

# A MATRIX-BASED RECURSION RELATION FOR $F_{F_n}$

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ABSTRACT. In 1977, Parberry introduced and proved a fifth-order and a sixth-order nonlinear recurrence relation for the sequence  $(F_{F_n} : n \in \mathbb{N}_0)$ , where  $F_n$  denotes the  $n$ th Fibonacci number. In this article, we prove an identity for  $F_{F_n}$  given by a Fibonacci-like recursion with matrix multiplication used in place of integer addition.

## 1. INTRODUCTION

The linear recurrence  $F_n = F_{n-1} + F_{n-2}$  used to define the famous and ubiquitous Fibonacci sequence  $(F_n : n \in \mathbb{N}_0)$  forms one of the most well-known recurrence relations in all of mathematics. This inspires the exploration of “Fibonacci-like” recurrences, and there has been a long history of research devoted to such recurrences, as in with the study of Lucas numbers, Tribonacci numbers, etc. We are interested in using matrix multiplication in place of integer addition to construct sequences that satisfy Fibonacci-like recurrences, and this has led us to discover an identity for the integer sequence

$$(F_{F_n} : n \in \mathbb{N}_0) \tag{1.1}$$

given by the Fibonacci sequence  $F$  composed with itself, which is one of the sequences indexed in the On-line Encyclopedia of Integer Sequences (OEIS) [6], as given by the entry labeled as A007570. Our new recursions for (1.1) significantly build upon the work of Parberry [8] considered in Section 1.1 below.

Every linear recursion can be expressed via the powers of a fixed matrix, and this leads us to consider a variant of this property. In particular, in place of the recurrence  $F_{n-2} + F_{n-1} = F_n$ , we consider recursions of the form  $M_{n-2}M_{n-1} = M_n$ , where the binary operation on the left side of this latter equation is matrix multiplication. For example, setting  $M_0$  as the  $2 \times 2$  identity matrix  $I_2$ , and letting  $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , and then, by enforcing the Fibonacci-like recurrence  $M_n = M_{n-2}M_{n-1}$ , it is almost immediate that

$$M_n = \begin{pmatrix} 1 & F_n \\ 0 & 1 \end{pmatrix}$$

for all  $n \in \mathbb{N}_0$ . Similarly, by letting  $M_0$  be as before, and using the same recursion, but with  $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , it is almost immediate that

$$M_n = \begin{pmatrix} 2^{F_n-1} & 2^{F_n-1} \\ 2^{F_n-1} & 2^{F_n-1} \end{pmatrix}$$

for all natural numbers  $n$ . A similar set-up, as in Theorem 2.1, has led us to experimentally discover a new recursion for (1.1), with the companion sequence

$$(F_{F_{n-1}} : n \in \mathbb{N}_0)$$

indexed as A130589 in the OEIS [6], and along with

$$(F_{F_{n+1}} : n \in \mathbb{N}_0),$$

which is listed in the OEIS [6] as A005370.

The recursion given as Theorem 2.1 appears to be new, but proving this result is nontrivial, and this result has not been given in any of the above referred OEIS sequences or in Parberry's work [8] on (1.1) or in subsequent references concerning A007570. The previously known recurrences for (1.1) [6, 8] are fundamentally different from Theorem 2.1, and we find it worthwhile to briefly review these previous results.

**1.1. Parberry's Recursions.** As in [8], we state that the problem of finding a recurrence for (1.1) dates back to Whitney's work [13], in 1966. Writing  $f_n = F_{F_n}$ , Parberry [8], in 1977, proved the identity

$$f_n = \left(5f_{n-2}^2 + (-1)^{F_{n+1}}\right)f_{n-3} + (-1)^{F_n} \left(f_{n-3} - (-1)^{F_{n+1}}f_{n-6}\right)f_{n-2}/f_{n-5}. \quad (1.2)$$

This is proved via the identity

$$F(a+b) = F(a)L(b) - (-1)^b F(a-b)$$

that Parberry had previously proved [7], letting the Lucas numbers be defined by  $L_{n+1} = L_n + L_{n-1}$ , where  $L_1 = 1$  and  $L_2 = 3$ . Similarly, Parberry [8] proved a much more complicated fifth-order recursion resembling the quotient in (1.2). These identities make the Fibonacci-like recurrence given as Theorem 2.1 all the more remarkable. In the OEIS entry A007570 [6], Chris Street also provided a recurrence resembling the quotient identity in (1.2), in contrast to our matrix identity given as Theorem 2.1 below.

