A METHOD FOR UNIFORMLY PROVING A FAMILY OF IDENTITIES

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Abstract. This paper presents a proof method and a result. The proof method presented is particularly suitable for uniformly proving families of identities satisfied by a family of recursive sequences. To illustrate the method, we study the family of recursive sequences $F_n^{(k)} = \sum_{i=1}^{k} F_{n-i}^{(k)}$, $n \geq 0$, $k \geq 2$, with n a parameter varying over integers, and k a parameter indexing members of the family. The main theorem states $F_n^{(k)} = \sum_{j=1}^{k} P_{k,j} F_{n-jk}^{(k)}$, with P a recursive triangle satisfying the triangle recursion $P_{i,j} = 2P_{i-1,j} - P_{i-1,j-1}$, with appropriate initial conditions. The proof of the theorem exploits the fact that characteristic polynomials of identities are divisible by the characteristic polynomial of the recursion generating the underlying sequence.

1. Introduction and Main Result

This paper presents a proof method and result. The proof method presented, in contrast to other familiar proof methods for identities, including the Binet form, generating functions, and matrix methods, is particularly suitable for proving a family of identities satisfied by a family of recursive sequences.

To motivate the method, we first state the result proven in this paper, which presents a family of identities satisfied by the following family of recursive sequences indexed by a parameter $k \geq 2$.

$$F_n^{(k)} = \sum_{i=1}^{k} F_{n-i}^{(k)}, \tag{1.1}$$

with n a parameter varying over all integers.

Comment 1.1. Four comments are now presented discussing i) initial values, ii) doubly infinite sequences, iii) minimal polynomials, and iv) the names associated with this family as well as with individual members.

In the sequel, we will use the abbreviation $F^{(k)}$ to refer to the sequence $\{F_n^{(k)}\}_{\{\text{all } n\}}$.

Initial values. The Online Encyclopedia of Integer Sequences, OEIS, in presenting the Tribonacci and Tetranacci sequences, [17, A000073], [17, A000078], indicates the following initial values:

$$F_i^{(k)} = 0, \qquad 0 \leq i \leq k-2, \qquad F_{k-1}^{(k)} = 1. \tag{1.2}$$

However, the literature notes two alternatives to (1.2): 1) [16] presents initial conditions with $k-2$ zeroes followed by two 1s, and 2) [2], while adhering to initial values of $k-1$ zeroes followed by a 1, starts the sequence at $F_{k-2}^{(k)}$ so that $F_1^{(k)} = 1$. In this paper, we follow (1.2), which is the OEIS approach. However, the results of this paper are true for any sequence satisfying (1.1), independent of initial conditions.

Doubly infinite sequence. As with any recursive sequence initially defined for nonnegative n, (1.1) allows extending the sequence to negative indices.

Minimal polynomials. For reference and later use, for $k \geq 2$, we let $r_k(X)$ indicate the minimal polynomial of $F^{(k)}$.

$$r_k(X) = X^k - \sum_{i=1}^{k} X^{k-i}. \tag{1.3}$$

The theory associated with $r_k(x)$ is well-known: i) $r_k(X)$ is irreducible over $\mathbb{Z}[x]$, and hence, ii) it is the *minimal polynomial* for $F^{(k)}$. iii) Furthermore, the $\mathbb{Z}[x]$-ideal generated by the minimal polynomial contains the characteristic equation associated with any identity satisfied by the sequences that the minimal recursion generates. Additionally, iv) $r_k(X)$ has simple roots; one root is close to two whereas the other roots lie in the unit circle [14, 15, 21]. However, in the sequel we will not need the entire theory.

Names. For $k = 2, 3, 4$ and for traditional initial values, (1.2), (1.1) corresponds to the Fibonacci, Tribonacci [17, A000073], and Tetranacci [17, A000078] sequences respectively.

However, there is no accepted standardized name for the family of sequences described by (1.1). When names do exist, they are also used to refer to other types of families of identities. Consequently, in this paper we will suffice with using the algebraic notation, $F^{(k)}$. Here are some details supporting this assertion.

An initial place to look for names for the family described by (1.1) would be *The Fibonacci Quarterly*. But, we find multiple terms such as *generalized Fibonacci numbers* [7, 11], *k-generalized Fibonacci numbers* [8], and *generalized Fibonacci k-sequences* [9]. Dresden [4] lists five accepted names for the family described by (1.1): *i) generalized Fibonacci numbers of order k, ii) Fibonacci numbers of order k, iii) Fibonacci k-step numbers, iv) Fibonacci k-sequences, and v) k-bonacci numbers.* If one looks for rarer names occurring in only one or two articles, other names pop up such as *the k-nacci numbers* [1]. But even here, a routine Google search using the search term "k-nacci" shows multiple papers.

It might appear from the preceding discussion that the description *generalized Fibonacci numbers* is a safe consensus for a name. But [5, 6], citing multiple references, point out that in generalizing the Fibonacci recursion, $F_n = aF_{n-1} + bF_{n-2}$, $a = b = 1$, some authors have altered the starting values, whereas others have preserved the first two terms of the sequence but changed the recurrence relation. In other words, the phrase *generalized Fibonacci numbers* can refer to different things in different articles.

