FIBONACCI ALONG EVEN POWERS IS (ALMOST) REALIZABLE

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Abstract. An integer sequence is called realizable if it is the count of periodic points of some map. The Fibonacci sequence (F_n) does not have this property, and the Fibonacci sequence sampled along the squares (F_{n^2}) also does not have this property. We show that the former is an irreparable feature of the Fibonacci sequence, whereas the latter is an arithmetic phenomenon related to the discriminant of the Fibonacci sequence by showing that (F_n) fails a congruence condition at infinitely many primes, whereas the sequence $(5F_{n^2})$ is realizable. More generally, we show that $(F_{n^{2k-1}})$ is not realizable in a particularly strong sense, whereas $(5F_{n^{2k}})$ is realizable, for any $k \geqslant 1$.

1. Introduction

Counting fixed points for iterates of maps provides a natural source of integer sequences. For example, the shift map σ on the 'golden mean' shift space

$$\Sigma = \{x = (x_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} \mid (x_k, x_{k+1}) \neq (1,1) \text{ for all } k \in \mathbb{Z}\}$$

defined by $\sigma\colon (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ has

$$\mathrm{Fix}_n(\sigma) = \#\{x \in \Sigma \mid \sigma^n(x) = x\} = \mathrm{Trace}\begin{pmatrix} 1\,1 \\ 1\,0 \end{pmatrix}^n = L_n$$

for all $n \in \mathbb{N}$, giving the Lucas sequence $(L_n) = (1, 3, 4, \ldots)$ (A convenient source for this result, and for more on this type of dynamical system, is the monograph by Lind and Marcus [6, Ch. 2]). The sequences we will discuss start naturally with the first term (not the zeroth), and so we use the notation $U = (U_n)$ for a sequence $(U_n)_{n\in\mathbb{N}}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

An integer sequence (U_n) is 'realizable' if there is some map $T\colon X \to X$ with the property that

$$U_n = \mathrm{Fix}_n(T) = \#\{x \in X \mid T^n(x) = x\}$$

for all $n \geqslant 1$. Puri and Ward [12] proved that the golden mean shift map illustrates a uniqueness phenomenon, by showing that if (U_n) is an integer sequence satisfying the Fibonacci recurrence $U_{n+2} = U_{n+1} + U_n$ for all $n \geqslant 1$ with $U_1 = a$ and $U_2 = b$, then (U_n) is realizable if and only if $b = 3a$ and $a \in \mathbb{N} \cup \{0\}$ (meaning that $(U_n) = (aL_n)$). To explain a more general setting within which this is a special case, we recall from [13] that a sequence (U_n) is realizable if and only if it satisfies two conditions:

1. the Dold condition from [2], that $\sum_{d\,|n} \mu\big(\tfrac{n}{d}\big) U_d \equiv 0$ modulo n for all $n \in \mathbb{N}$, and
2. the sign condition $\sum_{d\,|n} \mu\big(\tfrac{n}{d}\big) U_d \geqslant 0$ for all $n \in \mathbb{N}$.

All this means is that $\mathrm{Fix}_n(T) = U_n$ for all $n \geqslant 1$ if and only if the number of closed orbits of length n under T is given by

$$\mathrm{Orb}_n(T) = \frac{1}{n}\sum_{d\,|n} \mu\Big(\frac{n}{d}\Big) U_d, \tag{1}$$

and it must be the case that $\mathrm{Orb}_n(T)$ is a nonnegative integer for all $n \in \mathbb{N}$. The expression on the right side of (1) arises as follows. Writing $\mathrm{Orb}_d(T)$ for the number of sets of the form

$$\{x, T(x), T^2(x), \dots, T^d(x) = x\}$$

with $x \in X$ and cardinality d (that is, the closed orbits of length d under the action of T); we see that those d points contribute to the count of $\mathrm{Fix}_n(T)$ if and only if $d \big| n$, and that distinct closed orbits of any length are disjoint. It follows that $\mathrm{Fix}_n(T) = \sum_{d\,|n} d\mathrm{Orb}_d(T)$ for all $n \geqslant 1$, and the usual Möbius inversion formula (see [3, Th. 8.15], for example) then gives (1).

Minton [7] showed that any linear recurrence sequence satisfying the Dold condition must be a sum of traces of powers of algebraic numbers (and, in the case of binary recurrences, the sign condition is easily understood). This recovers the uniqueness result of [12], and much else besides.

