

WEIGHTED SUMS OF FIBONACCI AND LUCAS NUMBERS THROUGH COLORFUL TILINGS

GREG DRESDEN AND YU XIAO

ABSTRACT. We explore the number of different tilings of boards and bracelets with one color of squares, two colors of dominos, three colors of trominos, and so on. We give a visual proof of the surprising connections between these sequences and the Fibonacci and Lucas numbers, which then allows us to establish new identities and new proofs.

1. INTRODUCTION

The Fibonacci numbers F_n begin with $F_1 = F_2 = 1$, and then each subsequent number is the sum of the two previous numbers. We will also define $f_n = F_{n+1}$ because, as Benjamin and Quinn point out in [3], this f_n counts the number of ways to tile a strip of length n with squares and dominos. With this tiling interpretation for f_n , it is easy to give visual proofs of summation formulas such as $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$, which Benjamin and Quinn [3] proved by looking at the location of the last square in the tilings of a strip of length $2n + 1$.

But what about *weighted* sums of Fibonacci numbers? Baxter and Pudwell [1] proved this weighted-sum formula by induction:

$$1F_{2n-2} + 2F_{2n-4} + 3F_{2n-6} + 4F_{2n-8} + \cdots + (n-1)F_2 = F_{2n} - n. \quad (1.1)$$

A number of other weighted sums like this can be found in Koshy's book [5, chapter 25], where they are proved using algebra and also by a method involving derivatives (from N. Gauthier [4]). Benjamin, Crouch, and Sellers [2] proved (1.1) by a combinatorial argument, where they looked at the location of the *second* square in the tilings of a strip of length $2n - 1$. This illustrates that it is possible to find visual proofs for weighted sums.

In this paper, we count tilings with multiple tiles of many different colors. We define the new sequence a_n to be the number of different ways to tile a board of length n with one color of squares, two colors of dominos, three colors of trominos, and so on. We give a visual proof that $a_n = F_{2n}$, and we then give combinatorial proofs for weighted sums involving these a_n 's, which then become weighted sums involving F_{2n} 's. We then define b_n to be the number of *bracelet* tilings using these same multicolored tiles, and we prove visually that $b_n = L_{2n} - 2$, which then leads to formulas for weighted sums of these Lucas numbers. We finish with combinatorial proofs of new formulas for the following two expressions,

$$\sum_{k=1}^n k(k-1)F_{2n-2k} \quad \text{and} \quad \sum_{k=1}^n k^2 F_{2n-2k}. \quad (1.2)$$

Although Koshy has formulas for $\sum kF_k$ and $\sum k^2 F_k$, we have not seen formulas for the “reversed index” sums weighted by k^2 in (1.2) before now.

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2. THE COLORFUL SEQUENCES a_n AND b_n .

We number the cells of an n -board from left to right as cell 1, cell 2, \dots , cell n . (See Figure 1). As mentioned above, we define a_n to be the number of ways to tile a board of length n

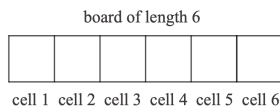


FIGURE 1. Numbering the cells of a 6-board.

with one color of squares, two colors of dominos, three colors of trominos, and so on. For convenience, we define $a_0 = 0$. We call a board with this kind of tiling an n -rainbow-board, and to avoid confusion we also define an n -monoboard to be a board of length n that is tiled with only one color of just squares and dominos. We easily calculate the first few values of a_n starting at $n = 1$ to be 1, 3, 8, 21, 55, \dots , suggesting that a_n could equal F_{2n} . We prove this in Theorem 3.2.

Next, we define b_n to be the number of different ways to tile a *bracelet* of length n with one color of squares, two colors of dominos, three colors of trominos, and so on. For convenience we define $b_0 = 0$. We henceforth call this kind of bracelet an n -rainbow-bracelet, and just as before we also use the term n -monobracelet to represent bracelets tiled with squares and dominos of just one color. We number the cells and tiles clockwise starting from the top of the bracelet, and we denote the tile covering cell 1 as the *first* tile. See Figure 2.

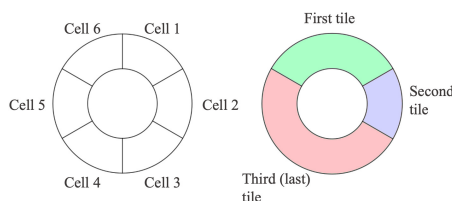


FIGURE 2. Numbering the cells and tiles of a 6-bracelet.