As indicated at the introduction of this comment, we find the best course is to state that we are proving a family of identities satisfied by a family of recursive sequences $F^{(k)}$, which for $k = 2, 3, 4$ are the familiar Fibonacci, Tribonacci, and Tetranacci sequences (with appropriate initial values).

We motivate the main result by considering three examples.

- The subsequence of Fibonacci numbers restricted to even indices, $\{G_n\}_{n\geq 0} = \{0, 1, 3, 8, 21, 55, \ldots\}$, satisfies the recursion $G_n = 3G_{n-1} - G_{n-2}$. Equivalently, $\{F_n\}_{n\geq 0}$ satisfies $F_n = 3F_{n-2} - F_{n-4}$. Note that this last identity is satisfied by the Fibonacci numbers restricted to either even or odd indices.
- Similarly, the subsequence of Tribonacci numbers, as defined by the OEIS, [17, A000073], restricted to indices divisible by 3, $\{G_n\}_{n\geq 0} = \{0, 1, 7, 44, 274, 1705, 10609, \ldots\}$, satisfies the recursion $G_n = 7G_{n-1} - 5G_{n-2} + G_{n-3}$. For purposes of this paper, we will address the corresponding identity on the Tribonacci numbers, $F_n^{(3)} = 7F_{n-3}^{(3)} - 5F_{n-6}^{(3)} + F_{n-9}^{(3)}$. As in the case of the Fibonacci numbers, it is simply a convenience to say that the identity holds for the subsequence of indices divisible by three; it actually also holds for any subsequence of indices at a fixed congruence modulo 3.

- The subsequence of the Tetranacci numbers, as defined by the OEIS, [17, A000078], restricted to indices divisible by 4, $\{G_n\}_{n\geq 0} = \{0, 1, 15, 208, 2872, 39648, 547337, \ldots\}$, satisfies the recursion, $G_n = 15G_{n-1} - 17G_{n-2} + 7G_{n-3} - G_{n-4}$, or equivalently, the Tetranacci numbers satisfy the recursion $F_n^{(4)} = 15F_{n-4}^{(4)} - 17F_{n-8}^{(4)} + 7F_{n-12}^{(4)} - F_{n-16}^{(4)}$.

These examples motivate defining a triangle P, presented in Table 1, whose second, third, and fourth rows (starting the count of rows from the 0th row) correspond to the coefficients in the recursions just listed.

Row Number-Column Number	0	1	2	3	4	5	6
0	–1	0	0	0	0	0	0
1	–1	1	0	0	0	0	0
2	–1	3	–1	0	0	0	0
3	–1	7	–5	1	0	0	0
4	–1	15	–17	7	–1	0	0
5	–1	31	–49	31	–9	1	0
6	–1	63	–129	111	–49	11	–1

Table 1: The recursive triangle P.

To formulate our main result, we instead define P recursively as follows.

Definition 1.2.

$$P_{i,j} = 2P_{i-1,j} - P_{i-1,j-1}. \tag{1.4}$$

The initial values for P are as given in Table 1 in row 0 and column 0. As with any triangle recursion, once initial values are given, the associated sequence can be extended to a doubly infinite array. This will be useful in the proof.

Having defined P, we can then formulate a main theorem, which states that the rows of P give the coefficients of the recursion satisfied by the subsequence of $\{F_n^{(k)}\}_{\{\text{all n}\}}$ restricted to indices divisible by k.

Theorem 1.3. *For $k \geq 2$, (and all n)*

$$F_n^{(k)} = \sum_{i=1}^{k} P_{k,i} F_{n-ki}^{(k)}. \tag{1.5}$$

2. Proof Method

This section motivates the need for a separate proof method. Traditional proof methods include the Binet form, generating functions, and matrix methods [13]. If we are proving an identity satisfied by a single recursive sequence, these methods suffice.

However, we are proving that for each k, (1.5) is satisfied by $F^{(k)}$. We therefore need a *uniform* method to prove the family of identities. The Binet form for each $F^{(k)}$ is different with different roots. Similar comments apply to generating functions and matrices.

The method by which we prove (1.5) exploits that the minimal polynomial of a recursive sequence generates an ideal containing the characteristic polynomials corresponding to all identities satisfied by the recursive sequence. This method will allow uniform proofs.

To make this precise, we first illustrate the flow of logic in the proof by discussing the proof for $k = 2$; that the subsequence of Fibonacci numbers restricted to even indices, $\{G_n\}_{n\geq 0} = \{0, 1, 3, 8, 21, 55, \ldots\}$, satisfies the recursion

$$G_n = 3G_{n-1} - G_{n-2}. \tag{2.1}$$

.

As just formulated, this appears to be a statement about a sequence $\{G_n\}_{n\geq 0}$. We can equivalently prove

$$F_n = 3F_{n-2} - F_{n-4}, \tag{2.2}$$

which is a statement about the Fibonacci numbers. We simply observe that (2.2) implies (2.1).

