined in the following lemma.

Lemma 1. (See [3, Theorem 2.1])) *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d$, and that $z = v_m = w_n$ has a solution for some integers m and n. Then, one of the following four cases holds:*

(1) *m and n are even with $z_0 = z_1$ and $(|z_0|, x_0; |z_1|, y_1) \in \{(1, 1; 1, 1), (cr - st, rs - at; cr - st, rt - bs)\}$.*
(2) *m is odd and n is even with $(|z_0|, x_0; |z_1|, y_1) = (t, r; cr - st, rt - bs)$ and $z_0 z_1 < 0$.*
(3) *m is even and n is odd with $(|z_0|, x_0; |z_1|, y_1) = (cr - st, rs - at; s, r)$ and $z_0 z_1 < 0$.*
(4) *m and n are odd with $(|z_0|, x_0; |z_1|, y_1) = (t, r; s, r)$ and $z_0 z_1 > 0$.*

Moreover, if $d > d_+$, then case (2) cannot occur.

Denote by $\{v_{z_0,m}\}$ the sequence $\{v_m\}$ with the initial term z_0, and by $\{w_{z_1,n}\}$ the sequence $\{w_n\}$ with the initial term z_1.

Lemma 2. (See [3, Lemma 2.3])) *$v_{cr-st,m} = v_{-t,m+1}$, $v_{st-cr,m+1} = v_{t,m}$ for all $m \ge 0$ and $w_{cr-st,n} = w_{-s,n+1}$, $w_{st-cr,n+1} = w_{s,n}$ for all $n \ge 0$.*

When we refer to the above two lemmas, it is enough to observe the following two cases:
Case 1. If $v_{2m} = w_{2n}$, then $z_0 = z_1 = \pm 1$ and $x_0 = y_1 = 1$.
Case 2. If $v_{2m+1} = w_{2n+1}$, then $z_0 = \pm t$, $z_1 = \pm s$, $x_0 = y_1 = r$, and $z_0 z_1 > 0$.

Also, we obtain

$$r = P_{2k+1}, \quad s = \frac{1}{2}Q_{2k+1}, \quad \text{and} \quad t = \frac{1}{2}Q_{2k+2}.$$

As mentioned above, we need to solve the system of generalized Pell equations

$$P_{2k}z^2 - 2P_{2k+2}x^2 = P_{2k} - 2P_{2k+2}, \tag{12}$$

$$P_{2k+2}z^2 - 2P_{2k+2}y^2 = P_{2k+2} - 2P_{2k+2}. \tag{13}$$

In [7], Dujella proved an unconditional relation between the indices m and n, which we state in the following lemma.

Lemma 3. (See [7, Lemma 3]) *If $v_m = w_n$, then $n - 1 \leq m \leq 2n + 1$.*

To get a gap principle between indices m, n, and k, we recall the following lemma.

Lemma 4. (See [6, Lemma 4]) *We have*

$$\begin{aligned} v_{2m} &\equiv z_0 + 2c(az_0m^2 + sx_0m) && \pmod{8c^2}, \\ v_{2m+1} &\equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] && \pmod{4c^2}, \\ w_{2n} &\equiv z_1 + 2c(bz_1n^2 + ty_1n) && \pmod{8c^2}, \\ w_{2n+1} &\equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] && \pmod{4c^2}. \end{aligned}$$

The next results will help us to determine a lower bound of m depending on k. So, in Case 1, we will prove the following result.

Lemma 5. *If $v_{2m} = w_{2n}$ has a solution for $m, n \geq 2$ and $z_0 = z_1 = \pm 1$, then*

$$m \geq \frac{\sqrt{1 + 2P_{2k+2}} - 1}{2}.$$

Proof. Using Lemma 4, we have

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}. \tag{14}$$

In our case, the congruence becomes

$$\pm P_{2k}m^2 + \frac{1}{2}Q_{2k+1}m \equiv \pm P_{2k+2}n^2 + \frac{1}{2}Q_{2k+2}n \pmod{2P_{2k+2}}.$$

It follows that

$$\pm P_{2k}m^2 + \frac{1}{2}Q_{2k+1}m \equiv \frac{1}{2}Q_{2k+2}n \pmod{P_{2k+2}}.$$

