ANTI-PALINDROMIC COMPOSITIONS

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Abstract. A palindromic composition of n is a composition of n that reads the same way forwards and backwards. In this paper, we define an anti-palindromic composition of n to be a composition of n that has no mirror symmetry among its parts. We then give a surprising connection between the number of anti-palindromic compositions of n and the so-called tribonacci sequence, a generalization of the Fibonacci sequence. We conclude by defining a new q-analogue of the Fibonacci sequence, which is related to certain equivalence classes of anti-palindromic compositions.

1. Introduction

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ be a sequence of positive integers such that $\sum \sigma_i = n$. The sequence σ is called a *composition* of n of length s. The numbers σ_i are called the *parts* of the composition. The number of compositions of n equals 2^{n-1}, whereas the number of compositions of n into s parts equals $\binom{n-1}{s-1}$. The empty composition is often considered the only composition of 0, having length equal to 0.

1.1. **Palindromic and Anti-palindromic Compositions.** If $\sigma_i = \sigma_{s-i+1}$ for all i, then σ is called a *palindromic* composition. It is well known [11] that if $pc(n)$ is the number of palindromic compositions, then $pc(n) = 2^{\lfloor \frac{n}{2} \rfloor}$. For instance, the $pc(5) = 4$ palindromic compositions of 5 are

$$(5), \quad (1,3,1), \quad (2,1,2), \quad \text{and} \quad (1,1,1,1,1).$$

Recent work of the authors [3, 12] generalize this result to compositions that are palindromic modulo m, where the condition $\sigma_i = \sigma_{s-i+1}$ is replaced with the weaker condition $\sigma_i \equiv \sigma_{s-i+1} \pmod{m}$.

If $\sigma_i \neq \sigma_{s-i+1}$ for all $i \neq \frac{s+1}{2}$, then we say σ is an *anti-palindromic* composition. Let $ac(n)$ be the number of anti-palindromic compositions of n. Then, the $ac(4) = 5$ anti-palindromic compositions of 4 are

$$(4), \quad (1,3), \quad (3,1), \quad (1,1,2), \quad \text{and} \quad (2,1,1).$$

Furthermore, let $ac(n,s)$ be the number of anti-palindromic compositions of n of length s, $ac_0(n)$ be the number of anti-palindromic compositions of n of even length, and $ac_1(n)$ be the number of anti-palindromic compositions of odd length (thus $ac(n) = ac_0(n) + ac_1(n)$).

Notice that for each anti-palindromic composition of n of length s, we can form $2^{\lfloor \frac{s}{2} \rfloor}$ *flip-equivalent* anti-palindromic compositions of n of length s by switching any number of the pairs σ_i and σ_{s-i+1} ($i \neq \frac{s+1}{2}$). For instance, the anti-palindromic compositions

$$(1,3,3,2,4), \quad (1,2,3,3,4), \quad (4,3,3,2,1), \quad \text{and} \quad (4,2,3,3,1)$$

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are all flip-equivalent compositions of 13 of length 5. The sets of flip-equivalent anti-palindromic compositions of n form a partition of the set of all anti-palindromic compositions of n, and we refer to each equivalence class as a *reduced anti-palindromic composition* of n of length s. Let $rac(n)$ equal the total number of reduced anti-palindromic compositions of n, and $rac(n,s)$ equal the number of reduced anti-palindromic compositions of n of length s. Furthermore, let $rac_0(n)$ and $rac_1(n)$ equal the total number of even and odd reduced anti-palindromic compositions of n, respectively. Clearly we have $rac(n) = rac_0(n) + rac_1(n)$. Because each equivalence class contains $2^{\lfloor \frac{s}{2} \rfloor}$ anti-palindromic compositions, it follows that

$$rac(n,s) = \frac{ac(n,s)}{2^{\lfloor \frac{s}{2} \rfloor}}.$$

Our primary results regarding the formulae for these functions come from observations made in Table 1, Table 2, and Table 3.

n	$ac_0(n)$	$ac_1(n)$	$ac(n)$	$rac_0(n)$	$rac_1(n)$	$rac(n)$
0	1	0	1	1	0	1
1	0	1	1	0	1	1
2	0	1	1	0	1	1
3	2	1	3	1	1	2
4	2	3	5	1	2	3
5	4	5	9	2	3	5
6	8	9	17	3	5	8
7	14	17	31	5	8	13
8	26	31	57	8	13	21
9	48	57	105	13	21	34
10	88	105	193	21	34	55

TABLE 1. Values of $ac_0(n)$, $ac_1(n)$, $ac(n)$, $rac_0(n)$, $rac_1(n)$, and $rac(n)$ for $n \leq 10$.