Definition 2.1. *If $H_m = \sum_{i=1}^{n} c_i H_{m-i}$ is an identity satisfied by a recursive sequence $\{H_m\}_{m\geq 0}$, then the characteristic polynomial of the identity is*

$$p(X) = X^n - \sum_{i=1}^{n} c_i X^{n-i}. \tag{2.3}$$

Equation (2.3) establishes a correspondence between recursive sequences and polynomials.

This definition of characteristic polynomial follows [18, Section 6.2]. As noted in [10], other definitions do exist. Definition 2.1 regards the characteristic polynomial as an attribute of the identity. Contrastively, the minimal polynomial is an attribute of the recursive sequence; roughly, it is the characteristic polynomial with leading coefficient 1 of smallest degree whose associated recursion is satisfied by the sequence. The following result is well-known, but is not commonly used in traditional sources [13] to prove identities. Webb [19, 20] has written some papers showing the usefulness of such methods. The proof is short and also given below.

Proposition 2.2. *A recursive identity is satisfied by a recursive sequence if and only if the characteristic polynomial of that identity is divisible by the minimal polynomial of the underlying recursion.*

Proof. Interpret the variable X as the difference operator [3]. Then the minimal polynomial, considered as an operator when applied to the sequence to which it is a minimal polynomial, annihilates that sequence; that is, the resulting sequence is identically 0. It immediately follows, by the associativity of polynomials, that any polynomial multiple of this minimal polynomial also annihilates the underlying recursive sequence, showing that the identity corresponding to this multiple of the minimal polynomial is satisfied by the recursive sequence. Because in $\mathbb{Z}[x]$, all ideals are generated by a single element, it follows that the characteristic polynomial associated with any identity must lie in the ideal generated by the minimal polynomial. □

It follows that to prove (2.2), it suffices to show that the characteristic polynomial of this identity is divisible by the minimal polynomial for the Fibonacci numbers, $r(X) = X^2 - X - 1$.

The flow of logic is concisely presented in Table 2.

Row Number	Item	Recursive-Identity Formulation	Characteristic-Polynomial Formulation
1 2	Fibonacci Numbers, Even-indexed Fibonacci numbers	$F_n = F_{n-1} + F_{n-2}$ $F_n = 3F_{n-2} - F_{n-4}$	$r(X) = X^2 - X - 1$ $p(X) = X^4 - 3X^2 + 1$
3	The Row-2 recursion is satisfied by the Fibonacci numbers		$p(X)$ is divisible by $r(X)$

Table 2: Polynomial formulation of theorem.

To prove that $p(X)$ is divisible by $r(X)$, it suffices to show that the quotient $p(X)/r(X) = q(X)$ lies in $\mathbb{Z}[x]$, that is, has integer coefficients.

For the case of general k, first, we define $p_k(X)$, $k \geq 2$, as follows.

$$p_k(X) = X^k - \sum_{i=1}^{k} P_{k,i} X^{k-i}. \tag{2.4}$$

Then, the characteristic polynomial of (1.5) is $p_k(X^k)$. $p_k(X)$ has descending powers of X, whereas $p_k(X^k)$ has powers that descend by multiples of k as required.

By the discussion above, to prove (1.5) it suffices to prove that the *corresponding* characteristic polynomial $p_k(X^k)$ is divisible by the minimal polynomial of $F^{(k)}$, $r_k(X)$. Alternatively, we must show that $\frac{p_k(X^k)}{r_k(X)}$ lies in $\mathbb{Z}[x]$.

We therefore proceed as follows.

Proposition 2.3. *Define rational functions $q_k(x)$, $k \geq 2$ by*

$$p_k(X^k) = r_k(X) q_k(X). \tag{2.5}$$

Then to prove (1.5)*, it suffices to prove that $q_k(X) \in \mathbb{Z}[X]$.*

Actually, in this paper, we prove the equivalent equation,

$$-p_k(X^k) = (-r_k(X)) q_k(X). \tag{2.6}$$

This is purely a convenience to avoid excessive minus signs in tables and equations.

To summarize, we will prove (2.6), which in turn implies (2.5), which in turn shows $\frac{p_k(X^k)}{r_k(X)}$ lies in $\mathbb{Z}[x]$, or equivalently that $p_k(X^k)$ is divisible by $r_k(X)$. By Proposition 2.2, this implies that (1.5), the identity corresponding to the characteristic polynomial $p_k(X^k)$, is satisfied by $F^{(k)}$, the recursive sequence corresponding to the minimal polynomial $r_k(X)$. In other words, proving (2.6) suffices to prove Theorem 1.3.

We close this section by explaining where the *work* for the proof lies. We must show, *for each* k, that $q_k(X)$ has integral coefficients. What we actually do is inductively describe the coefficients of $q_k(X)$, and then show that $q_k(X)(-r_k(X)) = -p_k(X^k)$.