As $Q_{2k+1} = 2P_{2k+2} - 2P_{2k+1} \equiv -2P_{2k+1} \pmod{P_{2k+2}}$, $P_{2k} \equiv -2P_{2k+1} \pmod{P_{2k+2}}$, and

$$Q_{2k+2} \equiv 2P_{2k+1} \pmod{P_{2k+2}},$$

we have

$$\pm P_{2k+1}(2m^2 \pm m \pm n) \equiv 0 \pmod{P_{2k+2}}.$$

Note that $\gcd(\pm P_{2k+1}, P_{2k+2}) = \gcd(P_{2k+1}, P_{2k+2}) = P_1 = 1$. Because P_{2k+1} and P_{2k+2} are relatively prime, we get

$$2m^2 \pm m \pm n \equiv 0 \pmod{P_{2k+2}}. \tag{15}$$

If $m, n \geq 2$, then

$$2m^2 + m + n \geq 2m^2 \pm m \pm n \geq 2m^2 - m - n > 0.$$

Hence, from (15), we obtain

$$2m^2 + m + n \geq P_{2k+2}.$$

By Lemma 3, we know that $m \geq n$. So we have $2m^2 + 2m - P_{2k+2} \geq 0$. This completes the proof of Lemma 5. □

Now considering Case 2, we get the following result.

Lemma 6. *If $v_{2m+1} = w_{2n+1}$ has a solution for $m, n \geq 1$ and $z_0 = \pm t$, $z_1 = \pm s$, with $z_0 z_1 > 0$, then*

$$m \geq \frac{\sqrt{9 + 8P_{2k+2}} - 3}{4}.$$

Proof. Considering the congruences modulo $4c^2$ in Lemma 4 for this case, we obtain

$$\pm astm(m+1) + rm \equiv \pm bstn(n+1) + rn \pmod{c}.$$

Because $st \equiv -1 \pmod{P_{2k+2}}$ and $a \equiv -2P_{2k+1} \pmod{P_{2k+2}}$, we deduce that

$$\pm P_{2k+1}[2m(m+1) \pm (m-n)] \equiv 0 \pmod{P_{2k+2}}.$$

As $\pm P_{2k+1}$ and P_{2k+2} are relatively prime, we see that

$$2m(m+1) \pm (m-n) \equiv 0 \pmod{P_{2k+2}}.$$

For $m, n \geq 1$, we have

$$2m(m+1) + (m-n) \geq 2m(m+1) \pm (m-n) \geq 2m(m+1) - m + n > 0.$$

Thus, we obtain $2m(m+1) + m - n \geq P_{2k+2}$. By Lemma 3, we know that $m \geq n$. Moreover, we have $m - n < m$ and hence, we get

$$2m^2 + 3m - P_{2k+2} \geq 0.$$

This completes the proof. □

3. Linear Forms in Three Logarithms

Using recurrences (10) and (11), we obtain

$$v_m = \frac{1}{2\sqrt{a}}\left[(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^m+(z_0\sqrt{a}-x_0\sqrt{c})(s-\sqrt{ac})^m\right], \tag{16}$$

$$w_n = \frac{1}{2\sqrt{b}}\left[(z_1\sqrt{b}+y_1\sqrt{c})(t+\sqrt{bc})^n+(z_1\sqrt{b}-y_1\sqrt{c})(t-\sqrt{bc})^n\right]. \tag{17}$$

In our case where $a = P_{2k}$, $b = P_{2k+2}$, and $c = 2P_{2k+2}$, we have $c = 2b$. Our idea is to use (16) and (17) to transform equation $v_m = w_n$ into an inequality for linear form in three logarithms of algebraic numbers. Recall the following lemma that will help us in that direction.

Lemma 7. [6, Lemma 5] *Assume that $c > 4b$. If $v_m = w_n$ and $m, n \neq 0$, then*

$$0 < m\log(s+\sqrt{ac}) - n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} < \frac{8}{3}ac(s+\sqrt{ac})^{-2m}.$$

Note that Dujella later proved the previous lemma (see (60) in [7]) by replacing the assumption $c > 4b$ by $c > b + \sqrt{c}$. In our particular case, we can easily verify that $c > b + \sqrt{c}$; we have everything ready for the application of the following famous result of Baker and Wüstholz [2].

Lemma 8. *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational integer coefficients $b_1, \dots, b_l$, we have*

$$\log \Lambda \geq -18(l+1)!\, l^{l+1}(32d)^{l+2} h'(\alpha_1)\dots h'(\alpha_l)\log(2ld)\log B,$$

where $B = \max(|b_1|, \dots, |b_l|)$, and where d is the degree of the number field generated by $\alpha_1, \dots, \alpha_l$.

Here,

$$h'(\alpha) = \max\left(h(\alpha), \frac{|\log \alpha|}{d}, \frac{1}{d}\right),$$

and $h(\alpha)$ denotes the standard Weil logarithmic height of α.

