INFINITE SUMS INVOLVING GIBONACCI POLYNOMIALS REVISITED

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Abstract. We explore four sums involving gibonacci polynomials and extract their Pell versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \text{ and } l_n(x) = \alpha^n(x) + \beta^n(x),$$

where $2\alpha(x) = x + \Delta$, $2\beta(x) = x - \Delta$, and $\Delta = \sqrt{x^2+4}$. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n.

1.1. Some Fundamental Identities.

Gibonacci polynomials g_n satisfy the following fundamental properties [2]:

a) $f_{n+1} + f_{n-1} = l_n$ (page 8);
b) $l_{n+1} + l_{n-1} = \Delta^2 f_n$ (page 57);
c) $f_{2n} = f_n l_n$ (page 56);
d) $l_{n+1}^2 - l_{n-1}^2 = \Delta^2 x f_{2n}$ (page 57);
e) $l_{n+2}^2 - l_{n-2}^2 = (x^3+2x)\Delta^2 f_{2n}$ (page 57);
f) $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ (page 36);
g) $l_{n+k}l_{n-k} - l_n^2 = (-1)^{n+k}\Delta^2 f_k^2$ (page 58).

Property (g) is the *Cassini-like* (or *Catalan-like*) identity for Lucas polynomials.

Property (d) implies that $l_{2n+2}^2 - l_{2n}^2 = \Delta^2 x f_{2(2n+1)}$ and property (e) implies that $l_{2n+4}^2 - l_{2n}^2 = (x^3+2x)\Delta^2 f_{2n+2}$. In addition, it follows by the Cassini-like identity (g) that

$$\begin{aligned} l_{2n+3}l_{2n-1} &= l_{2n+1}^2 - \Delta^2 x^2; & l_{2n+2}l_{2n} &= l_{2n+1}^2 + \Delta^2; \\ l_{2n+3}l_{2n+1} &= l_{2n+2}^2 - \Delta^2; & l_{2n+4}l_{2n} &= l_{2n+2}^2 + \Delta^2 x^2. \end{aligned}$$

These identities play a major role in our discourse.

With this background, we begin our explorations of gibonacci sums, where the numerators and denominators involve Fibonacci and Lucas polynomials, respectively.

2. Gibonacci Polynomial Sums

The first sum involves a special class of even-numbered Fibonacci polynomials and odd-numbered Lucas polynomials.

Theorem 2.1.

$$\sum_{n=1}^{\infty} \frac{\Delta^2 x f_{2(2n+1)}}{\left(l_{2n+1}^2 + \Delta^2\right)^2} = \frac{1}{(x^2+2)^2}. \tag{2.1}$$

Proof. Using recursion [2], we will first confirm that

$$\sum_{n=1}^{m} \frac{\Delta^2 x f_{2(2n+1)}}{\left(l_{2n+1}^2 + \Delta^2\right)^2} = \frac{1}{(x^2+2)^2} - \frac{1}{l_{2m+2}^2}. \tag{2.2}$$

To this end, we let A_m = LHS and B_m = RHS. Then,

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{2m}^2} - \frac{1}{l_{2m+2}^2} \\ &= \frac{l_{2m+2}^2 - l_{2m}^2}{l_{2m+2}^2 l_{2m}^2} \\ &= \frac{\Delta^2 x f_{2(2m+1)}}{\left(l_{2m+1}^2 + \Delta^2\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Consequently, $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 = \dfrac{\Delta^2 x f_6}{(l_3^2 + \Delta^2)^2} - \dfrac{l_4^2 - l_2^2}{l_4^2 l_2^2} = 0$. So, $A_m = B_m$, as desired.

Because $\lim_{m\to\infty} \dfrac{1}{l_m} = 0$, equation (2.2) yields the given result. □

It follows from equation (2.2) that

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 + 5\right)^2} &= \frac{1}{45} - \frac{1}{L_{2m+2}^2}; \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 + 5\right)^2} &= \frac{1}{45}. \end{aligned} \tag{2.3}$$

The following result involves even-numbered gibonacci polynomials.

Theorem 2.2.

$$\sum_{n=1}^{\infty} \frac{(x^3+2x)\Delta^2 f_{2(2n+2)}}{\left(l_{2n+2}^2 + \Delta^2 x^2\right)^2} = \frac{1}{(x^2+2)^2} + \frac{1}{(x^4+4x^2+2)^2}. \tag{2.4}$$

Proof. Using recursion [2], we will first establish that

$$\sum_{n=1}^{m} \frac{(x^3+2x)\Delta^2 f_{2(2n+2)}}{\left(l_{2n+2}^2 + \Delta^2 x^2\right)^2} = \frac{1}{(x^2+2)^2} + \frac{1}{(x^4+4x^2+2)^2} - \frac{1}{l_{2m+2}^2} - \frac{1}{l_{2m+4}^2}. \tag{2.5}$$

Letting A_m = LHS and B_m = RHS, we then get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{2m}^2} - \frac{1}{l_{2m+4}^2} \\ &= \frac{l_{2m+4}^2 - l_{2m}^2}{l_{2m+4}^2 l_{2m}^2} \\ &= \frac{(x^3+2x)\Delta^2 f_{2(2m+2)}}{\left(l_{2m+2}^2 + \Delta^2 x^2\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Consequently, $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 = \dfrac{(x^3+2x)\Delta^2 f_8}{(l_4^2 + \Delta^2 x^2)^2} - \left(\dfrac{1}{l_2^2} - \dfrac{1}{l_6^2}\right) = 0$. Thus, $A_m = B_m$, as desired.