n	$ac(n,0)$	$ac(n,1)$	$ac(n,2)$	$ac(n,3)$	$ac(n,4)$	$ac(n,5)$
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	0	0	0	0
3	0	1	2	0	0	0
4	0	1	2	2	0	0
5	0	1	4	4	0	0
6	0	1	4	8	4	0
7	0	1	6	12	8	4
8	0	1	6	18	20	12

TABLE 2. Values of $ac(n,s)$ for $n \leq 8$.

n	$rac(n,0)$	$rac(n,1)$	$rac(n,2)$	$rac(n,3)$	$rac(n,4)$	$rac(n,5)$
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	0	0	0	0
3	0	1	1	0	0	0
4	0	1	1	1	0	0
5	0	1	2	2	0	0
6	0	1	2	4	1	0
7	0	1	3	6	2	1
8	0	1	3	9	5	3

TABLE 3. Values of $rac(n,s)$ for $n \leq 8$.

1.2. **The k-bonacci Numbers.** Recall the nth Fibonacci number is given by $f_2(n) = 0$ for $n < 1$, $f_2(1) = 1$, and $f_2(n) = f_2(n-1) + f_2(n-2)$ for all $n \geq 2$.[1] The nth *tribonacci* number is given by $f_3(n) = 0$ for $n < 1$, $f_3(1) = 1$, and $f_3(n) = f_3(n-1) + f_3(n-2) + f_3(n-3)$ for $n \geq 2$. The sequence begins

$$0, 1, 1, 2, 4, 7, 13, 24, \ldots,$$

see OEIS [16, A000073]. It has been suggested that tribonacci numbers appear in Darwin's *Origins of Species* in a similar relation to elephant population growth as Fibonacci numbers bear to rabbit populations [13]. In general, we can define the nth k-bonacci number by $f_k(n) = 0$ for $n < 1$, $f_k(1) = 1$, and

$$f_k(n) = \sum_{i=1}^{k} f_k(n-i)$$

for $n > 1$. Connections between k-bonacci numbers for various k have been studied by Bravo and Luca [5]. In a paper by Benjamin, Chinn, Scott, and Simay [4], formulae for the k-bonacci are developed. For instance, we have

$$f_3(n+1) = \sum_{j=0}^{\lfloor n/4 \rfloor} (-1)^j \binom{n-3j}{j} \frac{n-2j}{n-3j} 2^{n-4j-1}.$$

The k-bonacci numbers also play a role in computing the probability of flipping exactly k consecutive heads in n flips of a fair coin [15].

1.3. **Formulae for Anti-palindromic Compositions.** Our first result gives a surprising connection between the tribonacci numbers and anti-palindromic compositions of even length.

Theorem 1. *For all $n \geq 1$, we have $ac_0(n) = 2 \cdot f_3(n-2)$.*

This theorem can be deduced by a careful inspection of the identity

$$\frac{\left(\frac{q}{1-q}\right)^2 - \frac{q^2}{1-q^2}}{1 - \left[\left(\frac{q}{1-q}\right)^2 - \frac{q^2}{1-q^2}\right]} = \frac{2q^3}{1-(q+q^2+q^3)}.$$

Indeed, the left side is

$$\sum_{n \geq 1} ac_0(n) q^n$$

[1]In some applications, the offset $f_2(0) = f_2(1) = 1$ is used.

because every even-length anti-palindromic composition is a sequence of pairs of distinct positive integers, and the right side is

$$\sum_{n\geq 1} 2\cdot f_3(n)q^{n+2}.$$

We will give an algebraic (Section 2.1) and combinatorial (Section 2.2) proof of this result. Note that $ac(0) = ac_0(0) = 1$, as the empty composition is vacuously anti-palindromic. Our next result gives the number of anti-palindromic compositions of n.

Theorem 2. *For all $n \geq 1$,*

$$ac(n) = f_3(n) + f_3(n-2).$$

We will prove Theorem 2 in Section 2.3, and also observe that (for $n \geq 2$)

$$ac_1(n) = f_3(n-1) + f_3(n-3).$$

In Section 2.4, we prove the following results, which give the formulae for $ac(n, s)$.

Theorem 3. *Let $s \geq 0$ be a fixed integer and*

$$G(q, s) = \sum_{n\geq 0} ac(n, s)q^n.$$

Then for $|q| < 1$,

$$G(q, s) = \frac{2^{\lfloor s/2\rfloor} q^{\lfloor 3s/2\rfloor}}{(1-q)^s(1+q)^{\lfloor s/2\rfloor}}.$$

For instance, $G(q, 0) = 1$,

$$G(q, 1) = \frac{q}{1-q} = q + q^2 + q^3 + \cdots ,$$

and

$$G(q, 2) = \frac{2q^3}{1-q-q^2+q^3} = 2q^3 + 2q^4 + 4q^5 + \cdots ,$$

which give the (verifiable) formulae $ac(0, 0) = 1$, $ac(n, 0) = 0$ for $n > 0$, $ac(n, 1) = 1$ for $n > 0$, and $ac(n, 2) = 2 \cdot \lfloor \frac{n-1}{2} \rfloor$ for $n > 1$. For $s \geq 2$, we have the following corollary.