We apply Lemma 8 to the following linear form in three logarithms

$$\Lambda = m\log(s+\sqrt{ac}) - n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}. \tag{18}$$

Put

$$\alpha_1 = s+\sqrt{ac}, \quad \alpha_2 = t+\sqrt{bc}, \quad \text{and} \quad \alpha_3 = \frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}.$$

Considering Case 1, we have $l = 3$, $d = 4$, $B = 2m$. Note that in this case, we have $c = s + t$ and $c = 2b$, so we obtain

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log c, \quad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log(c+\sqrt{bc}) < 0.58415\log c.$$

Furthermore, α_3 is a root of

$$a^2(c-b)^2x^4 + 4a^2b(c-b)x^3 + 2ab(3ab-ac-bc-c^2)x^2 + 4ab^2(c-a)x + b^2(c-a)^2 = 0$$

and all these roots are given by

$$\frac{\sqrt{ab}+\sqrt{bc}}{\sqrt{ab}+\sqrt{ac}}, \quad \frac{-\sqrt{ab}+\sqrt{bc}}{-\sqrt{ab}+\sqrt{ac}}, \quad \frac{\sqrt{bc}+\sqrt{ab}}{-\sqrt{ac}+\sqrt{ab}}, \quad \text{and} \quad \frac{-\sqrt{bc}+\sqrt{ab}}{\sqrt{ac}+\sqrt{ab}}.$$

As the absolute values of the conjugates of α_3 are greater than 1 and the leading coefficient of the above polynomial is at least 16, we have

$$h(\alpha_3) < \frac{1}{4}\log\left[a^2(c-b)^2\cdot\frac{(bc-ab)^2}{(ac-ab)^2}\right] = \frac{1}{4}\log\left(b^2(c-a)^2\right) < \log c.$$

Thus, we see that $h'(\alpha_3) < \log c$. Using Lemma 8, we obtain

$$\log\Lambda \geq -1.117\cdot 10^{15}\log(2m)\cdot\log^3 c. \tag{19}$$

Lemma 7 and the inequality

$$\log\left(\frac{8}{3}ac\right) < 2\log(2\sqrt{ac})$$

show that

$$\log\Lambda \leq (1-2m)\log c. \tag{20}$$

Combining (19) and (20), we obtain

$$\frac{2m-1}{\log(2m)} < 1.117\cdot 10^{15}\log^2 c. \tag{21}$$

Using Lemma 5, we observe that $c < 8m^2$. Moreover, $\log c < 2.501\log(2m)$ for $m \geq 2$. Thus, we have the inequality

$$\frac{2m-1}{\log^3(2m)} < 7 \cdot 10^{15}. \tag{22}$$

Solving the inequality and using Lemma 5, we get a bound on m and k that we summarize in the following result.

Lemma 9. *We have $m < 4 \cdot 10^{20}$ and $1 \leq k \leq 53$.*

Considering Case 2, we have $l = 3$, $d = 4$, $B = 2m + 1$. Note that in this case, α_3 is a root of

$$a^2(c-b)^2x^4 + 4a^2b(c-b)stx^3 + (4a^2b^2(-2c^2 + (a+b)c + 3) \\ - 2abc(a+b+c))x^2 + 4ab^2st(c-a)x + b^2(c-a)^2.$$

Using $b = \frac{1}{2}c$, we see that

$$\begin{aligned} h(\alpha_3) &\leq \frac{1}{4}\left[\log(a^2(c-b)^2) + 4\log\left(\frac{\max\{|\sqrt{b}(r\sqrt{c} \pm t\sqrt{a})|\}}{\min\{|\sqrt{a}(r\sqrt{c} \pm s\sqrt{b})|\}}\right)\right] \\ &= \frac{1}{4}\left[\log(a^2(c-b)^2) + 4\log\left(\frac{\sqrt{b}(r\sqrt{c} + t\sqrt{a})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})}\right)\right] \\ &= \frac{1}{4}\log\left(\frac{b^2(r\sqrt{c} + t\sqrt{a})^4(r\sqrt{c} + s\sqrt{b})^4}{(c-b)^2}\right) < \log(4c^3) < 3\log(1.6c). \end{aligned}$$

Thus, $h'(\alpha_3) < 3\log(1.6c)$. Combining Lemma 6 and Lemma 8, we get the following inequality

$$\frac{m}{\log^3(2m+1)} < 1.103 \cdot 10^{16}. \tag{23}$$

Solving inequality (23) and using Lemma 6, we get a bound on m and k that we summarize in the following result.

Lemma 10. *We have $m < 1.324 \cdot 10^{21}$ and $1 \leq k \leq 55$.*

4. Proof of Theorem 1

The goal of this section is to give a proof of Theorem 1. In Case 1 from Lemma 9, we know that $m < 4 \cdot 10^{20}$ and $1 \leq k \leq 53$. To solve the problem for the remaining cases $1 \leq k \leq 53$, we will use a Diophantine approximation algorithm, the so-called Baker-Davenport reduction method. The following lemma is a slight modification of the original version of Baker-Davenport reduction method (See [8, Lemma 5a] or [10, Lemma 9]).