Letting $m \to \infty$, equation (2.5) yields the given result. □

It follows from equation (2.5) that

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 + 5\right)^2} &= \frac{58}{6615} - \frac{1}{15L_{2m+2}^2} - \frac{1}{15L_{2m+4}^2} \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 + 5\right)^2} &= \frac{58}{6615}. \end{aligned} \tag{2.6}$$

A Gibonacci Delight: Equation (2.3), coupled with (2.6), yields

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{F_{2n}}{(L_n^2+5)^2} &= \sum_{n=1}^{\infty} \left[\frac{F_{2(2n+1)}}{\left(L_{2n+1}^2+5\right)^2} + \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2+5\right)^2}\right] \\ &= \frac{41}{1323}; \\ \sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2+5)^2} &= \frac{2}{27}. \end{aligned} \tag{2.7}$$

Using the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ from property (f), we can rewrite equation (2.7) in terms of Fibonacci numbers alone:

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{[5(F_n^2+1) + 4(-1)^n]^2} = \frac{2}{27}.$$

The next result involves odd-numbered Lucas polynomials.

Theorem 2.3.

$$\sum_{n=1}^{\infty} \frac{(x^3+2x)\Delta^2 f_{2(2n+1)}}{\left(l_{2n+1}^2 - \Delta^2 x^2\right)^2} = \frac{1}{x^2} + \frac{1}{(x^3+3x)^2}. \tag{2.8}$$

Proof. Employing recursion [2], we will first prove that

$$\sum_{n=1}^{m} \frac{(x^3+2x)\Delta^2 f_{2(2n+1)}}{\left(l_{2n+1}^2 - \Delta^2 x^2\right)^2} = \frac{1}{x^2} + \frac{1}{(x^2+3x)^2} - \frac{1}{l_{2m+1}^2} - \frac{1}{l_{2m+3}^2}. \tag{2.9}$$

Again, we let A_m = LHS and B_m = RHS. Then,

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{2m-1}^2} - \frac{1}{l_{2m+3}^2} \\ &= \frac{l_{2m+3}^2 - l_{2m-1}^2}{l_{2m+3}^2 l_{2m-1}^2} \\ &= \frac{(x^3+2x)\Delta^2 f_{2(2m+1)}}{\left(l_{2m+1}^2 - \Delta^2 x^2\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

As before, this implies that $A_m - B_m = A_1 - B_1 = \dfrac{(x^3+2x)\Delta^2 f_6}{(l_3^2 - \Delta^2 x^2)^2} - \left(\dfrac{1}{l_1^2} - \dfrac{1}{l_5^2}\right) = 0$. So, $A_m = B_m$, confirming the validity of formula (2.9).

Letting $m \to \infty$ in equation (2.9) yields the given result, as desired. □

Equation (2.9) yields

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 - 5\right)^2} &= \frac{17}{240} - \frac{1}{15L_{2m+1}^2} - \frac{1}{15L_{2m+3}^2} \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 - 5\right)^2} &= \frac{17}{240}. \end{aligned} \tag{2.10}$$

Theorem 2.4.

$$\sum_{n=1}^{\infty} \frac{\Delta^2 x f_{2(2n+2)}}{\left(l_{2n+2}^2 - \Delta^2\right)^2} = \frac{1}{(x^3+3x)^2}. \tag{2.11}$$

Proof. Using recursion [2], we will first establish that

$$\sum_{n=1}^{m} \frac{\Delta^2 x f_{2(2n+2)}}{\left(l_{2n+2}^2 - \Delta^2\right)^2} = \frac{1}{(x^3+3x)^2} - \frac{1}{l_{2m+3}^2}. \tag{2.12}$$

Letting A_m = LHS and B_m = RHS yields

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{2m+1}^2} - \frac{1}{l_{2m+3}^2} \\ &= \frac{l_{2m+3}^2 - l_{2m+1}^2}{l_{2m+3}^2 l_{2m+1}^2} \\ &= \frac{\Delta^2 x f_{2(2m+2)}}{\left(l_{2m+2}^2 - \Delta^2\right)^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

This yields $A_m - B_m = A_1 - B_1 = \dfrac{\Delta^2 x f_8}{(l_4^2 - \Delta^2)^2} - \left(\dfrac{1}{l_3^2} - \dfrac{1}{l_5^2}\right) = 0$. Consequently, $A_m = B_m$, as expected.