## 2. A MATRIX-BASED RECURSION RELATION FOR $F_{F_n}$

Our proof of Theorem 2.1 mainly relies on Binet's famous Fibonacci number formula

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},$$

letting  $\phi = \frac{1+\sqrt{5}}{2}$  denote the famous golden ratio constant [5].

We note that identities as in (2.2) are nontrivial in that computer algebra systems such as Mathematica are not able to verify or confirm such results, even with the use of commands such as `FunctionExpand`.

**Theorem 2.1.** *Let  $M_0 = I_2$ ,  $M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ , and  $M_n = M_{n-2}M_{n-1}$  for  $n \in \mathbb{N}_{\geq 2}$ . It then follows that*

$$M_n = \begin{pmatrix} F_{F_{n-1}} & F_{F_n} \\ F_{F_n} & F_{F_{n+1}} \end{pmatrix} \quad (2.1)$$

for all  $n \in \mathbb{N}_0$ .

*Proof.* The base cases for  $n = 0$  and  $n = 1$  are easily verified. Inductively, we may assume that (2.1) holds for  $n < m$ . So, rewriting the matrix product  $M_{m-2}M_{m-1}$  according to (2.1), and then evaluating this resultant product, we proceed to examine the entries that we obtain. We see that the (1, 1)-entry of the product under consideration, under our inductive hypothesis, is such that we need to show that

$$F_{F_{m-1}} = F_{F_{m-2}-1}F_{F_{m-1}-1} + F_{F_{m-2}}F_{F_{m-1}}. \quad (2.2)$$

So, we need to show that

$$\begin{aligned} & \frac{1}{5} \left( \phi^{\frac{\phi^{m-2} - (-\frac{1}{\phi})^{m-2}}{\sqrt{5}} - 1} - \left(-\frac{1}{\phi}\right)^{\frac{\phi^{m-2} - (-\frac{1}{\phi})^{m-2}}{\sqrt{5}} - 1} \right) \\ & \left( \phi^{\frac{\phi^{m-1} - (-\frac{1}{\phi})^{m-1}}{\sqrt{5}} - 1} - \left(-\frac{1}{\phi}\right)^{\frac{\phi^{m-1} - (-\frac{1}{\phi})^{m-1}}{\sqrt{5}} - 1} \right) + \\ & \frac{1}{5} \left( \phi^{\frac{\phi^{m-2} - (-\frac{1}{\phi})^{m-2}}{\sqrt{5}}} - \left(-\frac{1}{\phi}\right)^{\frac{\phi^{m-2} - (-\frac{1}{\phi})^{m-2}}{\sqrt{5}}} \right) \\ & \left( \phi^{\frac{\phi^{m-1} - (-\frac{1}{\phi})^{m-1}}{\sqrt{5}}} - \left(-\frac{1}{\phi}\right)^{\frac{\phi^{m-1} - (-\frac{1}{\phi})^{m-1}}{\sqrt{5}}} \right) \end{aligned}$$

evaluates as

$$\frac{\phi^{\frac{\phi^m - (-\frac{1}{\phi})^m}{\sqrt{5}} - 1} - \left(-\frac{1}{\phi}\right)^{\frac{\phi^m - (-\frac{1}{\phi})^m}{\sqrt{5}} - 1}}{\sqrt{5}}.$$

This is easily seen to be equivalent to the equality of

$$\frac{(\phi^2 + 1) \left( \phi^2 \left(-\frac{1}{\phi}\right)^{\frac{(\phi+1)\phi^m + (1-\phi)\phi^3(-\frac{1}{\phi})^m}{\sqrt{5}\phi^2}} + \phi^{\frac{(\phi+1)\phi^m + (1-\phi)\phi^3(-\frac{1}{\phi})^m}{\sqrt{5}\phi^2}} \right)}{5\phi^2}$$

and

$$\frac{\phi^{\frac{\phi^m - (-\frac{1}{\phi})^m}{\sqrt{5}}} + \phi^2 \left(-\frac{1}{\phi}\right)^{\frac{\phi^m - (-\frac{1}{\phi})^m}{\sqrt{5}}}}{\sqrt{5}\phi}.$$