Corollary 1. *Let a be a positive integer. If $s = 2a$, then*

$$ac(n, s) = \sum_{r+2t=n-3a} 2^a \binom{a+r-1}{r}\binom{a+t-1}{t},$$

and if $s = 2a + 1$, then

$$ac(n, s) = \sum_{r+2t=n-3a} 2^a \binom{a+r}{r}\binom{a+t-1}{t}.$$

By the observation in Section 1.1 regarding $rac(n, s)$ and $ac(n, s)$, we also have a formula for $rac(n, s)$ by dividing by the appropriate power of 2.

1.4. **An Observation Regarding the Fibonacci Numbers.** Recall the following two q-analogues of the Fibonacci numbers,

$$F_n(q) = \begin{cases} 0, & n = 0; \\ 1, & n = 1; \\ F_{n-1}(q) + q^{n-2}F_{n-2}(q), & n > 1; \end{cases}$$

and

$$\hat{F}_n(q) = \begin{cases} 0, & n = 0; \\ 1, & n = 1; \\ \hat{F}_{n-1}(q) + q^{n-1}\hat{F}_{n-2}(q), & n > 1. \end{cases}$$

These are referred to as q-analogues due to the property that $F_n(q) \to f_2(n)$ and $\hat{F}_n(q) \to f_2(n)$ as $q \to 1^-$. Properties of these two sequences of polynomials have been studied extensively, see for instance [2, 6, 7, 14].

We define a new q-analogue of the Fibonacci numbers, which will have a connection to the anti-palindromic compositions. Define

$$\phi_n(q) = \begin{cases} q, & n = 1; \\ q, & n = 2; \\ q + q^2, & n = 3; \\ \phi_{n-1}(q) + \phi_{n-2}(q) + (q^2 - 1)\phi_{n-3}(q), & n > 3. \end{cases}$$

Clearly, $\phi_n(q) \to f_2(n)$ as $q \to 1^-$ for all $n \geq 1$, and our final result gives a combinatorial description of the coefficients of these polynomials. For convenience, we set $\phi_0(q) = 1$.

Theorem 4. *The coefficient of q^s in the polynomial $\phi_n(q)$ equals $rac(n, s)$.*

We will give a proof of Theorem 4 in Section 2.5. The first few polynomials $\phi_n(q)$ are given below, where the coefficients can be compared with Table 2.

$$\begin{aligned} \phi_0(q) &= 1, & \phi_5(q) &= q + 2q^2 + 2q^3, \\ \phi_1(q) &= q, & \phi_6(q) &= q + 2q^2 + 4q^3 + q^4, \\ \phi_2(q) &= q, & \phi_7(q) &= q + 3q^2 + 6q^3 + 2q^4 + q^5, \\ \phi_3(q) &= q + q^2, & \phi_8(q) &= q + 3q^2 + 9q^3 + 5q^4 + 3q^5, \\ \phi_4(q) &= q + q^2 + q^3, & \phi_9(q) &= q + 4q^2 + 12q^3 + 8q^4 + 8q^5 + q^6. \end{aligned}$$

Also in Section 2.5, we deduce the following corollary.

Corollary 2. *For $n \geq 1$, we have $rac_0(n) = f_2(n-2)$, $rac_1(n) = f_2(n-1)$, and $rac(n) = f_2(n)$.*

We summarize our results regarding $ac(n)$ and $rac(n)$ for sufficiently large n below, illustrating the elegance of the formulae.

$$\begin{aligned} ac_0(n) &= 2 \cdot f_3(n-2), & rac_0(n) &= f_2(n-2), \\ ac_1(n) &= f_3(n-1) + f_3(n-3), & rac_1(n) &= f_2(n-1), \\ ac(n) &= f_3(n) + f_3(n-2), & rac(n) &= f_2(n). \end{aligned}$$