Starting at $n = 1$, we calculate the first few values for b_n to be 1, 5, 16, 45, \dots , suggesting that $b_n = L_{2n} - 2$. We prove this in Theorem 3.4.

To clarify discussions on coloring, we label the colors as c_1 , c_2 , c_3 , and so on. A square is color c_1 , a domino can be color c_1 or c_2 , and a k -mino can be color c_1 through c_k .

3. CONNECTION TO FIBONACCI AND LUCAS NUMBERS

We begin by establishing a recurrence formula for our sequence a_n . We will then show the connection between a_n and the Fibonacci numbers.

Theorem 3.1. *For $n \geq 2$, we have $a_n = 3a_{n-1} - a_{n-2}$.*

Proof. We will give a tiling proof for the related formula $3a_n = a_{n+1} + a_{n-1}$ for $n \geq 2$, which will then give us our theorem. We do so by creating a one-to-three correspondence between the a_n tilings of an n -rainbow-board, and the $a_{n+1} + a_{n-1}$ tilings of an $(n + 1)$ -rainbow-board or an $(n - 1)$ -rainbow-board.

Given a tiling of an n -rainbow-board, we make three copies. For the first copy, we add a square at the end to create an $(n+1)$ -board ending in a square. For the second copy, we extend the last tile from length k to length $k+1$ to create an $(n+1)$ -board. This last tile could be color c_1 through c_k , but not color c_{k+1} . Finally, we take the third copy and we condition on the color of the last tile (of length k). If this last tile has color c_m with $m < k$ (which implies $k > 1$), we shorten its length by 1 but keep the color c_m to give us all possible $(n-1)$ -boards. If instead it has color c_k , we extend its length by 1 and change its color to c_{k+1} . This gives us an $(n+1)$ -board whose last tile has length $k+1$ and color c_{k+1} .

Because we have covered all possible boards of length $n-1$ or $n+1$, we have established the correspondence. Figure 3 gives an example of how three copies of an n -board are turned into boards of length $n+1$ or $n-1$. \square

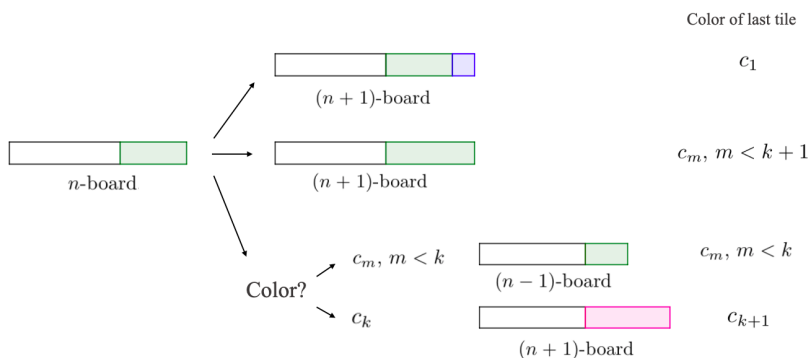


FIGURE 3. Demonstrating that $3a_n = a_{n+1} + a_{n-1}$.

Now that we have proved (in Theorem 3.1) that $a_n = 3a_{n-1} - a_{n-2}$, we note that the sequence $F_2, F_4, F_6, F_8, \dots$ satisfies exactly the same recurrence formula, and because both a_n and F_{2n} have the same initial values of 1 and 3, we can conclude that $a_n = F_{2n}$. However, it is enjoyable (and instructive) to prove this connection directly, by comparing different tilings.

Theorem 3.2. *For $n \geq 0$, we have $a_n = F_{2n}$.*

Proof. We give a tiling proof for the formula $a_n = f_{2n-1}$ for $n \geq 1$ by creating a one-to-one mapping from all tilings of an n -rainbow-board to all tilings of a $(2n-1)$ -monoboard.

Given one of the a_n tilings of an n -rainbow-board, we first turn each k -mino of color c_m into a $2k$ -monoboard composed of two squares and $k-1$ dominos; this $2k$ -monoboard will start with a square (covering cell 1), and then the second square will cover cell $2m$. See Figure 4 for an example. Then, we link all these $2k$ -monoboards to form a $2n$ -monoboard, and finally we remove the first square of the $2n$ -monoboard to create a $(2n-1)$ -monoboard. Figure 5 shows how a 6-rainbow-board is turned into an 11-monoboard.