3. The Coefficients of $q_k(X)$

To describe the coefficients of $q_k(x)$, we first define a matrix Q, presented in Table 3, which is obtained by applying the backward difference operator to P.

Definition 3.1. *The triangle Q is defined by*

$$Q_{i,j} = P_{i,j} - P_{i-1,j}. \tag{3.1}$$

Row Number-Column Number	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	2	−1	0	0	0	0
3	0	4	−4	1	0	0	0
4	0	8	−12	6	−1	0	0
5	0	16	−32	24	−8	1	0
6	0	32	−80	80	−40	10	−1

Table 3: The recursive triangle Q defined by Definition 3.1.

In the proof of Theorem 1.3, besides using (3.1), we will need the following two easily proven identities. These three identities allow us to convert sums in Q-elements into P-elements.

$$\sum_{r=c}^{t} Q_{r,c} = P_{t,c}, \qquad \text{by telescoping and (3.1),} \tag{3.2a}$$

$$\sum_{r=s}^{t} Q_{r,c} = P_{t,c} - P_{s-1,c}. \qquad \text{by the equation just stated.} \tag{3.2b}$$

Using (3.1), we can now completely describe the polynomial $q_k(x)$, $k \geq 2$.

$$q_k(x) = x^{k(k-1)} + \sum_{j=0}^{k-2} x^{kj} \left(-P_{k-1,k-1-j} + \sum_{i=k-1-j}^{k-1} Q_{i,k-1-j} x^{k-i} \right). \tag{3.3}$$

As explained in the last section, to prove (1.5), it suffices to show (2.6) is true, which is what we will spend the rest of the paper doing.

4. Some Literature

Before continuing the proof, we take note of certain related items appearing in the literature.

[17, A193845] presents a triangle similar to P with only positive entries. This triangle can be generated by power series, polynomials, or using fission and fusion [12]. It is easy to "fix" this OEIS triangle so that it agrees with P in sign.

For example, M. F. Hasler in the last comment of the *comment* section of sequence [17, A193845], dated Oct. 15, 2014, notes that the kth row of A193845 lists the coefficients of the polynomial $\sum_{i=0}^{k}(X+2)^i$, in order of increasing powers.

To modify this for the P triangle in Table 1, which has negative entries, we first replace $X+2$ with $2-X$. An example, for $k=3$, will illustrate the further required polynomial manipulations. We can verify that $(2-X)^3-(2-X)^2-(2-X)-(2-X)^0 = -X^3+5X^2-7X+1$, which is almost what we want except that the signs and exponents have to be adjusted. First, by (1.3), we can rewrite this as $r_3(2-X) = (2-X)^3-(2-X)^2-(2-X)-(2-X)^0 = -X^3+5X^2-7X+1$. To fix the sign, we can, by (2.4), write $X^3 r_3(2-\frac{1}{X}) = -1+5X-7X^2+X^3 = p_3(X)$. We can similarly write, for arbitrary k, $X^{k^2} r_k(2-\frac{1}{X^k}) = p_k(X^k)$.

This gives an alternate definition of triangle P, from which the recursion (1.4) can be derived. Expositionally, because all we need is the underlying recursion, it is simpler to directly define triangle P by (1.4).

Similar observations can be made about triangle Q in Table 3. A triangle similar to Q with only positive entries is found in [17, A038207]. N.-E. Fahssi, in the *comment* section of this sequence (with comment date, Apr. 13 2008), points out that the kth row of A038207 are the coefficients $(2+X)^k$.

We can adjust this idea to triangle Q, which has negative entries. The coefficients of $X(2-X)^k$ give the coefficients in the kth row of triangle Q. For example, when $k=2$, $X(2-X)^2 = 4X-4X^2+X^3$ corresponding to row 3 in Table 3.

These alternate definitions might provide an alternate proof of (1.5) using polynomials. A challenge in proving (2.6) by polynomials, or some similar method such as fusion and fission is that, as seen in (3.3), the coefficients of $q_k(X)$ are mixtures of Q and P triangle entries. The proof presented in this paper is straightforward and computational, and should have independent interest. We therefore leave the search for a purely polynomial proof of (2.6) as an open problem.

5. The Matrix M

We prove (2.6) computationally by performing the product long hand. To keep tabs on calculations, we store the information in a matrix $M = M_k$. Throughout the proof, Tables 4 through 7 presenting M_5 facilitate following arguments.

The matrix M, as illustrated in Tables 4 and 5 for the case $k = 5$, labels columns with X raised to a power. In the sequel, we will interchangeably refer to the *column with header* X^k as column X^k or column k.

The rows of M are labeled with the summands in $-r_k(X)$, (1.3). Similar to the way we refer to columns, in the sequel we will refer, for example, to the row with header $-X^k$ as either row k or row X^k (leaving out the sign). This should cause no confusion.