Lemma 11. *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 6M$ and let*

$$\eta = \| \mu q \| - M \cdot \| \kappa q \|,$$

where $\| \cdot \|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\eta)}{\log B} \leq m \leq M.$$

Then, we apply Lemma 11 with

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_3}{2\log \alpha_2}, \quad A = \frac{4ac}{3\log \alpha_2}, \quad B = \alpha_1^4$$

and $M = 4 \cdot 10^{20}$.

For the remaining proof, we use Mathematica to apply Lemma 11 for each positive integer k verifying $1 \le k \le 53$. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find one that satisfies the conditions. Under these conditions, the obtained results are as follows.

- If $z_0 = z_1 = -1$, then for $k = 1$ we obtain $m \le 5$ in the first convergent; when $k = 2$, we obtain $m \le 3$; and when $k = 3$, we obtain $m \le 2$ in the first convergent. Finally, when $k \ge 4$ we obtain $m \le 1$. The second application of Lemma 11, with $M = 2$ or $M = 3$ or $M = 5$, in all cases gives $m \le 1$.
- If $z_0 = z_1 = 1$, then we use the first convergent for $k = 1, 2, 3, 4$. But when $k = 1$, we obtain $m \le 5$; when $k = 2$, we obtain $m \le 3$; and when $k = 3$, we obtain $m \le 2$. When $k = 4$, we obtain $m \le 2$. When $5 \le k \le 12$, we obtain $m \le 1$. Finally, when $k \ge 13$, we obtain $m \le 0$. Again, when apply Lemma 11, with $M = 2$ or $M = 3$ or $M = 5$, we obtain in all cases $m \le 0$.

Then, we consider $m \le 1$ ($m = n = 0$ gives the trivial solution $d = 0$). So by Lemma 3, we have $m = n = 1$. When $z_0 = z_1 = 1$, we have $v_2 = Q_{2k+1}(2P_{2k+2} + \frac{1}{2}Q_{2k+1}) - 1$ and $w_2 = Q_{2k+2}(2P_{2k+2} + \frac{1}{2}Q_{2k+2}) - 1$ such that $v_2 < w_2$. When $z_0 = z_1 = -1$, we have $v_2 = Q_{2k+1}(2P_{2k+2} - \frac{1}{2}Q_{2k+1}) + 1$ and $w_2 = Q_{2k+2}(2P_{2k+2} - \frac{1}{2}Q_{2k+2}) + 1$. Because $Q_{2k+1} + Q_{2k+2} = 4P_{2k+2}$, we conclude that

$$z = v_2 = w_2 = \frac{1}{2}Q_{2k+1}Q_{2k+2} + 1. \tag{24}$$

Lemma 12. *Let $n \in \mathbb{N}$. We have*

$$\frac{1}{4}Q_{2n+1}Q_{2n+2} + 1 = 2P_{2n+1}P_{2n+2}. \tag{25}$$

Proof. Because $Q_n = 2(P_n + P_{n-1})$ and $P_{2n}P_{2n+2} + 1 = P_{2n+1}^2$, we have

$$\begin{aligned}
\frac{1}{4}Q_{2n+1}Q_{2n+2} + 1 &= \frac{1}{4}\left[2(P_{2n+1} + P_{2n})2(P_{2n+2} + P_{2n+1})\right] + 1 \\
&= (P_{2n+1} + P_{2n})(P_{2n+2} + P_{2n+1}) + 1 \\
&= P_{2n+1}P_{2n+2} + P_{2n+1}^2 + P_{2n}P_{2n+1} + (P_{2n}P_{2n+2} + 1) \\
&= P_{2n+1}(P_{2n+2} + 2P_{2n+1} + P_{2n}) = P_{2n+1}(P_{2n+2} + P_{2n+2}) \\
&= 2P_{2n+1}P_{2n+2}.
\end{aligned}$$

This completes the proof. □

Thus, using the equation (24) and Lemma 12, we obtain

$$\begin{aligned}
d &= \frac{z^2 - 1}{c} = \frac{(\frac{1}{2}Q_{2k+1}Q_{2k+2} + 1)^2 - 1}{2P_{2k+2}} \\
&= \frac{\frac{1}{4}Q_{2k+1}^2Q_{2k+2}^2 + Q_{2k+1}Q_{2k+2}}{2P_{2k+2}} = \frac{Q_{2k+1}Q_{2k+2}(\frac{1}{4}Q_{2k+1}Q_{2k+2} + 1)}{2P_{2k+2}} \\
&= \frac{Q_{2k+1}Q_{2k+2}(2P_{2k+1}P_{2k+2})}{2P_{2k+2}} = P_{2k+1}Q_{2k+1}Q_{2k+2}.
\end{aligned} \tag{26}$$