To show that we have successfully created all f_{2n-1} tilings of an $(2n-1)$ -monoboard exactly once, we will take all f_{2n-1} such tilings and turn them back into all a_n tilings of an n -rainbow-board. We proceed as follows. First, we add a square at the start of the $(2n-1)$ -monoboard to create a $2n$ -monoboard starting with a square. Then, starting at the third square (if it exists) from the left, break the monoboard immediately before alternate squares. This gives us shorter monoboards, each of which starts with a square, has exactly one other square (at some even-numbered cell from the left, call it cell $2m$), and has even length (call it $2k$). Next,

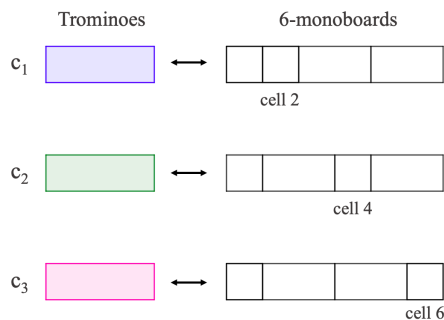


FIGURE 4. Trominos are turned into 6-monoboards.

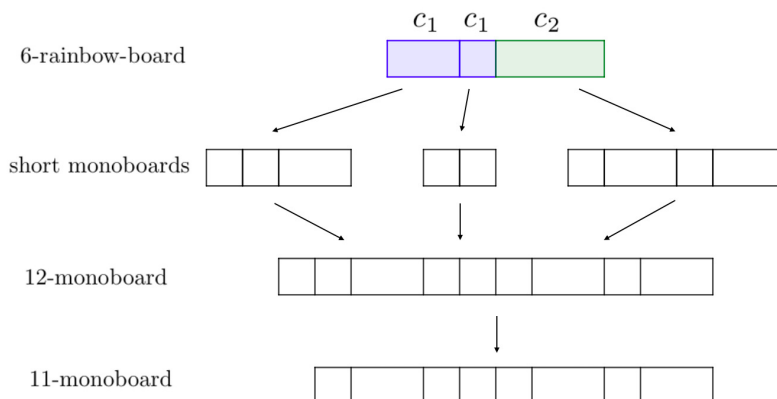


FIGURE 5. A 6-rainbow-board is turned into an 11-monoboard.

we take each short monoboard of length $2k$ with its second square at cell $2m$ and turn it into a k -mino of color c_m . Finally, we link all those k -minos to create a rainbow-board of length n .

Having established our one-to-one correspondence, we conclude that $a_n = f_{2n-1} = F_{2n}$, as desired. \square

Next, we establish a connection between the Lucas numbers and our colorful bracelet sequence b_n . We start by proving the following recurrence formula.

Theorem 3.3. *For $n \geq 2$, we have $b_n = 3b_{n-1} - b_{n-2} + 2$.*

Proof. We note that the “first tile” (the tile that overlays cell 1) of a bracelet tiling can have many different orientations. With this in mind, we will say that a bracelet has *phase* p to mean that cell p of the first tile is overlaid on cell 1 of the bracelet. (Remember that the cells and tiles are numbered clockwise.) Hence, if the first tile of a bracelet is a k -mino, it can be in k different phases p from $p = 1$ to $p = k$. Two examples of phases are given in Figure 6.

To prove the theorem, we will establish the formula $3b_n = b_{n+1} + b_{n-1} - 2$ by creating an *almost one-to-three* correspondence between the b_n tilings of an n -rainbow-bracelet and the $b_{n+1} + b_{n-1}$ tilings of an $(n+1)$ or $(n-1)$ rainbow-bracelet. As will become clear in a moment, it is important to note that none of the following actions will change the phase of the bracelet.

Given a tiling of an n -rainbow-bracelet with two or more tiles, we make three copies. For the first copy, we add a square after the last tile to create an $(n+1)$ -rainbow-bracelet that ends in a square. For the second copy, we extend the length of the last tile by one to create an

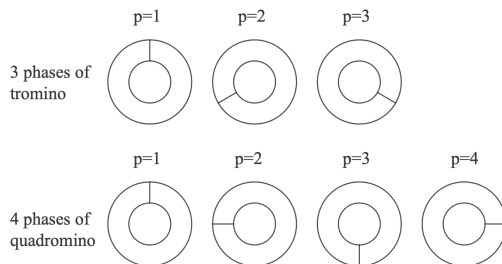


FIGURE 6. Three phases of 3-bracelets and four phases of 4-bracelets.