The Fibonacci sequence itself, $(F_n) = (1, 1, 2, 3, \dots)$, is not realizable. It fails the Dold condition in the following strong — and, in the sense of Theorem 2, irreparable — way.

Lemma 1. *The set of primes dividing a denominator of* $\frac{1}{n}\sum_{d\,|n} \mu\big(\frac{n}{d}\big) F_d$ *for some* $n \in \mathbb{N}$ *is infinite.*

Proof. We recall that F_p is equivalent modulo p to the Legendre symbol $\big(\frac{p}{5}\big)$ for any prime p (see Ribenboim [15, Eq. (IV.13), p. 60] or Lemmermeyer [5, Ex. 2.25, p. 73]). It follows that if p is an odd prime with $p \equiv \pm 2$ modulo 5, then $F_p \equiv -1$ modulo p, so the denominator of $\frac{1}{p}(F_p - 1) = \frac{1}{p}\sum_{d\,|p} \mu\big(\frac{p}{d}\big) F_d$ is p. □

A numerical observation is that this seems to be typical for integer linear recurrence sequences in the following sense. An integer linear recurrence sequence may be realizable (and, up to understanding the sign condition, Minton's results determine when this is the case), but if it fails to be realizable, then the denominators appearing in the associated sequence whose nonintegrality witnesses the failure of realizability are expected to be divisible by infinitely many primes.

In a different direction, Moss [8] showed that the property of realizability is preserved by a surprising diversity of 'time-changes'. That is, there are nontrivial maps $h\colon \mathbb{N} \to \mathbb{N}$ with the property that if (U_n) is a realizable sequence, then $(U_{h(n)})$ is also a realizable sequence. Examples of time-changes from [8] with this realizability-preserving property include the monomials, and in later work Jaidee, Moss, and Ward [4], showed that the monomials are the only polynomials with this property, and that there are, nonetheless, uncountably many maps with this property.

The unexpected phenomena we wish to discuss here is that some of these time-changes that preserve realizability seem to 'repair' the failure to be realizable for the Fibonacci sequence along the squares — up to a finite set of primes. At this stage, we understand neither the reason for this, nor its full extent among linear recurrence sequences.

Theorem 2. *The sequence* (F_{n^2}) *is not realizable, but the sequence* $(5F_{n^2})$ *is.*

The negative part of Theorem 2 may be seen from the observation

$$\tfrac{1}{5}\sum_{d\,|5} \mu\big(\tfrac{5}{d}\big) F_{d^2} = \tfrac{1}{5}\,(F_{25} - F_1) = \tfrac{75024}{5},$$

which shows that (F_{n^2}) fails the Dold congruence for realizability. The positive part of Theorem 2 consists of a direct proof that the sequence $(5F_{n^2})$ satisfies the Dold conditions and the sign condition, and this will require several steps. Lemma 1 (strictly speaking, its proof), Theorem 2, and a result from [4] together give the following description of the behavior of the Fibonacci sequence along powers.

Corollary 3. *If j is odd, then the set of primes dividing denominators of $\frac{1}{n}\sum_{d\,|n}\mu\big(\frac{n}{d}\big)F_{d^j}$ for $n \in \mathbb{N}$ is infinite. If j is even, then the sequence (F_{n^j}) is not realizable, but the sequence $(5F_{n^j})$ is.*

2. Modular Periods of the Fibonacci Sequence

It will be convenient to use Dirichlet convolution notation, so that for sequences $f = (f_n)$ and $g = (g_n)$, we write

$$(f * g)_n = \sum_{d\,|n} f_d g_{n/d}$$

for all $n \geqslant 1$. The two conditions for realizability of a sequence $U = (U_n)$ can then be stated as $(\mu * U)_n \equiv 0$ modulo n and $(\mu * U)_n \geqslant 0$ for all $n \geqslant 1$.

The argument involves working modulo various natural numbers, and we adopt the convention that a representative of an equivalence class modulo $m \in \mathbb{N}$ is always chosen among the representatives $\{0, 1, \ldots, m-1\}$. The sequence (F_n) is automatically periodic modulo m, and we define $\ell(m)$ to be its period. That is,

$$\ell(m) = \min\{d \in \mathbb{N} \mid F_{n+d} \equiv F_n \pmod{m} \text{ for all } n \in \mathbb{N}\}.$$

The quantity $\ell(m)$ is well studied; a convenient source for the type of results we need is the paper of Wall [17], whose Theorems 5, 6, and 7 give the following result.