Using this terminology, we define M as follows:

$$M_{r,c} = \text{coefficient of } X^c \text{ in } X^r q_k(X), \qquad 0 \le r \le k-1 \tag{5.1a}$$

$$= \text{coefficient of } X^{c-r} \text{ in } q_k(X), \qquad c \ge r,\ r \ne k \tag{5.1b}$$

$$M_{k,c} = \text{coefficient of } X^c \text{ in } -X^k q_k(X), \tag{5.1c}$$

$$= \text{coefficient of } X^{c-k} \text{ in } -q_k(X), \qquad c \ge k \tag{5.1d}$$

For $k = 5$, Tables 4 and 5 present all non-leading coefficients of $q_k(X)$ using Q and P with subscripts, whereas Tables 6 and 7 present the associated numerical values. The leading coefficient for the X^{20} column is simply 1 in Tables 5 and 7. Tables 4 through 7 have a terminal *sum* row indicating in each column the sum of the entries above it in that column. The top *case* row is explained in the next section.

Case	A	B	B	B	B	C	E	B	B	B	C	E	E	B	B
$row - col$	X^0	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}	X^{11}	X^{12}	X^{13}	X^{14}
X^0	$-P_{4,4}$	$Q_{4,4}$				$-P_{4,3}$	$Q_{4,3}$	$Q_{3,3}$			$-P_{4,2}$	$Q_{4,2}$	$Q_{3,2}$	$Q_{2,2}$	
X^1		$-P_{4,4}$	$Q_{4,4}$				$-P_{4,3}$	$Q_{4,3}$	$Q_{3,3}$			$-P_{4,2}$	$Q_{4,2}$	$Q_{3,2}$	$Q_{2,2}$
X^2			$-P_{4,4}$	$Q_{4,4}$				$-P_{4,3}$	$Q_{4,3}$	$Q_{3,3}$			$-P_{4,2}$	$Q_{4,2}$	$Q_{3,2}$
X^3				$-P_{4,4}$	$Q_{4,4}$				$-P_{4,3}$	$Q_{4,3}$	$Q_{3,3}$			$-P_{4,2}$	$Q_{4,2}$
X^4					$-P_{4,4}$	$Q_{4,4}$				$-P_{4,3}$	$Q_{4,3}$	$Q_{3,3}$			$-P_{4,2}$
$-X^5$						$P_{4,4}$	$-Q_{4,4}$				$P_{4,3}$	$-Q_{4,3}$	$-Q_{3,3}$		
SUM	$P_{5,5}$					$P_{5,4}$					$P_{5,3}$				

Table 4: Coefficients of of $-r_5(X) \times q_5(X)$ using Q, P, columns 0–14.

Case	C	E	E	E	B	D	D	D	D	D	A
$row - col$	X^{15}	X^{16}	X^{17}	X^{18}	X^{19}	X^{20}	X^{21}	X^{22}	X^{23}	X^{24}	X^{25}
X^0	$-P_{4,1}$	$Q_{4,1}$	$Q_{3,1}$	$Q_{2,1}$	$Q_{1,1}$	1					
X^1		$-P_{4,1}$	$Q_{4,1}$	$Q_{3,1}$	$Q_{2,1}$	$Q_{1,1}$	1				
X^2	$Q_{2,2}$		$-P_{4,1}$	$Q_{4,1}$	$Q_{3,1}$	$Q_{2,1}$	$Q_{1,1}$	1			
X^3	$Q_{3,2}$	$Q_{2,2}$		$-P_{4,1}$	$Q_{4,1}$	$Q_{3,1}$	$Q_{2,1}$	$Q_{1,1}$	1		
X^4	$Q_{4,2}$	$Q_{3,2}$	$Q_{2,2}$		$-P_{4,1}$	$Q_{4,1}$	$Q_{3,1}$	$Q_{2,1}$	$Q_{1,1}$	1	
$-X^5$	$P_{4,2}$	$-Q_{4,2}$	$-Q_{3,2}$	$-Q_{2,2}$		$P_{4,1}$	$-Q_{4,1}$	$-Q_{3,1}$	$-Q_{2,1}$	$-Q_{1,1}$	-1
SUM	$P_{5,2}$					$P_{5,1}$					-1

Table 5: Coefficients of $-r_5(X) \times q_5(X)$ using Q, P, columns 15–25.

$row-col$	X^0	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}	X^{11}	X^{12}	X^{13}	X^{14}
X^0	1	–1				–7	6	1			17	–12	–4	–1	
X^1		1	–1				–7	6	1			17	–12	–4	–1
X^2			1	–1				–7	6	1			17	–12	–4
X^3				1	–1				–7	6	1			17	–12
X^4					1	–1				–7	6	1			17
$-X^5$						–1	1				7	–6	–1		
SUM	1					–9					31				

Table 6: Numerical coefficients of $-r_5(X) \times q_5(X)$, columns 0–14.

$row-col$	X^{15}	X^{16}	X^{17}	X^{18}	X^{19}	X^{20}	X^{21}	X^{22}	X^{23}	X^{24}	X^{25}
X^0	–15	8	4	2	1	1					
X^1		–15	8	4	2	1	1				
X^2	–1		–15	8	4	2	1	1			
X^3	–4	–1		–15	8	4	2	1	1		
X^4	–12	–4	–1		–15	8	4	2	1	1	
$-X^5$	–17	12	4	1		15	–8	–4	–2	–1	–1
SUM	–49					31					–1

Table 7: Numerical coefficients of $-r_5(X) \times q_5(X)$, columns 15–25.