$(n + 1)$ -rainbow-bracelet that does not end in a square. This last tile, of length $k + 1$, could be color c_1 through c_k , but not color c_{k+1} . Finally, we take the third copy and we condition on the color of the last tile (of length k). If this last tile has color c_m with $m < k$, we shorten its length by 1 but keep the color c_m to give us an $(n - 1)$ -bracelet. If instead this last tile has color c_k , we extend its length by 1 (to length $k + 1$) and change its color to c_{k+1} to give us an $(n + 1)$ -bracelet whose last tile is length $k + 1$ and color c_{k+1} . Figure 7 shows how an n -rainbow-bracelet is turned into three rainbow-bracelets of length $n + 1$ or $n - 1$.

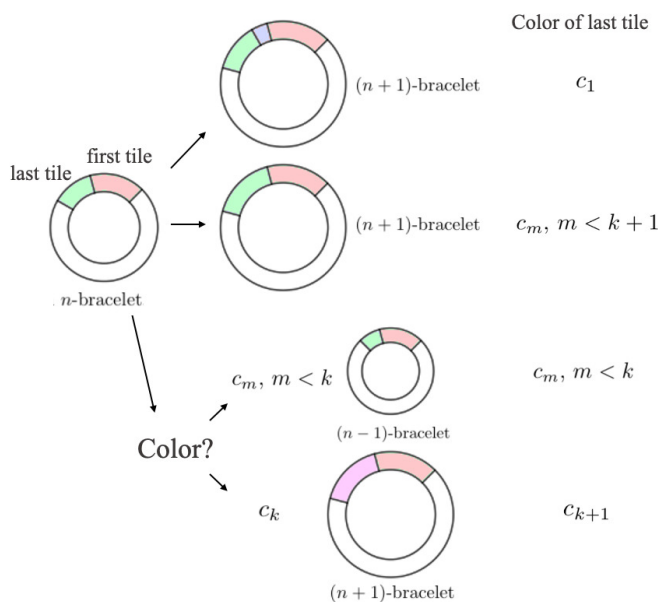


FIGURE 7. Almost one-to-three correspondence between Set 1 and Set 2.

As mentioned above, this is an *almost* one-to-three correspondence. There are two cases where the above correspondence fails. First, our correspondence is not completely valid if our tiling of the n -rainbow-bracelet is made of just one tile of length n , color c_m with $m < n$, and phase n ; we cannot shorten the length by one as directed by the instructions above because that would produce an $(n - 1)$ -bracelet of phase n . There are $n - 1$ such faulty bracelets, one for each color c_m with $m < n$. And second, our correspondence fails to map onto the rainbow-bracelets comprised of a single tile of length $n + 1$ and phase $n + 1$; there are $n + 1$ such uncovered bracelets (with colors c_1 through c_{n+1}). Summing up, this gives us $3b_n - (n - 1) = b_{n-1} + b_{n+1} - (n + 1)$, which simplifies to our desired formula. \square

Now that we have proved in Theorem 3.3 that $b_n = 3b_{n-1} - b_{n-2} + 2$, we note that the sequence $L_{2n} - 2$ satisfies exactly the same recurrence formula, and because b_n and $L_{2n} - 2$ have the same initial values of 1 and 5, we can conclude that $b_n = L_{2n} - 2$. However, it is enjoyable (and instructive) to prove this connection directly, by comparing different tilings.

Theorem 3.4. *For $n \geq 0$, we have $b_n = L_{2n} - 2$.*

Proof. Not surprisingly, this proof differs in only a few details from the proof of Theorem 3.2. We provide an overview, and leave the details to the reader.

We wish to find a one-to-one correspondence between all b_n tilings of an n -rainbow-bracelet and all the $L_{2n} - 2$ tilings of a $2n$ -monobracelet with at least one square; the two missing tilings are the all-domino “in-phase” and “out-of-phase” tilings.

Given one of the b_n tilings of an n -rainbow-bracelet, we again convert each k -mino of color c_m in that tiling into a $2k$ -monoboard composed of 2 squares and $k - 1$ dominos, where the first square will cover cell 1 and the second square will cover cell $2m$. Then, we link these short monoboards to form a bracelet of length $2n$. Finally, we rotate this $2n$ -monobracelet so that cell $2p - 1$ of the first short monoboard (converted from the first k -mino of the rainbow-bracelet) covers cell 1 of the $2n$ -monobracelet, where p is the phase of that first k -mino from the old rainbow-bracelet. Figure 8 shows how an n -rainbow-bracelet is turned into a $2n$ -monobracelet.