In Case 2, we again apply Lemma 11 with

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_3}{\log \alpha_2}, \quad A = \frac{8ac}{3\log \alpha_2}, \quad B = \alpha_1^2,$$

and $M = 1.324 \cdot 10^{21}$. For the remaining values of k, we apply the Baker-Davenport reduction, which gives us the same result. At the end, we see that we can only have a solution $z = v_1 = w_1$, which will (depending on the sign of z_0) give us the extension with $d = d_+$ or $d = d_-$. Moreover, we see that $d = d_- = 0$ and, from Lemma 12, we have

$$\begin{aligned} d = d_+ &= a + b + c + 2abc + 2rst \\ &= P_{2k} + 3P_{2k+2} + 4P_{2k+2}(P_{2k+1}^2 - 1) + \frac{1}{2}P_{2k+1}Q_{2k+1}Q_{2k+2} \\ &= 4P_{2k+1}P_{2k+2} - P_{2k+2} + P_{2k} + \frac{1}{2}P_{2k+1}Q_{2k+1}Q_{2k+2} \\ &= 2P_{2k+1}(2P_{2k+1}P_{2k+2} - 1) + \frac{1}{2}P_{2k+1}Q_{2k+1}Q_{2k+2} \\ &= P_{2k+1}Q_{2k+1}Q_{2k+2}. \end{aligned} \tag{27}$$

By combining (26) and (27), we obtain the same result. Thus, we have finished the proof of Theorem 1.

5. Proof of Theorem 2

5.1. **Proof of Theorem 2 in the Case $c = c_1^+$.** In this subsection, we use a method similar to that of Theorem 1. Here, we put $a = P_{2k}$, $b = 2P_{2k+2}$, and $c = c_1^+ = 2P_{2k} + 3P_{2k+2}$. As we mentioned in Section 2, to extend a Diophantine triple $\{a, b, c\}$ to a Diophantine quadruple $\{a, b, c, d\}$, we will transform the problem of solving the system of simultaneous Pellian equations (4)–(5) into solving finitely many Diophantine equations of the form $z = v_m = w_n$. Note that the initial terms of the sequences (v_m) and (w_n) are almost completely determined in Lemma 1 and Lemma 2. Also, we see that

$$r = P_{2k} + P_{2k+1}, \ s = 2P_{2k} + P_{2k+1}, \text{ and } t = P_{2k} + P_{2k+3}.$$

Lemma 13. *If $v_{2m} = w_{2n}$ has a solution for $m, n \geq 1$ and $z_0 = z_1 = \pm 1$, then*

$$m \geq \frac{\sqrt{49 + \frac{7c}{\gamma}} - 7}{14},$$

where $\gamma \in \{4, 6\}$.

Proof. From Lemma 4, we see that

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}.$$

In our case, we have

$$6s \equiv 7P_{2k} \pmod{c} \quad \text{and} \quad 4t \equiv 7P_{2k+2} \pmod{c}.$$

Thus, we get the congruence

$$P_{2k}(\pm 24m^2 + 28m) \equiv P_{2k+2}(\pm 48n^2 + 42n) \pmod{c}. \tag{28}$$

Because

$$2P_{2k} \equiv -3P_{2k+2} \pmod{c},$$

congruence (28) becomes

$$-3P_{2k+2}(12m^2 \pm 14m) \equiv P_{2k+2}(48n^2 \pm 42n) \pmod{c}.$$

We deduce that

$$-(12m^2 \pm 14m) \equiv (16n^2 \pm 14n) \pmod{\frac{c}{\gcd(c, 3P_{2k+2})}}. \tag{29}$$

For all integers x and y, we have $\gcd(x, y) = \gcd(x, x - y)$. Thus, we get

$$\begin{aligned}\gamma = \gcd(c, 3P_{2k+2}) &= \gcd(2P_{2k} + 3P_{2k+2}, 3P_{2k+2}) \\ &= \gcd(2P_{2k}, 3P_{2k+2}) \in \{4, 6\}.\end{aligned}$$

Therefore, we obtain

$$(12m^2 \pm 14m + 16n^2 \pm 14n) \equiv 0 \pmod{\frac{c}{\gamma}}. \tag{30}$$

For $m, n \geq 2$ and by Lemma 3, we observe that

$$28m^2 + 28m \geq 12m^2 \pm 14m + 16n^2 \pm 14n \geq 12m^2 - 14m + 16n^2 - 14n > 0.$$

Hence, we get

$$28m^2 + 28m \geq \frac{c}{\gamma},$$

which implies

$$28m^2 + 28m - \frac{c}{\gamma} \geq 0.$$

This completes the proof of Lemma 13. □

Now, we prove the following lemma.

Lemma 14. *If $v_{2m+1} = w_{2n+1}$ has a solution for $m, n \geq 1$ and $z_0 = \pm t$, $z_1 = \pm s$, with $z_0 z_1 > 0$, then*

$$m \geq \frac{\sqrt{225 + \frac{56c}{\gamma}} - 15}{28},$$

where $\gamma \in \{2, 4\}$.