As can be seen from the X^0 rows of Tables 6 and 7,

$$\begin{aligned} q_5(X) = X^{20} &+ \Big(X^{19} + 2X^{18} + 4X^{17} + 8X^{16} - 15X^{15}\Big) \\ &+ \Big(-X^{13} - 4X^{12} - 12X^{11} + 17X^{10}\Big) + \Big(X^7 + 6X^6 - 7X^5\Big) + \Big(-X + 1\Big). \end{aligned} \tag{5.2}$$

Inspecting the expression inside the summation sign of (3.3), for any j, $0 \le j \le k-2$,

$$X^{kj}\left(-P_{k-1,k-1-j} + \sum_{i=k-1-j}^{k-1} Q_{i,k-1-j}X^{k-i}\right), \tag{5.3}$$

we note certain patterns in each such block. More specifically, the jth block (from left to right) starts with a P term, followed by Q terms, followed by 0 terms. In (5.2), the jth block, $0 \le k \le 3$, is indicated by the use of bigger parentheses.

Formally, for each j, $0 \le j \le k-2$,

- $M_{0,jk} = -P_{k-1,k-1-j}$,
- $M_{0,jk+m} = Q_{k-m,k-1-j}$ for $1 \le m \le j+1$,
- $M_{0,jk+m} = 0$ for $j+2 \le m \le k-1$.

Additionally, geometrically, each successive row after the first row shifts the row above it by one column to the right as seen in Tables 4 and 5 (with the last row having negative signs). This shifting is formally justified by (5.1). In the sequel, we will simply refer to (5.1) for justifications, it being understood that (5.1) is geometrically clarified by these bullets.

6. Overview of Proof

To prove (2.6) using the matrix M, we must show that for each column of M, the sum of coefficients in rows 0 through k equals the corresponding coefficient in $-p_k(X^k)$, found in the bottom *sum* row.

That is, we must show

$$\sum_r M_{r,jk} = P_{k,j}, \qquad 0 \leq j \leq k, \tag{6.1a}$$

$$\sum_r M_{r,jk+m} = 0, \qquad 0 \leq j \leq k-1,\ 1 \leq m \leq k-1. \tag{6.1b}$$

To prove (2.6), we must prove (6.1). We prove (6.1) separately on the following five groups of columns. The proof of (6.1) for each group of columns will occupy one of the next five sections.

A. Columns k^2 and 0.
B. Columns $jk+m$, $0 \leq j \leq k-2$, $j+1 \leq m \leq k-1$.
C. Columns jk, $1 \leq j \leq k-2$.
D. Columns $jk+m$, $j = k-1$, $0 \leq m \leq k-1$.
E. Columns $jk+m$, for

$$0 \leq j \leq k-2, 1 \leq m \leq j. \tag{6.2}$$

For the reader's convenience, the letters denoting cases are included in Tables 4 and 5. As shown, these five cases are mutually exclusive and exhaust all columns. Throughout the proof of these five cases, presented in the next five sections, we i) review Tables 4 and 5 to discover patterns, and, using (5.1), ii) state the pattern discovered for $k = 5$, for general k.

7. Proof of (6.1) for Case A, Columns 0 and k^2

By (1.4) and the initial conditions for triangle P presented in Table 1, we have for $k \geq 1$,

$$-P_{k-1,k-1} = P_{k,k}.$$

It follows that for column 0, the *sum* row equals the sum of entries above it, verifying (6.1) for column X^0. For the column X^{k^2}, (6.1) reduces to verification that $-X^k \times X^{k(k-1)} = -X^{k^2}$.

8. Proof of (6.1) for Case B, Columns $jk+m$, $0 \leq j \leq k-2$, $j+1 \leq m \leq k-1$

For the case $k = 5$, Table 4 shows that the three elements of block 1 at columns 5, 6, and 7 of row X^0 diagonally descend so that these three elements comprise the entries in columns 7, 8, and 9. Similarly, the four elements in block 2 at columns 10, 11, 12, and 13 in row X^0 diagonally descend so that these four elements comprise the entries in columns 13 and 14.

This pattern is true for general k. By (5.1), for the jth block, $1 \leq j \leq k-2$, the set of column elements for columns $jk+m$, $0 \leq j \leq k-2$, $j+1 \leq m \leq k-1$ is equal to the set of entries in the 0th row of that block.

Hence, for $0 \le j \le k-2$, $j+1 \le m \le k-1$,

$$\left.\begin{aligned}
&\text{the sum of entries in the } (jk+m)\text{th column}\\
&= \sum_{r=0}^{k} M_{r,jk+m}\\
&= -P_{k-1,k-1-j} + \sum_{r=k-1-j}^{k-1} Q_{r,k-1-j} = 0, \qquad \text{by (3.2)(a).}
\end{aligned}\right\}$$

Notice that by (1.4), (3.1), and the initial conditions, row 0 and column 0 of Table 1, $Q_{r,c} = 0$ if $r < c$. These 0 entries correspond to blank cells in Tables 4 through 7.

