RATIOS OF GENERALIZED FIBONACCI NUMBERS

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Abstract. We give a sufficient condition for a solution X_n of a linear recurrence relation, of any order, and with positive coefficients, to be such that X_{n+1}/X_n converges to a limit as $n \to \infty$.

1. Introduction

The Fibonacci sequence F_0, F_1, … is the solution of the recurrence relation $X_{n+2} = X_n + X_{n+1}$, where $(X_0, X_1) = (0, 1)$, and it is well known that $F_{n+1}/F_n \to (1+\sqrt{5})/2$. This result has been generalized in many ways (indeed, far too many to list here); for example, some authors have focussed on tribonacci numbers, tetranacci numbers, and so on, while others have been concerned with integer solutions generated from different initial values. Our concern here is that if we focus on the limit of F_{n+1}/F_n, or any of its immediate generalizations, but fail to place them in a general context, we may, by omission, give the misleading impression that, in this respect, the Fibonacci sequence and its allied sequences are a special case. They are not, and the existence of $\lim_{n\to\infty} X_{n+1}/X_n$ holds in much greater generality, and requires essentially nothing about the Fibonacci sequence, nor its initial values, nor indeed integer sequences. It seems worthwhile, therefore, to emphasize this by establishing a general result of this type with a simple proof.

The tribonacci numbers, tetranacci numbers, and so on, arise as particular solutions of the difference equation

$$X_{n+q} = X_n + X_{n+1} + \cdots + X_{n+q-1}, \tag{1.1}$$

where here (and elsewhere) q is an integer with $q \geq 2$. We shall not stop to define them here, for we shall pass immediately to the more general recurrence relation

$$X_{n+q} = a_0 X_n + a_1 X_{n+1} + \cdots + a_{q-1} X_{n+q-1}, \tag{1.2}$$

where the a_i are any positive numbers, and we shall prove the following result.

Theorem 1.1. *Given any positive numbers a_0, …, a_{q-1}, there is a positive number λ, and a proper subspace Π of $\mathbb{R}^q$, such that if the real sequence X_0, X_1, … satisfies* (1.2)*, and if the vector $(X_0, \dots, X_{q-1})$ of initial values is not in Π, then $X_{n+1}/X_n \to \lambda$ as $n \to \infty$.*

Theorem 1.1 shows that $X_{n+1}/X_n \to \lambda$ as $n \to \infty$ for essentially all choices $(X_0, \dots, X_{q-1})$ of the initial terms, and it is easy to identify λ. The two functions x and $a_0/x^{q-1} + \cdots + a_{q-2}/x + a_{q-1}$ are increasing, and decreasing, respectively on the interval $(0, +\infty)$, and their graphs cross at a unique point that, by definition, is the point (λ, λ). Thus, λ can be defined as the unique positive solution of $x^q = a_0 + a_1 x + \cdots + a_{q-1} x^{q-1}$. Special cases of Theorem 1.1 have appeared in [1] (which discusses the case $a_0 = \cdots = a_{q-1}$), and in [3] (which discusses the case in which the a_i are integers with $1 \leq a_0 \leq \cdots \leq a_{q-1}$). For general background material, we recommend [2] and the references contained therein.

2. The Proof of Theorem 1.1

We know that the solutions of (1.2) are determined by the roots of the equation

$$z^q = a_0 + a_1 z + \cdots + a_{q-1} z^{q-1} \tag{2.1}$$

with the initial terms $X_0, \ldots, X_{q-1}$. Let $f(z) = z^q - \big(a_0 + a_1 z + \cdots + a_{q-1} z^{q-1}\big)$. Then, $f(x)$ is real when x is real, and, as we have seen above, f has a unique positive zero, namely λ. Also, because $f(x) \to +\infty$ as $x \to +\infty$, we see that $f(x) > 0$ when $x > \lambda$. Next, for each complex z,

$$\begin{aligned} |f(z)| &\geq |z|^q - \big|a_0 + a_1 z + \cdots + a_{q-1} z^{q-1}\big| \\ &\geq |z|^q - \big(a_0 + a_1|z| + \cdots + a_{q-1}|z|^{q-1}\big) \\ &= f(|z|); \end{aligned}$$

however, we can say a little more than this. It is obvious that for complex numbers $w_1, \ldots, w_m$ we have

$$|w_1 + \cdots + w_m| \leq |w_1| + \cdots + |w_m|$$

with a strict inequality unless all of the w_i have the same argument. It follows from this that if z is complex, but not positive, then $|f(z)| > f(|z|)$ (note the strict inequality here). We conclude that if $|z| \geq \lambda$, then $|f(z)| > 0$ unless $z = \lambda$. Thus, the equation (2.1) has roots, say $\mu_1, \ldots, \mu_{q-1}$ and λ, where, for each i, $|\mu_i| < \lambda$.