2. Proofs of Theorems

2.1. **Algebraic Proof of Theorem 1.** Because $ac_0(1) = ac_0(2) = 0$, $ac_0(3) = ac_0(4) = 2$, and $ac_0(5) = 4$, we see that the theorem is true for $n < 6$. Assume now that $n \geq 6$. Clearly, we can construct an anti-palindromic composition of n from one of two fewer parts by inserting j at the beginning and k at the end (making sure $j \neq k$), where if the inner composition is a composition of m, then $j + k$ must equal $m - n$. Hence,

$$ac_0(n) = \sum_{m=0}^{n-3} (n - m - 1 - \chi(n-m))\, ac_0(m),$$

where $\chi(j) = 1$ if j is even and 0 if j is odd. The term $(n - m - 1 - \chi(n - m))$ accounts for the number of j and k. Hence,

$$\begin{aligned}
ac_0(n) - ac_0(n-1) &= \sum_{m=0}^{n-3} (n - m - 1 - \chi(n-m))\, ac_0(m) \\
&\quad - \sum_{m=0}^{n-4} (n - 1 - m - 1 - \chi(n-1-m))\, ac_0(m) \\
&= (2 - \chi(3))\, ac_0(n-3) + \sum_{m=0}^{n-4} (n - m - 1 - \chi(n-m))\, ac_0(m) \\
&\quad - \sum_{m=0}^{n-4} (n - 1 - m - 1 - \chi(n-1-m))\, ac_0(m) \\
&= 2ac_0(n-3) + 2\sum_{m=0}^{n-4} \chi(n-m-1) ac_0(m).
\end{aligned}$$

Thus,

$$ac_0(n) - ac_0(n-1) - 2ac_0(n-3) = 2\sum_{m=0}^{n-4} \chi(n-m-1) ac_0(m).$$

Let $r(n) = ac_0(n) - ac_0(n-1) - 2ac_0(n-3)$. Then,

$$\begin{aligned}
r(n) + r(n-1) &= 2\sum_{m=0}^{n-4} \chi(n-m-1) ac_0(m) \\
&\quad + 2\sum_{m=0}^{n-5} \chi(n-m-2) ac_0(m) \\
&= 2\sum_{m=0}^{n-5} ac_0(m)
\end{aligned}$$

because $\chi(n) + \chi(n-1) = 1$ and $\chi(3) = 0$. Therefore,

$$r(n) + r(n-1) - (r(n-1) + r(n-2)) = 2ac_0(n-5),$$

and simplifying, we obtain

$$ac_0(n) - m_1(n-1) - ac_0(n-2) - ac_0(n-3) = 0.$$

This is the defining recurrence for $f_3(n)$, and because $2 \cdot f_3(n-2) = ac_0(n)$ for $n > 6$, we see that by induction, $ac_0(n) = 2 \cdot f_3(n-2)$ for all $n \geq 1$.

2.2. **Combinatorial Proof of Theorem 1.** We begin with a lemma regarding the tribonacci numbers.

Lemma 1. *For $n \geq 2$, the tribonacci number $f_3(n)$ equals the number of compositions of $n-1$ with parts equal to 1, 2, or 3.*

Proof. First note that $f_3(2) = 1$, $f_3(3) = 2$, and $f_3(4) = 4$. Because the compositions of 1, 2, and 3 only consist of parts equal to 1, 2, or 3, and the number of compositions of n is equal to 2^{n-1}, the lemma holds for $n \leq 4$. Now for $n > 4$, each composition of $n-1$ into parts equal to 1, 2, or 3 is formed by taking a composition of $n-4$, $n-3$, or $n-2$ and adjoining a 3, 2, or 1, respectively. Thus, the number of compositions of $n-1$ into parts equal to 1, 2, or 3 is equal to $f_3(n-3) + f_3(n-2) + f_3(n-1) = f_3(n)$. □

We will now show that for $n \geq 3$, the number of compositions of $n-3$ into parts equal to 1, 2, or 3 equals the number of anti-palindromic compositions of n. Because $ac_0(1) = 0 = 2 \cdot f_3(-1)$ and $ac_0(2) = 0 = 2 \cdot f_3(0)$, this will establish the theorem.

Proof of Theorem 1. For $n = 2$, we see that $ac(2) = 2 \cdot f_3(0) = 0$, so for any $n \geq 3$, start with a composition σ of $n-3$ into parts equal to 1, 2, or 3. The key will be to use σ to construct a sequence of pairs of distinct positive integers with sum equal to n.

Now recall a *partition* of n is a composition of n where the parts are written in nonincreasing order. Let $\sigma + \tau$ denote sequence concatenation, as in $(1,2)+(4,5) = (1,2,4,5)$. For our choice of σ, we can find partitions λ_1, λ_2, …, λ_r with parts equal to 1 or 2 (or the empty partition, $\varnothing$) such that

$$\sigma = \lambda_1 + \sigma_2 + \lambda_2 + \cdots + \sigma_r + \lambda_r,$$

where each σ_j is either equal to the composition (3) or equal to the composition $(1,2)$.