Clearly, the given result now follows from equation (2.12), as desired. □

It follows from equation (2.12) that

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2-5\right)^2} &= \frac{1}{80}-\frac{1}{5L_{2m+3}^2} \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2-5\right)^2} &= \frac{1}{80}. \end{aligned} \tag{2.13}$$

Another Gibonacci Delight: Equation (2.10), coupled with (2.13), yields

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{F_{2n}}{(L_n^2-5)^2} &= \sum_{n=1}^{\infty} \left[\frac{F_{2(2n+1)}}{\left(L_{2n+1}^2-5\right)^2}+\frac{F_{2(2n+2)}}{\left(L_{2n+2}^2-5\right)^2}\right] \\ &= \frac{1}{12}; \\ \sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2-5)^2} &= \frac{1}{3}. \end{aligned} \tag{2.14}$$

Using the identity $L_n^2-5F_n^2=4(-1)^n$, we can rewrite equation (2.14) also in terms of Fibonacci numbers alone:

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{[5(F_n^2-1)+4(-1)^n]^2}=\frac{1}{3}.$$

2.1. **Fibonacci Consequences.** Using property (f), we can rewrite equations (2.1), (2.4), (2.8), and (2.11), in terms of Fibonacci polynomials alone:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Delta^2 x f_{2(2n+1)}}{\left[\Delta^2(f_{2n+1}^2+1)-4\right]^2} &= \frac{1}{(x^2+2)^2}; \\ \sum_{n=1}^{\infty} \frac{(x^3+2x)\Delta^2 f_{2(2n+2)}}{\left[\Delta^2(f_{2n+2}^2+x^2)+4\right]^2} &= \frac{1}{(x^2+2)^2}+\frac{1}{(x^4+4x^2+2)^2}; \\ \sum_{n=1}^{\infty} \frac{(x^3+2x)\Delta^2 f_{2(2n+1)}}{\left[\Delta^2(f_{2n+1}^2-x^2)-4)\right]^2} &= \frac{1}{x^2}+\frac{1}{(x^3+3x)^2}; \\ \sum_{n=1}^{\infty} \frac{\Delta^2 x f_{2(2n+2)}}{\left[\Delta^2(f_{2n+2}^2-1)+4\right]^2} &= \frac{1}{(x^3+3x)^2}, \end{aligned}$$

respectively.

They yield

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(5F_{2n+1}^2+1\right)^2} &= \frac{1}{45}; & \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(5F_{2n+2}^2+9\right)^2} &= \frac{58}{6615}; \\ \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(5F_{2n+1}^2-9\right)^2} &= \frac{17}{240}; & \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(5F_{2n+2}^2-1\right)^2} &= \frac{1}{80}, \end{aligned}$$

respectively.

Finally, we explore the Pell implications of the infinite sums in Theorems 2.1 through 2.4.

3. Pell Versions

Using the relationship $b_n(x) = g_n(2x)$, equations (2.1), (2.4), (2.8), and (2.11) yield the following sums:

$$\sum_{n=1}^{\infty} \frac{x(x^2+1)p_{2(2n+1)}}{\left[q_{2n+1}^2 + 4(x^2+1)\right]^2} = \frac{1}{32(2x^2+1)^2};$$

$$\sum_{n=1}^{\infty} \frac{(x^2+1)(2x^3+x)p_{2(2n+2)}}{\left[q_{2n+2}^2 + 16x^2(x^2+1)\right]^2} = \frac{1}{64(2x^2+1)^2} + \frac{1}{64(8x^4+8x^2+1)^2};$$

$$\sum_{n=1}^{\infty} \frac{(x^2+1)(2x^3+x)p_{2(2n+1)}}{\left[q_{2n+1}^2 - 16x^2(x^2+1)\right]^2} = \frac{1}{64x^2} + \frac{1}{64(4x^3+3x)^2};$$

$$\sum_{n=1}^{\infty} \frac{x(x^2+1)p_{2(2n+2)}}{\left[q_{2n+2}^2 - 4(x^2+1)\right]^2} = \frac{1}{32(4x^3+3x)^2},$$

respectively.

It then follows that

$$\sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{\left(Q_{2n+1}^2 + 2\right)^2} = \frac{1}{36}; \qquad \sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{\left(Q_{2n+2}^2 + 8\right)^2} = \frac{149}{31212};$$

$$\sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{\left(Q_{2n+1}^2 - 8\right)^2} = \frac{25}{588}; \qquad \sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{\left(Q_{2n+2}^2 - 2\right)^2} = \frac{1}{196},$$

respectively.

4. Acknowledgment

The author thanks the reviewer for a careful reading of the article, and for constructive suggestions and encouraging words.

References

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MSC2020: Primary 11B37, 11B39, 11B83, 11C08

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