Theorem (Wall [17])**.** *If p is an odd prime, then*

$$\ell(p)\big|p-1 \text{ if } p \equiv \pm 1 \pmod{10} \tag{2}$$

and

$$\ell(p)\big|2(p+1) \text{ if } p \equiv \pm 3 \pmod{10}. \tag{3}$$

If p is a prime with $\ell(p) \neq \ell(p^2)$, then

$$\ell(p^n) = p^{n-1}\ell(p)$$

for all $n \in \mathbb{N}$. Moreover, if t is the largest integer with $\ell(p^t) = \ell(p)$, then

$$\ell(p^n) = p^{n-t}\ell(p) \tag{4}$$

for all $n \in \mathbb{N}$ with $n \geqslant t$.

From now on, in this section, p will always denote a prime, and k an integer with $k \geqslant 2$.

Clearly, (2) and (3) show that $\ell(p)\big|2(p^2-1)$ for an odd prime p, but a little more is true. We claim that

$$\ell(p)\big|p^2-1 \tag{5}$$

for any prime p.

For $p = 2$, it is easy to check that $\ell(p) = 3$. For an odd prime $p \equiv \pm 1$ modulo 10, (2) shows that $\ell(p)\big|p^2-1$. For an odd prime $p \equiv \pm 3$ modulo 10, $p-1$ is even so p^2-1 is a multiple of $2(p+1)$, and hence, $\ell(p)\big|p^2-1$ by (3).

By definition, $F_{n+\ell(p^d)} \equiv F_n$ modulo p^d and $F_{n+\ell(p^{d+1})} \equiv F_n$ modulo p^{d+1} for any $d \in \mathbb{N}$, so $F_{n+\ell(p^{d+1})} \equiv F_n$ modulo p^d and hence,

$$\ell(p^d) \leqslant \ell(p^{d+1}) \tag{6}$$

for any $d \in \mathbb{N}$.

Lemma 4. *For $n \in \mathbb{N}$, there is some $s = s(n)$ with $0 \leqslant s < n$ such that $\ell(p^n) = p^s \ell(p)$.*

Proof. If $\ell(p) \neq \ell(p^2)$, then (4) gives $s = n - 1$.

Suppose therefore, that $\ell(p) = \ell(p^2)$. If $\ell(p) = \ell(p^n)$ for all $n \in \mathbb{N}$, then we may set $s = 0$. If $\ell(p) \neq \ell(p^n)$ for some $n \in \mathbb{N}$, then let $t \in \mathbb{N}$ be the largest integer with $\ell(p^t) = \ell(p)$. By (6) we then have $\ell(p) = \ell(p^j)$ for $j = 1, \ldots, t$, and so we can use (4) to define $s = 0$ if $n \leqslant t$, and $s = n - t$ if $n > t$. □

By Lemma 4, we have

$$\frac{p^n(p^2-1)}{\ell(p^n)} = \frac{p^{n-s}(p^2-1)}{\ell(p)},$$

so (5) shows that

$$\ell(p^n) \big| p^n(p^2-1) \tag{7}$$

for any $n \in \mathbb{N}$.

We now define sequences $u = (u_n)$ and $v = (v_n)$ by

$$u_n = (F_n \pmod{p^{2k}})$$

and

$$v_n = (F_n \pmod{p^{2(k-1)}}). \tag{8}$$

Lemma 5. *For any integer $c \geqslant 0$, we have*

$$F_{p^{2k}+c} \equiv F_{p^{2(k-1)}+c}$$

modulo p^k.

Proof. By definition,

$$F_{p^{2k}+c} \equiv u_{p^{2k}+c} \pmod{p^{2k}}$$

and

$$F_{p^{2(k-1)}+c} \equiv v_{p^{2(k-1)}+c} \pmod{p^{2(k-1)}}$$

For any integer $j \geqslant 0$ we have

$$v_{p^{2(k-1)}+c} = v_{p^2(k-1)+c+j\ell(p^{2(k-1)})}.$$

By (7), we may set $j = \frac{p^{2(k-1)}(p^2-1)}{\ell(p^{2(k-1)})}$, so

$$v_{p^{2(k-1)}+c+j\ell(p^{2(k-1)})} = v_{p^{2(k-1)}+c+p^{2(k-1)}(p^2-1)} = v_{p^{2k}+c}.$$