For example, take the composition

$$\sigma = (2,3,1,1,2,2,1,1,1,2,1,3)$$

of 20. Then we can decompose σ as

$$\begin{aligned}
\lambda_1 &= (2),\\
\sigma_2 &= (3),\\
\lambda_2 &= (1),\\
\sigma_3 &= (1,2),\\
\lambda_3 &= (2,1,1),\\
\sigma_4 &= (1,2),\\
\lambda_4 &= (1),\\
\sigma_5 &= (3),\\
\lambda_5 &= \varnothing.
\end{aligned}$$

It is not difficult to see that this decomposition is unique; the only way a segment in the composition that is a partition with parts equal to 1 or 2 terminates is with the segment (3) or the segment $(1,2)$.

Now, given the decomposition $\sigma = \lambda_1 + \sigma_2 + \lambda_2 + \cdots + \sigma_r + \lambda_r$, form a sequence of pairs (s_1, λ_1), (s_2, λ_2), …, (s_r, λ_r), where $s_1 = +3$, $s_j = +3$ if $\sigma_j = (3)$, and $s_j = -3$ if $\sigma_j = (1,2)$. For our example shown above, we have the pairs

$$(+3,(2)),\ (+3,(1)),\ (-3,(2,1,1)),\ (-3,(1)),\ (+3,\varnothing).$$

For each pair (s_j, λ_j), we now form a new pair (b_j, c_j) in the following way. Start with $b_j = 2$ and $c_j = 1$. For each 2 in the partition λ_j, increase both b_j and c_j by one. For each 1 in the partition λ_j, increase b_j by one. We now have pairs (b_j, c_j) of positive integers such that $b_j > c_j$. Now if $s_j = +3$, we are done. If $s_j = -3$, we switch the numerical values of b_j and c_j so that $b_j < c_j$, and then we are done.

Finally, form the anti-palindromic composition $\tau = (\tau_1, \tau_2, \ldots, \tau_{2r})$ by setting $\tau_j = b_j$ and $\tau_{2r-j+1} = c_j$. Notice that although we started with a composition of $n-3$, this is a composition of n; the addition of 3 came from inserting $s_1 = +3$. In our toy example, we have

$$\tau = (3, 3, 2, 1, 2, 1, 3, 5, 1, 2).$$

We have now embedded the compositions of $n-3$ made up of parts equal to 1, 2, or 3 into the anti-palindromic compositions of n. We still need to embed a second, disjoint copy. To do this, we return to the pairs (s_j, λ_j) and make a new collection of pairs (s'_j, λ_j) by setting $s'_j = -s_j$. Now, following the same procedure as before, we construct an anti-palindromic word τ' that is the *reverse* of τ. Again looking at our example from before, we have

$$\tau' = (2, 1, 5, 3, 1, 2, 1, 2, 3, 3).$$

To show that these two embedded sets are disjoint, notice that for a composition τ formed by using $s_1 = +3$, we have $\tau_1 > \tau_{2r}$, and that for a word τ' formed by using $s_1 = -3$, we have $\tau_1 < \tau_{2r}$.

Showing this process reverses and that we can send the pairs $\{\tau, \tau'\}$ of an anti-palindromic composition of n and its reverse back to a composition of $n-3$ with parts equal to 1, 2, or 3 is straightforward, which the reader can verify. □

2.3. **Proof of Theorem 2.** In this section, we develop the formula for $ac(n)$. We start by proving some initial observations regarding $ac_0(n)$, $ac_1(n)$, $ac(n)$, and $ac(n, s)$.

Proposition 1. *For all $n \geq 3$, we have*

$$ac_0(n) = f_3(n-1) + f_3(n-5).$$

Proof. This is just two applications of the defining recurrence for $f_3(n)$, recalling that $f_3(n) = 0$ for $n < 1$.

$$\begin{aligned} f_3(n-1) + f_3(n-5) &= f_3(n-2) + f_3(n-3) + f_3(n-4) + f_3(n-5) \\ &= f_3(n-2) + f_3(n-2) \\ &= 2 \cdot f_3(n-2) \\ &= ac_0(n) \end{aligned}$$ □

Proposition 2. *We have $ac(0, 0) = 1$, $ac(0, 1) = 0$, and for all $n \geq 0$ and $s \geq 0$,*

$$ac(n, 2s) + ac(n, 2s+1) = ac(n+1, 2s+1).$$

Proof. When $n = 0$, there is only one composition (the empty composition) that has length 0.

Now, any anti-palindromic composition σ of $n+1$ of length $2s+1$ has a central part σ_{s+1}. If $\sigma_{s+1} = 1$, this composition can be formed from an anti-palindromic composition of n of length $2s$ by adding a central part equal of 1. If $\sigma_{s+1} > 1$, this composition can be formed from an anti-palindromic composition of n of length $2s+1$ by adding 1 to the central part. Therefore, $ac(n, 2s) + ac(n, 2s+1) = ac(n+1, 2s+1)$. □

Proposition 3. *For $n \geq 0$ and $s \geq 0$,*

$$ac(n, 2s+1) = \sum_{j=0}^{n-1} ac(j, 2s),$$

where, in the case $n = 0$, we take the empty sum to be 0.