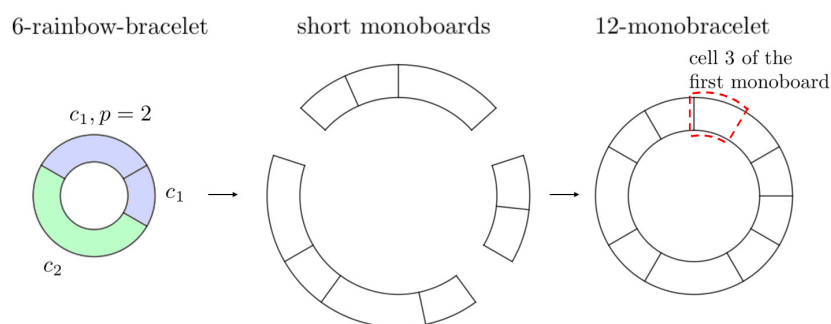


FIGURE 8. A 6-rainbow-bracelet is turned into a 12-monobracelet.

We check that every $2n$ -monobracelet other than the two all-domino tilings is created exactly once by converting each of them back to unique n -rainbow-bracelets, as follows. First, we need to dissect our $2n$ -monobracelet into shorter monoboards. To do this, we look for the location of the first square, counting clockwise from cell 1 at the top of our bracelet; we denote by a the location of this first square, and if a is odd, we cut the monobracelet immediately before alternate squares starting at this *first* square, but if a is even, we do the same but starting at the *second* square. This gives us a number of short monoboards, each of which starts with a square, has exactly one other square (at some position $2m$ relative to the beginning of each monoboard), and has even length (call it $2k$). Figure 9 gives an example. Each such monoboard is now converted into a k -mino of color c_m , and then all the k -minos are assembled back into a rainbow-bracelet of length n . The first short monoboard (the one that covered cell 1 of our $2n$ -monoboard) becomes the first k -mino of our n -rainbow-bracelet.

Finally, we must select the phase of our new n -rainbow-bracelet. Recall that if a was odd, we cut the $2n$ -monobracelet immediately before the first square. If $a = 1$, then the first short monoboard (which by definition covers cell 1 of the $2n$ -monobracelet) from our dissection actually began at cell 1, and so we naturally assign its associated k -mino to have phase 1. Otherwise, for $a > 1$ odd, this first short monoboard (the one that covers cell 1) could only

cover this first cell of the $2n$ -monobracelet at position 3, 5, 7, ... relative to the beginning of the short monoboard. We denote this position by $2p - 1$ for $p > 1$. Likewise, if a was even, we cut the monobracelet immediately before the *next* square, and so this first square (at an even location) would be the second square of the short monoboard, and so again the short monoboard would cover the first cell of the $2n$ -monobracelet at some position $2p - 1$ for $p > 1$. In both cases, we assign our first k -mino in the n -rainbow bracelet to have phase p for $p > 1$. (Because the color of this first k -mino is determined by the location of its second square, there is no duplication in this mapping; we leave the details to the reader.) Figure 9 and Figure 10 show how $2n$ -monobracelets are turned into n -rainbow-bracelets.

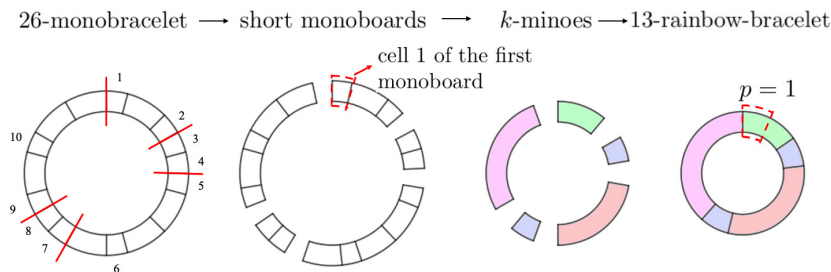


FIGURE 9. A 26-monobracelet is turned into a 13-rainbow-bracelet.