If the μ_i are distinct, then the general solution of (1.2) is $X_n = A_1\mu_1^n + \cdots + A_{q-1}\mu_{q-1}^n + B\lambda^n$, and, providing that $B \neq 0$, we see that $X_{n+1}/X_n \to \lambda$ as $n \to \infty$. However, it is not necessary to know that the roots of (2.1) are distinct for it is known that, as λ is a simple zero of f, the general solution of (1.2) can be written as

$$X_n = \sum_{j=1}^{t} P_j(n)\mu_j^n + B\lambda^n, \tag{2.2}$$

where $\mu_1, \ldots, \mu_t$ are the distinct zeros of f (excluding the zero λ), and P_j is a polynomial whose degree is strictly less than the multiplicity of the zero μ_j. Thus, we have shown that, in all cases, if we write the general solution of (1.2) in the form (2.2), then $X_{n+1}/X_n \to \lambda$ as $n \to \infty$ when $B \neq 0$. Now each solution $(X_0, X_1, \ldots)$ of (1.2) determines, and is determined by, the vector $(X_0, \ldots, X_{q-1})$ of initial values, and the map $\Theta\colon (X_0, \ldots, X_{q-1}) \mapsto (X_0, X_1, \ldots)$ is an isomorphism of the vector space of initial values (which is $\mathbb{R}^q$) onto the vector space $\mathcal{S}$ of solutions of (1.2). The set of solutions for which $B = 0$ is a proper subspace, say $\mathcal{S}_0$, of $\mathcal{S}$, and if we let Π be the subspace $\Theta^{-1}(\mathcal{S}_0)$ of $\mathbb{R}^q$, our proof is complete.

3. The Fibonacci Case

We end with a brief discussion of the solutions of (1.1). The discussion below can be found in the literature, but our proof, which is based on the geometry of complex polynomials, seems simpler than other proofs. Obviously the results derived for Theorem 1.1 hold in this case. The auxiliary equation of (1.1) is $g(z) = 0$, where $g(z) = z^q - \big(1 + z + \cdots + z^{q-1}\big)$, and from our discussion above, we know that g has zeros $\mu_1, \ldots, \mu_{q-1}$ and λ, where $\lambda > 0$ and $|\mu_i| < \lambda$ for each i. We shall now give a simple proof of the following (known) result.

Lemma 3.1. *The auxiliary equation $g(z) = 0$ has $q-1$ distinct solutions, say $\mu_1, \ldots, \mu_{q-1}$, with $|\mu_i| < 1$ for each i, and one solution, say λ, in the real interval $(1, 2)$.*

Proof. Because $g(1) \neq 0$, it is enough (and more convenient) to study the polynomial h, where

$$h(z) = (z-1)g(z) = z^q(z-2) + 1, \quad h'(z) = (q+1)z^{q-1}(z - r_q), \quad r_q = 2q/(q+1).$$

Clearly $1 < r_q < 2$, and by considering $h'(x)$, we see that h is decreasing on the interval $(1, r_q)$, and increasing on $(r_q, 2)$. As $h(1) = 0$ and $h(2) = 1$, this shows that $h(r_q) < 0$, that $r_q < \lambda < 2$, and that $\lambda \to 2$ as $q \to \infty$ (because $r_q \to 2$ as $q \to \infty$). It is also immediate that h has no repeated zeros. Indeed, if w were such a zero, we would have $h(w) = h'(w) = 0$. But as $h'(w) = 0$, we see that w is 0 or r_q, and $h(r_q) < 0 < h(0)$. In particular, the q zeros of g are distinct.

It remains to show that, for each i, $|\mu_i| < 1$. Choose any R such that $1 < R < \lambda$, and let $C_R = \{z\colon |z| = R\}$. As $h(z) - 1 = z^q(z-2)$, it is clear that $|h(z) - 1|$ attains its minimum value on C_R at the point of C_R that is closest to 2, namely R; thus, if $z \in C_R$, then $|h(z) - 1| \geq |h(R) - 1|$. However, $h(R) < 0$; thus, if $z \in C_R$, then $|h(z) - 1| > 1$. Thus, by Rouchè's theorem, $h(z) - 1$ and $h(z)$ (that is, $h(z) - 1$ and $h(z) - 1 + 1$) have the same number of zeros inside C_R. Clearly, $h(z) - 1$ has exactly q zeros inside C_R; thus, $h(z)$ has exactly q zeros inside C_R. This shows that g has $q - 1$ zeros inside C_R, and if we let $R \to 1$, we see that g has exactly $q - 1$ zeros in the closed unit disc. Now suppose that w is a zero of h on the unit circle. Then $|w| = 1$ and $|w - 2| = |w^q(w-2)| = |h(w) - 1| = 1$, so that $w = 1$. We conclude that g has no zeros on the unit circle; thus, the zeros $\mu_1, \ldots, \mu_{q-1}$ of g lie in $\{z\colon |z| < 1\}$. ☐

Finally, we now know that the general solution of (1.1) is, say, $X_n = A_1\mu_1^n + \cdots + A_{q-1}\mu_{q-1}^n + B\lambda^n$, and this shows that X_n converges to 0 or ∞ according as B is, or is not, zero. If $(X_0, \ldots, X_{q-1})$ is such that $B \neq 0$, then $X_{n+1}/X_n \to \lambda$ as $n \to \infty$.

References

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