Proof. Let $n > 0$. Then by applying Proposition 2 n times, we have

$$\begin{aligned} ac(n, 2s+1) &= ac(n-1, 2s+1) + ac(n-1, 2s) \\ &= ac(n-2, 2s+1) + ac(n-2, 2s) + ac(n-1, 2s) \\ &\vdots \\ &= ac(0, 2s+1) + \sum_{j=0}^{n-1} ac(j, 2s). \end{aligned}$$

Because $ac(0, 2s+1) = 0$ for all $s \geq 0$, the result follows. □

Proposition 4. *For all $n \geq 0$,*

$$ac(n) = ac_1(n+1).$$

Proof. If $n = 0$, we see that $ac(0) = ac_1(1) = 1$. If $n > 0$, by definition we have

$$\begin{aligned} ac(n) &= \sum_{s \geq 0} ac(n, s) \\ &= \sum_{j \geq 0} (ac(n, 2j) + ac(n, 2j+1)) \\ &= \sum_{j \geq 0} ac(n+1, 2j+1) \end{aligned}$$

by Proposition 2. But this last expression is equal to $ac_1(n+1)$. □

Proposition 5. *For $n \geq 0$,*

$$ac_1(n) = \sum_{j=0}^{n-1} ac_0(j),$$

where, in the case $n = 0$, we take the empty sum to be 0.

Proof. For $n > 0$, we have by Proposition 4 that

$$\begin{aligned} ac_1(n) &= ac(n-1) \\ &= ac_0(n-1) + ac_1(n-1). \end{aligned}$$

Now if $n = 1$, we are done because $ac_1(0) = 0$. If $n > 1$, we can again apply Proposition 4 to get

$$ac_1(n) = ac_0(n-1) + ac(n-2).$$

Repeating the same argument $n-2$ more times gives the result. □

Proposition 6. *For all $n \geq 0$, we have*

$$\sum_{j=0}^{n} f_3(j) = \frac{f_3(n) + f_3(n+2) - 1}{2}.$$

Proof. We give a proof by mathematical induction. For $n = 0$,

$$f_3(0) = 0 = \frac{f_3(0) + f_3(2) - 1}{2}.$$

Now for $n > 0$, suppose the proposition holds for all $k < n$. Then,

$$\begin{aligned}
\sum_{j=0}^{n} f_3(j) &= \sum_{j=0}^{n-1} f_3(j) + f_3(n) \\
&= \frac{f_3(n-1) + f_3(n+1) - 1}{2} + f_3(n) \\
&= \frac{f_3(n+2) - f_3(n) - 1}{2} + f_3(n) \\
&= \frac{f_3(n) + f_3(n+2) - 1}{2}.
\end{aligned}$$

□

Proposition 7. *For all $n \geq 2$,*

$$ac_1(n) = f_3(n-3) + f_3(n-1).$$

Proof. By Proposition 5, Theorem 1, and Proposition 6, we have

$$\begin{aligned}
ac_1(n) &= \sum_{j=0}^{n-1} ac_0(j) \\
&= 2\sum_{j=0}^{n-1} f_3(j-2) + ac_0(0) \\
&= 2\sum_{j=0}^{n-3} f_3(j) + ac_0(0) \\
&= f_3(n-3) + f_3(n-1) - 1 + ac_0(0).
\end{aligned}$$

Because $ac_0(0) = 1$, the result follows. □

Proof of Theorem 2. Theorem 2 now immediately follows from Proposition 7, because $ac(1) = 1 = f_3(1) + f_3(-1)$, and for $n \geq 2$,

$$\begin{aligned}
ac(n) &= ac_0(n) + ac_1(n) \\
&= 2 \cdot f_3(n-2) + f_3(n-3) + f_3(n-1) \\
&= f_3(n) + f_3(n-2).
\end{aligned}$$

□

2.4. **Proof of Theorem 3 and Corollary 1.** In this section, we develop the formulae for $ac(n, s)$ by deriving the ordinary generating function $G(q, s)$ for a fixed $s \geq 0$. We split the proof into cases when s is even and odd.