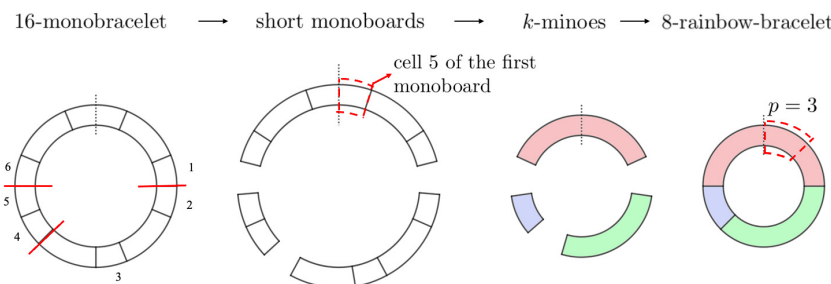


FIGURE 10. A 16-monobracelet is turned into an 8-rainbow-bracelet.

We conclude that $b_n = L_{2n} - 2$, as desired. \square

4. COLORFUL NEW PROOFS AND NEW IDENTITIES FOR FIBONACCI AND LUCAS NUMBERS

Because $a_n = F_{2n}$ and $b_n = L_{2n} - 2$, we can establish new identities (or at the least, new proofs of old identities) for the Fibonacci and Lucas numbers by way of identities for a_n and b_n .

The following theorem was first proved by induction in [1], and then by tiling with squares and dominos in [2]. Here, we give a new proof.

Theorem 4.1. *For F_n the Fibonacci numbers, we have $\sum_{k=1}^n k \cdot F_{2n-2k} = F_{2n} - n$.*

Proof. We will prove the colorful formula $a_n = n + \sum_{k=1}^{n-1} k \cdot a_{n-k}$. Consider the number of ways to tile an n -rainbow-board. On the one hand, this is a_n by definition. On the other

hand, we consider the last tile in any such tiling: there are a_{n-1} tilings that end in a square, and $2a_{n-2}$ tilings that end in a domino (because there are two possible colors for that last domino), and $3a_{n-3}$ tilings that end in a tromino, and so on, ending with $(n-1)a_1$ tilings that end in an $(n-1)$ -mino, and n tilings that are made of a single n -mino. Figure 11 shows the possible cases.

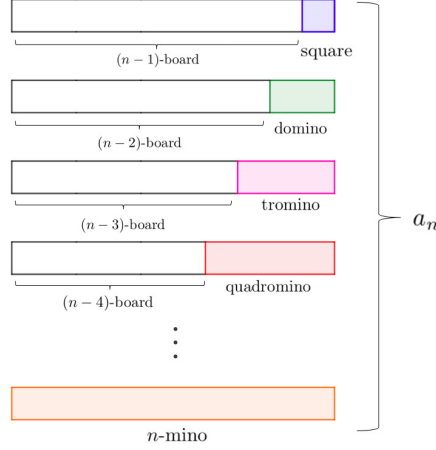


FIGURE 11. Condition on the last tile of an n -board.

Adding up all the terms gives us our desired formula. \square

As an aside, we note that we can use Theorem 4.1 to prove its “companion identity” as seen in [2],

$$\sum_{k=1}^n k \cdot F_{2n+1-2k} = F_{2n+1} - 1.$$

We start with

$$F_{2n} = n + \sum_{k=1}^n k \cdot F_{2n-2k} = n + \sum_{k=1}^{n-1} k \cdot F_{2n-2k},$$

and if we replace n with $n+1$, we get

$$F_{2n+2} = n+1 + \sum_{k=1}^n k \cdot F_{2n+2-2k}.$$

If we subtract the two formulas above, we get

$$F_{2n+2} - F_{2n} = 1 + \sum_{k=1}^n k \cdot F_{2n+2-2k} - \sum_{k=1}^n k \cdot F_{2n-2k}.$$

A bit of simplification turns this into our “companion identity”,

$$F_{2n+1} = 1 + \sum_{k=1}^n k \cdot F_{2n+1-2k}.$$

In [2], Benjamin, Crouch, and Sellers suggested that a formula similar to Theorem 4.1 but for Lucas numbers could be obtained by using squares and dominos on a bracelet of length n .

In keeping with the theme of this section, we have found just such a formula by using colorful tilings on a rainbow-bracelet of length $2n$, as we show next.