It follows that $v_{p^{2(k-1)}+c} = v_{p^{2k}+c}$ and so

$$F_{p^{2(k-1)}+c} \equiv v_{p^{2k}+c} \pmod{p^{2(k-1)}}.$$

Clearly $p^{2(k-1)} \big| u_n - v_n$ for all $n \in \mathbb{N}$, so $p^k \big| u_n - v_n$ for all $n \in \mathbb{N}$ because $k \geqslant 2$. In particular,

$$p^k \big| u_{p^{2k}+c} - v_{p^{2k}+c}$$

and hence,

$$p^k \big| u_{p^{2k}+c} - v_{p^{2(k-1)}+c}.$$

Thus, $F_{p^{2k}+c} \equiv F_{p^{2(k-1)}+c}$ modulo p^k, as required. □

Note that in the proof above we saw that $p^{2(k-1)} \big| u_n - v_n$, so for $k \geqslant 3$ we have $p^{k+1} \big| u_n - v_n$. It follows that

$$F_{2^{2k}+c} \equiv F_{2^{2(k-1)}+c} \pmod{2^{k+1}} \tag{9}$$

for any $k \geqslant 3$.

3. Properties of the Sequence $(5F_{n^2})$

In this section, p again denotes a prime, k a positive integer, $\phi = (\phi_n)$ denotes the sequence defined by $\phi_n = 5F_{n^2}$ for all $n \in \mathbb{N}$, and $L = (L_n)$ denotes the Lucas sequence. Because L is a realizable sequence, it satisfies the Dold congruences so we have

$$L_p \equiv 1 \pmod{p}. \tag{10}$$

The next result appeared as an exercise due to Desmond [1], with a solution using an earlier result of Ruggles [16].

Lemma 6 (Desmond)**.** *For a nonnegative integer n, we have $F_{np} \equiv F_n F_p$ modulo p.*

Proof. The case $p = 2$ or $n \leqslant 1$ is clear, so suppose that p is odd and $n \geqslant 2$, and assume the statement holds for all $n \leqslant m$ for some $m \geqslant 2$. Recall that $F_{r+s} = F_r L_s + (-1)^{s+1} F_{r-s}$ (see, for example, Ribenboim [14, Eq. (2.8)]). It follows that

$$F_{mp+p} = F_{mp} L_p + (-1)^{p+1} F_{mp-p} = F_{mp} L_p + F_{(m-1)p},$$

and so

$$F_{(m+1)p} \equiv F_{mp} + F_{(m-1)p} \pmod{p}$$

by (10). The inductive assumption gives

$$F_{(m+1)p} \equiv F_m F_p + F_{m-1} F_p \pmod{p},$$

and then the relation $F_m F_p + F_{m-1} F_p = F_{m+1} F_p$ completes the proof by induction. □

By Lemma 6, we have

$$F_{np^2} \equiv F_{np} F_p \equiv F_n (F_p)^2 \pmod{p}.$$

Because $F_p^2 \equiv \left(\frac{p}{5}\right) \equiv 1$ modulo p if $p \neq 5$, we deduce that

$$F_{np^2} \equiv F_n \pmod{p}$$

for $p \neq 5$. It follows that

$$5F_{np^2} \equiv 5F_n \pmod{p} \tag{11}$$

for any prime p, because it is trivial for $p = 5$.

Lemma 7. *For nonnegative integers n and k, we have*

$$5F_{np^{2k}} \equiv 5F_{np^{2(k-1)}} \pmod{p^k}.$$

Proof. For $k = 1$, this follows from (11). If $k > 1$, then Lemma 5 with $c = (n-1)p^{2k}$ gives

$$F_{np^{2k}} \equiv F_{p^{2(k-1)}+(n-1)p^{2k}} \pmod{p^k}. \tag{12}$$

By definition (8), we have

$$F_{p^{2(k-1)}+(n-1)p^{2k}} \equiv v_{p^{2(k-1)}+(n-1)p^{2k}} \pmod{p^{2(k-1)}},$$

and for $j \geqslant 0$, we have

$$v_{np^{2(k-1)}} = v_{p^{2(k-1)}+(n-1)p^{2(k-1)}+j\ell(p^{2(k-1)})}.$$

Taking $j = \frac{(n-1)p^{2(k-1)}(p^2-1)}{\ell(p^{2(k-1)})}$, which is integral by (7), gives