9. Proof of (6.1) for Case C, Columns jk, $1 \le j \le k-2$

Table 4 shows that the three elements in the 0th row of block 1 at columns 5, 6, and 7 diagonally descend to column 10, the beginning column of the 2nd block, except that i) the $-P$ term from column 5 now has a positive sign and ii) column 10 has an additional $-P$ term in row X^0.

By (5.1), for general k, column jk, $1 \le j \le k-2$, has i) all Q terms from the 0th row of the $(j-1)$st block, ii) a P term from column $(j-1)k$ with a positive sign, and iii) the P term in row 0, column jk.

Hence, for $1 \le j \le k-2$,

$$\left.\begin{aligned}
&\text{the sum of elements in column } kj\\
&= P_{k-1,k-j} + \sum_{r=k-j}^{k-1} Q_{r,k-j} - P_{k-1,k-j-1}\\
&= 2P_{k-1,k-j} - P_{k-1,k-j-1}, \qquad \text{by (3.2)(a),}\\
&= P_{k,k-j}. \qquad \text{by (1.4).}
\end{aligned}\right\}$$

10. Proof of (6.1) for Case D, Columns $jk+m$, $j = k-1$, $0 \le m \le k-1$

For the case $k = 5$, Table 7 motivates proving (6.1) using well-known identities about powers of 2.

For general k, by (1.4), (3.1), and by the initial conditions presented in Tables 1 and 3, a straightforward inductive argument shows that

$$P_{n,1} = 2^n - 1, \qquad Q_{n,1} = 2^{n-1}, \qquad n \ge 1.$$

By inspection of Table 5 for the case $k = 5$, and by (5.1), for general k, we see that column $jk+m$, $0 \le m \le k-1$ has a 1 entry and elements $Q_{1,1}, \ldots, Q_{k-1-m,1}$. There are now two cases to consider according to the value of m.

<u>Case $m = 0$</u>. If $m = 0$, column $jk+m$ also has entry $P_{k-1,1}$. Hence, the following proves (6.1) for column $jk+m$, $m = 0$ in the $(k-1)$st block.

$$\sum_{r=0}^{k} M_{r,k(k-1)} = 1 + \sum_{r=1}^{k-1} Q_{r,1} + P_{k-1,1} = 1 + \sum_{r=1}^{k-1} 2^{r-1} + (2^{k-1} - 1) = P_{k,1}.$$

Case $m = 1$. If $m \neq 0$, column $jk + m$ also has entry $-Q_{k-m,1}$. Hence, the following proves (6.1) for column $jk + m$, $m \neq 0$ in the $(k-1)$st block.

$$\sum_{\substack{r=0\\m\neq 0}}^{k} M_{r,k(k-1)+m} = 1 + \sum_{r=1}^{k-1-m} Q_{r,1} - Q_{k-m,1} = 1 + \sum_{r=1}^{k-1-m} 2^{r-1} - 2^{k-m-1} = 0.$$

11. Proof of (6.1) for Case E, (6.2), Columns $jk + m$ for $0 \leq j \leq k-2$, $1 \leq m \leq j$

For each column in Case E whose sum we are trying to show equal to 0, we will first show that the column entries naturally divide into four groups (with some groups empty). The proof is then completed by showing that the sum of the four group-sums equals 0.

1st group. As can be motivated by Tables 4 and 5, and as justified for general k by (5.1), all columns satisfying (6.2) have a $-P$ term which descends diagonally from the column in that block divisible by k.

For example, column 6 has a $-P_{4,3}$ term that diagonally descends from column 5; columns 11 and 12 have a $-P_{4,2}$ term that diagonally descends from column 10; and columns 16, 17, 18 each have a $-P_{4,1}$ term that diagonally descends from column 15.

Thus, for general k, this first group of the elements in a column satisfying (6.2) is defined as the singleton P term that diagonally descends from the column of that block divisible by k.

$$\text{Sum of the Singleton Element in 1st Group} = -P_{k-1,k-1-j}. \tag{11.1}$$

2nd group. In the X^5 row of Tables 4 and 5, any column satisfying (6.2) has a negative Q term that descends from column $(j-1)k$ in the $(j-1)$st block. When $j = 0$, this group is vacuously empty; for $j = 1$, column 6 has an entry $-Q_{4,4}$, which diagonally descends from column 5; for $j = 2$, columns 11 and 12 have entries of $-Q_{4,3}$, $-Q_{3,3}$ respectively, which diagonally descend from column 10; and for $j = 3$, columns 16, 17, and 18 have entries $-Q_{4,2}$, $-Q_{3,2}$, and $-Q_{2,2}$ respectively, which diagonally descend from column 15.