Proof. Again, using Lemma 4, we have

$$\pm astm(m + 1) + rm \equiv \pm bstn(n + 1) + rn \pmod{c}. \tag{31}$$

Note that in this case we have $c = s + t$, which implies that

$$st \equiv -1 \pmod{c}.$$

Also,

$$4r \equiv -P_{2k+2} \pmod{c} \quad \text{and} \quad 2a \equiv -3P_{2k+2} \pmod{c},$$

so that congruence (31) implies

$$P_{2k+2}\left[\pm 6m(m + 1) - m \pm 8n(n + 1) + n\right] \equiv 0 \pmod{c}.$$

Thus, we obtain

$$\pm 6m(m + 1) - m \pm 8n(n + 1) + n \equiv 0 \pmod{\frac{c}{\gcd(c, P_{2k+2})}}. \tag{32}$$

Here, we have

$$\begin{aligned}\gamma = \gcd(c, P_{2k+2}) &= \gcd(2P_{2k} + 3P_{2k+2}, P_{2k+2}) \\ &= \gcd(P_{2k+2}, 2P_{2k}) \in \{2, 4\}.\end{aligned}$$

It follows that

$$6m(m + 1) + 8n(n + 1) \pm (-m + n) \equiv 0 \pmod{\frac{c}{\gamma}},$$

with $\gamma \in \{2, 4\}$. If $m, n \geq 1$, then

$$6m(m+1) + 8n(n+1) \pm (-m+n) \geq 6m(m+1) + 8n(n+1) - m + n > 0.$$

So we deduce that $14m(m+1) + m - n \geq \frac{c}{\gamma}$. Because $m - n \leq m$, we have

$$14m^2 + 15m - \frac{c}{\gamma} \geq 0.$$

This ends the proof of the lemma. □

The proof of the following lemma is similar to that of Lemma 9 and Lemma 10. Thus, we will omit it.

Lemma 15.

(1) *If $z = v_m = w_n$ has a solution $m, n \geq 1$ in Case 1, then $m \leq 4.05 \cdot 10^{21}$ and $1 \leq k \leq 58$.*
(2) *If $z = v_m = w_n$ has a solution $m, n \geq 1$ in Case 2, then $m \leq 8.214 \cdot 10^{21}$ and $1 \leq k \leq 58$.*

Moreover, one can see that

$$\begin{aligned} c - (b + \sqrt{c}) &= 2P_{2k} + P_{2k+2} - \sqrt{2P_{2k} + 3P_{2k+2}} \\ &= \frac{2P_{2k}(2P_{2k} - 1) + P_{2k+2}(P_{2k+2} - 3) + 4P_{2k}P_{2k+2}}{2P_{2k} + P_{2k+2} + \sqrt{2P_{2k} + 3P_{2k+2}}} \\ &> 0 \quad \text{for } k \geq 1. \end{aligned}$$

Thus, we use Lemma 7 to apply Lemma 11.

So, in Case 1, we have proved that the equation $z = v_{2m} = w_{2n}$ has no solution for $n \geq 1$ and $k > 58$. For the remaining values of k, we get the same statement using the Baker-Davenport reduction. For the first application of Lemma 11, we obtain $m \leq 8$ in all cases. The second application of Lemma 11 with $M = 8$, in all cases, gives $m \leq 3$. A third application does not change this bound.

In the end, we only have a solution $v_0 = w_0$, which will give us the extension of Diophantine triple with $d = 0$, which is not a real extension to a quadruple.

In Case 2, we proved that there is no solution to the equation $v_{2m+1} = w_{2n+1}$ for $n \geq 1$ and $k > 58$. For the remaining values of k, we apply the Baker-Davenport reduction, which gives us the same result. In the end, we only have a solution $v_1 = w_1$, which will (depending on the sign of z_0) give us the extension with $d = d_+$ or $d = d_-$.

5.2. **Proof of Theorem 2 in the Case $c = c_2^{\pm}, c_3^{\pm}$.** The aim of this subsection is to prove Theorem 2 in the case where $c = c_2^{\pm}, c_3^{\pm}$. Now, we will give the lower bounds of the indices m and n in the equation $v_m = w_n$, if m and n have the same parity. We can check that all solutions of $v_m = w_n$ with smaller indices ($n \leq m \leq 2$) will give the extension of the Diophantine triple $\{a, b, c\}$ to a quadruple with $d = d_+ = c_{\nu+1}^{\pm}$ or $d = d_- = c_{\nu-1}^{\pm}$, where $c_\nu^{\pm}$ is defined in Proposition 1. So, to prove there are no other extensions, we have to show that $v_m = w_n$ for $m \geq n \geq 2$ does not have a solution for $c = c_2^{\pm}, c_3^{\pm}$. From Proposition 1, we have

$$\begin{aligned} c_2^{\pm} &= 4ab(a + b \pm 2r) + 4(a + b \pm r), \\ c_3^{\pm} &= 16a^2b^2(a + b \pm 2r) + 8ab(3a + 3b \pm 4r) + 3(3a + 3b \pm 2r). \end{aligned}$$

In the proof, we use the assumption $b > 4000$ of [3, Lemma 2.2], which is satisfied in our case for $k \geq 5$, and that $11.6a < b < 11.8a$.