Suppose $s = 2a$, where $a \geq 0$. An anti-palindromic composition of n of length $2a$ consists of a sequence of a ordered pairs of distinct positive integers. If $d(n)$ is the number of distinct pairs of positive integers that sum to n, then

$$D(q) := \sum_{n\geq 0} d(n)q^n = \left(\frac{q}{1-q}\right)^2 - \frac{q^2}{1-q^2} = \frac{2q^3}{(1-q^2)(1-q)}.$$

To see why this is the case, notice that
$$\left(\frac{q}{1-q}\right)^2 = \left(q^{1+1}\right) + \left(q^{1+2} + q^{2+1}\right) + \left(q^{1+3} + q^{2+2} + q^{3+1}\right) + \cdots$$
and
$$\frac{q^2}{1-q^2} = q^{1+1} + q^{2+2} + q^{3+3} + \cdots ,$$
so we are taking all pairs of positive integers and subtracting the repeated pairs.

To form a sequence of a such pairs, we multiple $D(q)$ by itself a times, showing that
$$G(q,2a) = [D(q)]^a = \frac{2^a q^{3a}}{(1-q^2)^a(1-q)^a}.$$
To prove the first half of Corollary 1, recall that for $a > 0$,
$$\frac{1}{(1-q^2)^a} = \sum_{n\geq 0} \binom{a-1+n}{n} q^{2n}$$
and
$$\frac{1}{(1-q)^a} = \sum_{n\geq 0} \binom{a-1+n}{n} q^{n}.$$
Multiplying these two series and reindexing gives the result.

Now suppose $s = 2a+1$, where $a \geq 0$. An anti-palindromic composition of n of length $2a+1$ still consists of a ordered pairs of distinct positive integers, with an additional central part. Therefore,
$$G(q,2a+1) = G(q,2a)\cdot\frac{q}{1-q} = \frac{2^a q^{3a}}{(1-q^2)^a(1-q)^a}.$$
The second half of Corollary 1 follows in the same manner as the first half, once we observe that
$$\frac{1}{(1-q)^{a+1}} = \sum_{n\geq 0} \binom{a+n}{n} q^{n}.$$

2.5. **Proof of Theorem 4 and Corollary 2.** We begin with a lemma.

Lemma 2. *For $n \geq 3$ and $s \geq 2$, we have*
$$ac(n,s) = ac(n-1,s) + ac(n-2,s) + 2\cdot ac(n-3,s-2) - ac(n-3,s).$$

Proof. If σ is an anti-palindromic composition of $n \geq 3$ of length $s \geq 2$, let $m_\sigma := \sigma_1 + \sigma_s$. Observe that $m_\sigma \geq 3$ and
$$\delta(m_\sigma) \leq |\sigma_1 - \sigma_s| \leq m_\sigma - 2,$$
where $\delta(m_\sigma) = 1$ if m_σ is odd and $\delta(m_\sigma) = 2$ if m_σ is even.

Let us first count the number of anti-palindromic compositions of n of length s with $m_\sigma = 3$. Each one of these compositions can be formed by taking an anti-palindromic composition of $n-3$ of length $s-2$ and adjoining a 1 at the beginning and a 2 at the end, or a 2 at the beginning and a 1 at the end. Therefore, the number of anti-palindromic compositions of n of length s with $m_\sigma = 3$ equals $2\cdot ac(n-3,s-2)$.

Next, we count the number of anti-palindromic compositions of n of length s with $m_\sigma > 3$. Now for any anti-palindromic composition τ of $n-1$ of length s, we can form an anti-palindromic composition of n of length s by adding 1 to τ_1 if $\tau_1 > \tau_s$, or adding 1 to τ_s if $\tau_s > \tau_1$. Now in this way, we have constructed all the anti-palindromic compositions of n of length s with $m_\sigma > 3$ and $|\sigma_1 - \sigma_s| > \delta(m_\sigma)$.

For any composition γ of $n-2$ of length s, form an anti-palindromic composition of n of length s by adding 1 to γ_1 and 1 to γ_s. In this way, we have constructed all the anti-palindromic compositions of n of length s with $m_\sigma > 3$ and $|\sigma_1 - \sigma_s| \le m_\sigma - 4$.

Therefore, the total number of anti-palindromic compositions of n of length s with $m_\sigma > 3$ and $\delta(m_\sigma) \le |\sigma_1 - \sigma_s| \le m_\sigma - 2$ equals $apc(n-1, s) + apc(n-2, s)$ minus the anti-palindromic compositions of n of length s with $m_\sigma > 3$ and $\delta(m_\sigma) < |\sigma_1 - \sigma_s| \le m_\sigma - 4$, as we have counted these compositions exactly twice. To prove the lemma, we now must show that the number of compositions that we counted twice equals $apc(n-3, s)$.