$$v_{p^{2(k-1)}+(n-1)p^{2(k-1)}+j\ell(p^{2(k-1)})} = v_{p^{2(k-1)}+(n-1)p^{2k}},$$

so $v_{p^{2(k-1)}+(n-1)p^{2k}} = v_{np^{2(k-1)}}$. It follows that

$$F_{p^{2(k-1)}+(n-1)p^{2k}} \equiv F_{np^{2(k-1)}} \pmod{p^{2(k-1)}}.$$

Because $k > 1$, this gives

$$F_{p^{2(k-1)}+(n-1)p^{2k}} \equiv F_{np^{2(k-1)}} \pmod{p^k},$$

and hence,

$$F_{np^{2k}} \equiv F_{np^{2(k-1)}} \pmod{p^k}$$

by (12). □

A similar argument using (9) shows that if n is a positive integer and $k \geqslant 3$, then

$$F_{n2^{2k}} \equiv F_{n2^{2(k-1)}} \pmod{2^{k+1}}.$$

The modular arguments thus far are aimed at establishing the Dold condition. The sign condition is satisfied because of the rapid rate of growth in the sequence, which is more than sufficient by the following remark of Puri [11].

Lemma 8. *If (A_n) is an increasing sequence of nonnegative real numbers with $A_{2n} \geqslant nA_n$ for all $n \in \mathbb{N}$, then $(\mu * A)_n \geqslant 0$ for all $n \in \mathbb{N}$.*

Proof. In the even case, we have

$$(\mu * A)_{2n} = \sum_{d\,|2n} \mu(2n/d)A_d \geqslant A_{2n} - \sum_{k=1}^{n} A_k \geqslant A_{2n} - nA_n \geqslant 0,$$

because the largest divisor of $2n$ is n. Similarly, in the odd case we have

$$(\mu * A)_{2n+1} \geqslant A_{2n+1} - \sum_{k=1}^{n} A_k \geqslant A_{2n} - nA_n \geqslant 0,$$

because the largest divisor of $2n+1$ is smaller than n, proving the lemma. □

Proof of the Positive Part of Theorem 2. We wish to show that $n \big| (\mu * \phi)_n$ and $(\mu * \phi)_n \geqslant 0$ for all $n \in \mathbb{N}$. For $n = 1$, this is clear. If $n = p^k$, then

$$(\mu * \phi)_n = \sum_{d\,|p^k} \mu(d)\phi_{p^k/d} = \phi_{p^k} - \phi_{p^{k-1}} = 5F_{p^{2k}} - 5F_{p^{2(k-1)}},$$

which is clearly nonnegative, and Lemma 7 shows that it is divisible by n.

For the general case, we will work with one prime at a time using Lemma 7. Suppose that $n = p_1^{k_1} \cdots p_m^{k_m}$ with $m \geqslant 2$, $k_1, \dots, k_m \in \mathbb{N}$, and distinct primes $p_1, \dots, p_m$. Select one of these primes p_i, and to reduce the notational complexity write $p^k = p_i^{k_i}$. Writing $s = n/p_i^{k_i}$, we have

$$(\mu * \phi)_n = \sum_{d\,|p^k s} \mu(d)\phi_{p^k s/d} = \sum_{d\,|s} \big(\phi_{p^k s/d} - \phi_{p^{k-1} s/d}\big) = \sum_{d\,|s} \big(5F_{(s/d)^2 p^{2k}} - 5F_{(s/d)^2 p^{2(k-1)}}\big).$$

Lemma 7 therefore shows that $p_i^{k_i} \big| (\mu * \phi)_n$, and by using this for each prime dividing n, we deduce that $n \big| (\mu * \phi_n)$ as required.