By (5.1), for general k, this second group of elements in a column satisfying (6.2) is defined as the negative Q term that diagonally descends from column $(j-1)k$ in the $(j-1)$st block. By (3.1), for each m and j satisfying (6.2), the sum of the singleton element in this group is the following.

$$-Q_{k-m,k-j} = -\Big(P_{k-m,k-j} - P_{k-m-1,k-j}\Big). \tag{11.2}$$

3rd group. For each m satisfying (6.2), we define this third group as the Q entries in row 0 of the jth block diagonally descending to column $jk + m$ (this includes the jth block entry already in that column in row 0).

For example, Tables 4 and 5 show that column 6 has a singleton $Q_{4,3}$; column 11 has a singleton $Q_{4,2}$, and column 12 has $Q_{3,2}$, $Q_{4,2}$; column 16 has a singleton $Q_{4,1}$, column 17 has $Q_{3,1}$, $Q_{4,1}$ and column 18 has $Q_{2,1}$, $Q_{3,1}$, $Q_{4,1}$.

Using (5.1), for general k, the set of Q entries in row 0 of the jth block diagonally descending to column $jk + m$, $0 \leq m \leq j$ equals $\{Q_{k-m,k-j-1}, \ldots, Q_{k-1,k-j-1}\}$. By (3.2)(b), the sum of elements in this group is as follows:

$$\sum_{r=k-m}^{k-1} Q_{r,k-j-1} = P_{k-1,k-j-1} - P_{k-m-1,k-j-1}. \tag{11.3}$$

4th group. For each fixed j, k, m satisfying (6.2), we define this fourth group to consist of the Q entries diagonally descending from row 0 in the $(j-1)$st block.

This last group requires more care to address because, for some columns, the group is empty. For example, Column 16 in Table 5 has $Q_{2,2}$ and $Q_{3,2}$, whereas column 17 has singleton $Q_{2,2}$. Column 18 has no entries for this group; but we can alternatively describe this empty group as all Qs between $Q_{2,2}$ and $Q_{1,2}$, which would result in no Qs because there are no integers between 2 and 1.

For general k, by(5.1), this group consists of the following (possibly) empty set of elements: $\{Q_{k-j,k-j}, \ldots, Q_{k-1-m,k-j}\}$, where we follow the usual convention that the set is empty if $k-1-m < k-j$.

Hence, by (3.2)(a), the sum of the entries in this fourth group is the following.

$$\sum_{r=k-j}^{k-1-m} Q_{r,k-j} = P_{k-m-1,k-j}. \tag{11.4}$$

It is straightforward to check, for Tables 4 and 5, that the sum is vacuous for $k = 5$ at columns 6 and 18, and that the P value on the right side of (11.4) is also 0 as required.

To complete the proof for this case E, we note that for any column $jk+m$ satisfying (6.2), i) the four groups enumerated above are mutually exclusive, ii) each entry in column $jk+m$ is in some group, and iii) by (1.4) the sum of the right sides of (11.1) through (11.4) is 0.

This completes the proof for case E.

12. Completion of Proof of Theorem 1.3

To see that the proof of Theorem 1.3 is complete, we again review the flow of logic.

- The last five sections have proven (6.1), which for Tables 4 and 5 state that the *sum* row of M contains the sum of all column entries above it, with the column entries being obtained by the multiplication of $q_5(X)$, the top row, by $-r_5(X)$ the left-most column.
- Equation (6.1) in turn proves (2.6), $-p_k(X^k) = -r_k(X) \times q_k(X)$. The long-hand multiplication is illustrated for the case $k = 5$ in Tables 4 and 5.
- Equation (2.6) is equivalent to (2.5), $p_k(X^k) = r_k(X)q_k(X)$.
- Equation (2.5) directly implies that $r_k(X)$, the characteristic polynomial for $\{F_n^{(k)}\}_{\text{all } n}$, divides $p_k(X^k)$, the characteristic polynomial for $\{F_{kn}^{(k)}\}_{\text{all } n}$.
- Hence, by Table 2 or Proposition 1.4, which creates a *dictionary* between polynomial identities and recursions, $F_n = \sum_{i=1}^{k} P_{k,i}F_{n-ki}$, the identity associated with the polynomial $p_k(X^k)$ is satisfied by the sequence $\{F_n^{(k)}\}_{\text{all } n}$.
- This last bullet is the statement of Theorem 1.3, which is therefore proven.

13. Conclusion

In this paper, we have proven a family of identities for the family of recursive sequences $F^{(k)}$, $k = 2, 3, \ldots$ restricted to a fixed modulus modulo k. This paper has provided a proof-method tool that exploits the divisibility properties of characteristic polynomials and was facilitated by a matrix M showing the long-hand multiplication. We believe this tool may prove useful for proving families of identities in other infinite families of recursions. A typical application of the methods of this paper could involve starting with some second order recursion such as the Pell, Pell-Lucas, or Jacobsthal sequence, generalizing this sequence to a family of recursive sequences similar to the $F^{(k)}$, $k = 2, 3, \ldots$, and uniformly proving identities satisfied by the subsequences whose indices are divisible by k.

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