Let ρ be an anti-palindromic composition of $n-3$ of length s. Form an anti-palindromic composition of n of length s by adding 2 to ρ_1 and 1 to ρ_s if $\rho_1 > \rho_s$, or 1 to ρ_1 and 2 to ρ_s if $\rho_s > \rho_1$. In this way, we have constructed all of the anti-palindromic compositions of n of length s with $\delta(m_\sigma) < |\sigma_1 - \sigma_s| \le m_\sigma - 4$. □

Proof of Theorem 4. The theorem can be verified for all n and s with $n + s < 5$:

$$\begin{aligned}
\phi_0(q) &= rac(0,0)\cdot q^0 + rac(0,1)\cdot q^1 + rac(0,2)\cdot q^2 = 1\cdot q^0 + 0\cdot q^1 + 0\cdot q^2,\\
\phi_1(q) &= rac(1,0)\cdot q^0 + rac(1,1)\cdot q^1 + rac(1,2)\cdot q^2 = 0\cdot q^0 + 1\cdot q^1 + 0\cdot q^2,\\
\phi_2(q) &= rac(2,0)\cdot q^0 + rac(2,1)\cdot q^1 + rac(2,2)\cdot q^2 = 0\cdot q^0 + 1\cdot q^1 + 0\cdot q^2,\\
\phi_3(q) &= rac(3,0)\cdot q^0 + rac(3,1)\cdot q^1 + rac(3,2)\cdot q^2 = 0\cdot q^0 + 1\cdot q^1 + 1\cdot q^2.
\end{aligned}$$

Let $[q^s]\phi_n(s)$ be the coefficient of q^s in the polynomial $\phi_n(s)$. Now for $n \ge 3$ and $s \ge 2$, using the defining recurrence for $\phi_n(q)$, we have

$$\begin{aligned}
[q^s]\phi_n(q) &= [q^s]\phi_{n-1}(q) + [q^s]\phi_{n-2}(q) + [q^{s-2}]\phi_{n-3}(q) - [q^s]\phi_{n-3}(q)\\
&= rac(n-1,s) + rac(n-2,s) + rac(n-3,s-2) - rac(n-3,s)
\end{aligned}$$

by induction. Using the relationship between $rac(n,s)$ and $ac(n,s)$,

$$\begin{aligned}
[q^s]\phi_n(q) &= \frac{ac(n-1,s)}{2^{\lfloor \frac{s}{2}\rfloor}} + \frac{ac(n-2,s)}{2^{\lfloor \frac{s}{2}\rfloor}} + \frac{ac(n-3,s-2)}{2^{\lfloor \frac{s-2}{2}\rfloor}} - \frac{ac(n-3,s)}{2^{\lfloor \frac{s}{2}\rfloor}}\\
&= \frac{ac(n-1,s)}{2^{\lfloor \frac{s}{2}\rfloor}} + \frac{ac(n-2,s)}{2^{\lfloor \frac{s}{2}\rfloor}} + \frac{2\cdot ac(n-3,s-2)}{2^{\lfloor \frac{s}{2}\rfloor}} - \frac{ac(n-3,s)}{2^{\lfloor \frac{s}{2}\rfloor}}\\
&= \frac{ac(n,s)}{2^{\lfloor \frac{s}{2}\rfloor}}
\end{aligned}$$

by Lemma 2. Therefore, $[q^s]\phi_n(s) = rac(n,s)$. □

Proof of Corollary 2. Notice that by Theorem 4, we have

$$rac(n) = \sum_{s\ge 0} rac(n,s) = \phi_n(1) = f_2(n).$$

As for $rac_0(n)$, we have $rac_0(1) = rac_0(2) = 0$, $rac_0(3) = 1$, and for $n \ge 4$,

$$rac_0(n) = \sum_{s\ge 0} rac(n,2s) = \frac{\phi_n(1) + \phi_n(-1)}{2},$$

again using Theorem 4. By the definition of $\phi_n(q)$, this equals

$$\frac{\phi_{n-1}(1) + \phi_{n-1}(-1)}{2} + \frac{\phi_{n-2}(1) + \phi_{n-2}(-1)}{2} = rac_0(n-1) + rac_0(n-2).$$

This is the defining recurrence relation for the Fibonacci numbers; thus, we conclude that $rac_0(n) = f_2(n-2)$.

Similarly for $rac_1(n)$, we have $rac_1(0) = 1$, $rac_1(2) = rac_1(3) = 1$, and for $n \geq 4$,

$$rac_1(n) = \sum_{s \geq 0} rac(n, 2s+1) = \frac{\phi_n(1) - \phi_n(-1)}{2}$$

by Theorem 4. By the definition of $\phi_n(q)$, this equals

$$\frac{\phi_{n-1}(1) - \phi_{n-1}(-1)}{2} + \frac{\phi_{n-2}(1) - \phi_{n-2}(-1)}{2} = rac_1(n-1) + rac_1(n-2).$$

This is the defining recurrence relation for the Fibonacci numbers; thus, we conclude that $rac_1(n) = f_2(n-1)$. □

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