Remark 1. In the case $c = c_3^{\pm}$, one has $k = 1$ by part (4) of Proposition 1. Note that each of the two triples $\{2, 24, c_3^{\pm}\}$ can be extended to a $D(1)$-quadruple in a unique way by [3, Lemma 2.2].

Lemma 16. [14, Lemma 2] *If $b \geq 8$ and $v_{2m} = w_{2n}$ has solutions for $m \geq 3$ and $n \geq 2$, then $m > 0.48b^{-1/2}c^{1/2}$.*

We also have the following result.

Lemma 17. *If $v_{2m+1} = w_{2n+1}$ has solutions for $n \geq 2$, then $m^2 > 0.125b^{-1}c^{0.5}$.*

Proof. In the case of odd indices, from Lemma 4, inserting the conditions of Case 2, we have

$$\pm 2astm(m+1) + r(2m+1) \equiv \pm 2bstn(n+1) + r(2n+1) \pmod{c}. \tag{33}$$

Because $(st)^2 \equiv 1 \pmod{c}$, we conclude that $st \equiv \pm 1 \pmod{c'}$ for some c', which is a divisor of c and $c' \geq \sqrt{c}$. Note that the $\pm$ sign means one of the congruences is true. Hence, we get the congruence

$$\pm 2am(m+1) + r(2m+1) \equiv \pm 2bn(n+1) + r(2n+1) \pmod{c'}. \tag{34}$$

Now, assume the opposite, that is, $m^2 \leq 0.125b^{-1}c^{0.5}$. Then, we see that both sides of the congruence relation (34) are less than c' and they have the same sign. More precisely, we have

$$\max\big(2am(m+1), r(2m+1), 2bn(n+1), r(2n+1)\big) \leq 2bm(m+1)$$

and

$$2bm(m+1) < 4bm^2 \leq \frac{c'}{2}.$$

Therefore, we get

$$|\pm 2am(m+1) + r(2m+1)| < c' \quad \text{and} \quad |\pm 2bn(n+1) + r(2n+1)| < c'.$$

Note that in the case of the sign "-", the two quantities $\pm 2am(m+1) + r(2m+1)$ and $\pm 2bn(n+1) + r(2n+1)$ are negative, and in the case of the sign "+", they are positive. Thus, we actually have the equations

$$\pm 2am(m+1) + r(2m+1) = \pm 2bn(n+1) + r(2n+1)$$

and

$$bn(n+1) - am(m+1) = r(m-n).$$

Combining Lemma 3 and the inequalities $11.6a < b < 11.8a$, we get

$$bn(n+1) - am(m+1) > 11.6an(n+1) - a(2n+1)(2n+2) > 7.6an^2$$

and

$$r(m-n) \leq r(n+1) < 5.2an \leq 5.2an^2.$$

This leads to a contradiction. □

Now, we use another theorem for the lower bounds of linear forms in logarithms from Matveev [12], which is quoted below.

Lemma 18. *Denote by $\alpha_1, \ldots, \alpha_j$ algebraic numbers, not 0 or 1, by $\log\alpha_1, \ldots, \log\alpha_j$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_j)$, and by $b_1, \ldots, b_j$ integers. Define $B = \max\{|b_1|, \ldots, |b_j|\}$, and $A_i = \max\{Dh(\alpha_i), |\log\alpha_i|, 0.16\}$ $(1 \leq i \leq j)$, where $h(\alpha)$ denotes the absolute logarithmic Weil height of α. If the number*

$$\Lambda = b_1 \log\alpha_1 + \cdots + b_n \log\alpha_j$$

does not vanish, then

$$|\Lambda| \geq \exp\{-C(j,\chi) D^2 A_1 \cdots A_j \log(eD) \log(eB)\},$$

where $\chi = 1$ *if* $\mathbb{K} \subset \mathbb{R}$ *and* $\chi = 2$ *otherwise; and*

$$C(j,\chi) = \min\left\{\frac{1}{\chi}\left(\frac{1}{2}ej\right)^{\chi} 30^{j+3} j^{3.5}, 2^{6j+20}\right\}.$$

Lemma 19. *Assume that* $c > b^{5/3}$ *and* $b > 4000$.