Theorem 4.2. *For L_n the Lucas numbers, we have $\sum_{k=1}^n k \cdot L_{2n-2k} = L_{2n} + (n-2)$.*

Proof. We will first prove the colorful formula $b_n = n^2 + \sum_{k=1}^{n-1} kb_{n-k}$. Consider the number of ways to tile an n -rainbow-bracelet. On the one hand, this is b_n by definition. On the other hand, we can condition on the last tile (the one immediately to the left of the first tile) in the bracelet. There are b_{n-1} tilings whose last tile is a square, and $2b_{n-2}$ tilings whose last tile is a domino (because there are two colors for that last domino), and $3b_{n-3}$ tilings of an n -bracelet whose last tile is a tromino, and so on, ending with $(n-1)b_1$ tilings of an n -bracelet whose last tile is an $(n-1)$ -mino, and n^2 tilings of an n -bracelet tiled with a single n -mino, because there are n colors and n phases for that n -mino. In total, this gives us $n^2 + \sum_{k=1}^{n-1} kb_{n-k}$ such tilings, giving us our colorful formula.

If we now replace each b_i with $L_{2i} - 2$, we have

$$L_{2n} - 2 = n^2 + \sum_{k=1}^{n-1} k(L_{2n-2k} - 2),$$

and after simplifying we obtain our theorem. \square

Just as we saw after the proof of Theorem 4.1, there is also a “companion identity” for the weighted sums of Lucas numbers. The formula is

$$\sum_{k=1}^n k \cdot L_{2n+1-2k} = L_{2n+1} - (2n+1),$$

and we leave the details to the reader.

We conclude with a colorful tiling proof that will give us the following weighted sum.

Theorem 4.3. *For $n \geq 1$, we have $\sum_{k=1}^n k(k-1)F_{2n-2k} = 2F_{2n-1} - (n^2 - n + 2)$.*

As seen with Theorems 4.1 and 4.2, there is also a “companion formula” for the sum in Theorem 4.3, and after a bit of work we find that

$$\sum_{k=1}^n k(k-1)F_{2n+1-2k} = 2(F_{2n} - n).$$

Furthermore, we note that if we add together Theorems 4.1 and 4.3, then after a bit of simplifying we obtain the following formula that, as far as we can tell, has not yet appeared in the literature:

$$\sum_{k=1}^n k^2 F_{2n-2k} = L_{2n} - (n^2 + 2).$$

We suspect that there might also be a direct tiling proof for this, but we leave the details to the reader.

Proof of Theorem 4.3. We will first prove the colorful formula

$$b_n = a_n + n^2 - n + \sum_{k=2}^{n-1} k(k-1)a_{n-k},$$

and then we will convert the a_n s and b_n s into Fibonacci and Lucas numbers. To do this, we consider the number of (colorful) ways to tile an n -rainbow-bracelet. On the one hand, this is b_n by definition. On the other hand, we can condition on the first tile of the bracelet (the tile that covers cell 1). If the first tile is “in phase”, then we break the bracelet at the top and unroll it to form a strip; there are a_n such tilings. If the first tile is an out-of-phase domino, we can remove the domino to get an $(n - 2)$ -board. There are two colors of dominos, and so there are $2 \cdot a_{n-2}$ of these tilings. Likewise, if the first tile is an out-of-phase tromino, we remove it to get an $(n - 3)$ -board. There are three colors of trominos, and two out-of-phase phases for those trominos, and so we have $3 \cdot 2 \cdot a_{n-2}$ such tilings. We continue with this line of reasoning until we reach the case where the first tile is an out-of-phase n -mino; there are n colors and $n - 1$ out-of-phase phases, giving us $n(n - 1)$ such tilings.

Summing up, we have that

$$b_n = a_n + n(n - 1) + \sum_{k=2}^{n-1} k(k - 1)a_{n-k},$$

and if we replace a_n and b_n with F_{2n} and $L_{2n} - 2$ respectively, we find that

$$L_{2n} = F_{2n} + n(n - 1) + 2 + \sum_{k=2}^{n-1} k(k - 1)F_{2n-2k}.$$

If we now use the identity $L_{2n} - F_{2n} = 2F_{2n-1}$, we obtain our desired formula. \square

5. CONCLUSION

We were delighted to discover that by counting colorful tilings we could produce new theorems for weighted sums. We can only imagine that uncountably many more identities are out there, waiting to be discovered.

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DEPARTMENT OF MATHEMATICS, WASHINGTON & LEE UNIVERSITY, LEXINGTON, VA 24450
Email address: dresdeng@wlu.edu

SHENZHEN MIDDLE SCHOOL, SHENZHEN, GUANGDONG, CHINA
Email address: angelica.a@126.com