For the sign condition, we use Lemma 8 and Binet's formula. Clearly, ϕ is an increasing sequence. Writing $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, we have

$$\phi_{2n} = 5F_{4n^2} = \sqrt{5}\big(\alpha^{2n^2} + \beta^{2n^2}\big)\big(\alpha^{n^2} + \beta^{n^2}\big)\big(\alpha^{n^2} - \beta^{n^2}\big)$$

and

$$n\phi_n = n\sqrt{5}\big(\alpha^{n^2} - \beta^{n^2}\big).$$

Thus, to show the growth condition used in Lemma 8, it is enough to show that

$$\big(\alpha^{2n^2} + \beta^{2n^2}\big)\big(\alpha^{n^2} + \beta^{n^2}\big) \geqslant n$$

for $n \in \mathbb{N}$. Clearly,

$$\big(\alpha^{2n^2} + \beta^{2n^2}\big)\big(\alpha^{n^2} + \beta^{n^2}\big) > G(n) = \alpha^{2n^2}\big(\alpha^{n^2} - 1\big)$$

for all $n \in \mathbb{N}$. We check that $G(1) = \alpha > 1$ and for $n \geqslant 2$ we have

$$G(n) > \alpha^{2n^2}(\alpha - 1) = \alpha^{2n^2-1} > n.$$

Thus, $\phi_{2n} \geqslant n\phi_n$ for all $n \in \mathbb{N}$, completing the proof. □

Proof of Corollary 3. Assume first that j is odd, and recall that if $p \equiv \pm 2$ modulo 5 is an odd prime, then $F_p \equiv -1$ modulo p (as in the proof of Lemma 1). By Lemma 6 it follows that $F_{p^j} \equiv -1$ modulo p, so the denominator of $\frac{1}{p}(F_{p^j} - 1) = \frac{1}{p}\sum_{d\,|p}\mu\big(\frac{p}{d}\big)F_{d^j}$ is p.

For j even, Lemma 6 shows that $F_{5^j} \equiv F_{25} \equiv 0$ modulo 5, so $(1/5)\big(F_{5^j} - F_1\big)$ has denominator 5, showing that (F_{n^j}) is not realizable.

Finally, by [4, Thm. 5] we know that for any $k \in \mathbb{N}$, the map $h(n) = n^k$ preserves realizability. That is, if (U_n) is a realizable sequence, then (U_{n^k}) is also. Thus, the positive part of Theorem 2 shows that $(5F_{n^{2k}})$ is realizable for any $k \in \mathbb{N}$. □

4. Remarks

(1) The correspondence between a pair (X, T), denoting a map $T\colon X \to X$ with the property that $\mathrm{Fix}_n(T) < \infty$ for all $n \geqslant 1$, and the associated sequence $(\mathrm{Fix}_n(T))$ or $(\mathrm{Orb}_n(T))$ is 'functorial' with regard to many natural operations (we refer to the work of Pakapongpun and Ward [9, 10] for an explanation of this cryptic comment, and for results in this direction). The time-changes studied in [4] do not seem to have any such property. For example, we do not have any reasonable way to start with a pair (X, T) and set-theoretically 'construct' another pair (X', T') with the property that $\mathrm{Fix}_n(T') = \mathrm{Fix}_{n^2}(T)$ for all $n \geqslant 1$. We have even less ability — indeed, have no starting point — to 'construct' some reasonable pair (X, T) with $\mathrm{Fix}_n(T) = 5F_{n^2}$ for all $n \geqslant 1$, particularly if the permutation of a countable set implicitly constructed in the proof is not viewed as reasonable. A general result from Windsor [18] shows that there must be a C^∞ map of the 2-torus with this property, but we know nothing more meaningful about such a map beyond that it must exist.

(2) For integers P, Q, we may define the Lucas sequence $(U_n(P,Q))$ and companion Lucas sequence $(V_n(P,Q))$ by

$$\frac{x}{1 - Px + Qx^2} = \sum_{n=0}^{\infty} U_n(P,Q)x^n$$

and

$$\frac{2 - Px}{1 - Px + Qx^2} = \sum_{n=0}^{\infty} V_n(P,Q)x^n.$$

Binet's formulas show that the sequence $(V_n(P,Q))$ always satisfies the Dold condition, but that $(U_n(P,Q))$ can only do so if the discriminant $P^2 - 4Q = \pm 1$. Thus, for example, the sequence

$$\big(U_n(\pm(2k+1), k^2+k)\big)$$

satisfies the Dold condition for any $k \in \mathbb{Z}$. Theorem 2 states that $(5U_{n^2}(1,-1))$ is realizable, and we expect that similar arguments may be used to prove the following.

Conjecture. *For $P, Q \in \mathbb{Z}$ the sequence $\big((P^2-4Q)U_{n^2}(P,Q)\big)$ satisfies the Dold condition, and so is realizable when it satisfies the sign condition.*

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