1) *If* $v_{2m} = w_{2n}$ *has a solution for* $m, n \geq 1$, *then*
$$\frac{2m-1}{\log(2em)} < 1.906 \cdot 10^{14} \log^2 c. \tag{35}$$
2) *If* $v_{2m+1} = w_{2n+1}$ *has a solution for* $m, n \geq 1$, *then*
$$\frac{m}{\log(2em+e)} < 2.859 \cdot 10^{14} \log^2 c. \tag{36}$$

Proof. We apply the above lemma with $j = 3$ and $\chi = 1$ to equation (18).

1) Here, we take $D = 4$, $b_1 = 2m$, $b_2 = -2n$, $b_3 = 1$. From the computations done in Section 3, we put
$$h(\alpha_1) = \frac{1}{2}\log\alpha_1 < 0.749 \log c,\ h(\alpha_2) = \frac{1}{2}\log\alpha_2 < 0.749\log c \text{ and } h(\alpha_3) < \log c.$$
Therefore, we take
$$A_1 = A_2 = 2.996 \log c \quad \text{and} \quad A_3 = 4\log c.$$
Using Matveev's result, we have
$$\log|\Lambda| > -1.906 \cdot 10^{14} \cdot \log(2em) \cdot \log^3 c. \tag{37}$$
Combining (20) and (37), we get the first part of the lemma.
2) The proof is similar to the proof in the first part. Here, one can observe that for $c > b^{5/3}$
$$\begin{aligned} h(\alpha_3) &\leq \frac{1}{4}\left[\log(a^2(c-b)^2) + 4\log\left(\frac{\max\{|\sqrt{b}(r\sqrt{c} \pm t\sqrt{a})|\}}{\min\{|\sqrt{a}(r\sqrt{c} \pm s\sqrt{b})|\}}\right)\right] \\ &= \frac{1}{4}\left[\log(a^2(c-b)^2) + 4\log\left(\frac{\sqrt{b}(r\sqrt{c} + t\sqrt{a})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})}\right)\right] \\ &= \frac{1}{4}\log\left(\frac{b^2(r\sqrt{c}+t\sqrt{a})^4(r\sqrt{c}+s\sqrt{b})^4}{(c-b)^2}\right) < \log(c^3) = 3\log c. \end{aligned}$$
Again, using Matveev's result and the inequality $\log|\Lambda| < -2m\log c$, we obtain the second part of the lemma.

□

For the remainder of our proof, we combine the lower bounds for indices m and n, together with the result obtained using Baker's theory of linear forms in logarithms to prove the main theorem for large values of k.

In the case of even indices, from Lemma 16 and inequality (35) of Lemma 19, we get the inequality
$$\frac{2 \cdot 0.48 b^{-0.5} c^{0.5} - 1}{\log(2e \cdot 0.48 b^{-0.5} c^{0.5})} < 1.906 \cdot 10^{14} \log^2 c. \tag{38}$$

In the case of odd indices, from Lemma 17 and inequality (36) of Lemma 19, we get the inequality

$$\frac{2 \cdot 0.125^{0.5} b^{-0.5} c^{0.25}}{\log(2e \cdot 0.125^{0.5} b^{-0.5} c^{0.25} + e)} < 2.859 \cdot 10^{14} \log^2 c. \tag{39}$$

Therefore using Maple, the solutions obtained for inequalities (38) and (39) are summarized in the following lemma.

Lemma 20.

(1) *Case of $c = c_2^-$: If $v_{2m} = w_{2n}$ has a solution for $m \geq n \geq 2$, then $m \leq 8.894 \cdot 10^{19}$ and $k \leq 26$. If $v_{2m+1} = w_{2n+1}$ has a solution for $m \geq n \geq 2$, then $m \leq 5.421 \cdot 10^{21}$ and $k \leq 114$.*

(2) *Case of $c = c_2^+$: If $v_{2m} = w_{2n}$ has a solution for $m \geq n \geq 2$, then $m \leq 8.373 \cdot 10^{19}$ and $k \leq 25$. If $v_{2m+1} = w_{2n+1}$ has a solution for $m \geq n \geq 2$, then $m \leq 5.346 \cdot 10^{21}$ and $k \leq 113$.*

Thus, to solve our main problem, we use the well-known Baker-Davenport reduction method (see Lemma 11), taking into account Lemma 20. For this, we also need the inequality that follows from $v_m = w_n$ (that is in Lemma 7). In the case of even indices, we have $z_0 = z_1 = \pm 1$, $x_0 = y_1 = 1$, and in the case of odd indices, we have $z_0 = \pm t$, $z_1 = \pm s$, $x_0 = y_1 = r$, and $z_0 z_1 > 0$. For the first application of Lemma 11, we obtain $m \leq 5$ in all cases. The second application of Lemma 11 with $M = 5$ gives, in all cases, $m \leq 2$. A third application does not change this bound. This finishes the proof of Theorem 2.

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