From the equality

$$\frac{\phi^2 + 1}{5\phi^2} = \frac{1}{\sqrt{5}\phi}, \tag{2.3}$$

in order to prove (2.2), it remains to prove that

$$\phi^2 \left(-\frac{1}{\phi}\right)^{\frac{(\phi+1)\phi^m + (1-\phi)\phi^3(-\frac{1}{\phi})^m}{\sqrt{5}\phi^2}} + \phi^{\frac{(\phi+1)\phi^m + (1-\phi)\phi^3(-\frac{1}{\phi})^m}{\sqrt{5}\phi^2}}$$

equals

$$\phi^{\frac{\phi^m - (-\frac{1}{\phi})^m}{\sqrt{5}}} + \phi^2 \left(-\frac{1}{\phi}\right)^{\frac{\phi^m - (-\frac{1}{\phi})^m}{\sqrt{5}}}.$$

Comparing the powers of  $-\frac{1}{\phi}$  and of  $\phi$ , the desired formula in (2.2) then easily follows.

Again, under our inductive hypothesis, let us consider the  $(1, 2)$ -entry of the product  $M_{m-2}M_{m-1}$ . In this case, we need to prove that

$$F_{F_m} = F_{F_{m-2-1}}F_{F_{m-1}} + F_{F_{m-2}}F_{F_{m-1+1}}. \quad (2.4)$$

In this case, it remains to prove that

$$\begin{aligned} & \frac{1}{5} \left( \phi \frac{\phi^{m-1} - \left(-\frac{1}{\phi}\right)^{m-1}}{\sqrt{5}} - \left(-\frac{1}{\phi}\right) \frac{\phi^{m-1} - \left(-\frac{1}{\phi}\right)^{m-1}}{\sqrt{5}} \right) \\ & \left( \phi \frac{\phi^{m-2} - \left(-\frac{1}{\phi}\right)^{m-2}}{\sqrt{5}} - 1 - \left(-\frac{1}{\phi}\right) \frac{\phi^{m-2} - \left(-\frac{1}{\phi}\right)^{m-2}}{\sqrt{5}} - 1 \right) + \\ & \frac{1}{5} \left( \phi \frac{\phi^{m-2} - \left(-\frac{1}{\phi}\right)^{m-2}}{\sqrt{5}} - \left(-\frac{1}{\phi}\right) \frac{\phi^{m-2} - \left(-\frac{1}{\phi}\right)^{m-2}}{\sqrt{5}} \right) \\ & \left( \phi \frac{\phi^{m-1} - \left(-\frac{1}{\phi}\right)^{m-1}}{\sqrt{5}} + 1 - \left(-\frac{1}{\phi}\right) \frac{\phi^{m-1} - \left(-\frac{1}{\phi}\right)^{m-1}}{\sqrt{5}} + 1 \right) \end{aligned}$$

equals

$$\frac{\phi \frac{\phi^m - \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}} - \left(-\frac{1}{\phi}\right) \frac{\phi^m - \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}}}{\sqrt{5}}. \quad (2.5)$$

Simplifying this second-to-last expression, it remains to show that

$$\frac{(\phi^2 + 1) \left( \phi \frac{(\phi+1)\phi^m + (1-\phi)\phi^3 \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}\phi^2} - \left(-\frac{1}{\phi}\right) \frac{(\phi+1)\phi^m + (1-\phi)\phi^3 \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}\phi^2} \right)}{5\phi}$$

equals (2.5). According to the golden ratio formula in (2.3), it remains to prove that

$$\phi \frac{(\phi+1)\phi^m + (1-\phi)\phi^3 \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}\phi^2} - \left(-\frac{1}{\phi}\right) \frac{(\phi+1)\phi^m + (1-\phi)\phi^3 \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}\phi^2}$$

equals

$$\phi \frac{\phi^m - \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}} - \left(-\frac{1}{\phi}\right) \frac{\phi^m - \left(-\frac{1}{\phi}\right)^m}{\sqrt{5}}.$$

By comparing the above powers, we may easily establish that the desired identity for  $F_{F_n}$  holds. The remaining identities, which correspond to the  $(2, 1)$ - and  $(2, 2)$ -entries of  $M_{m-2}M_{m-1}$ , are such that

$$F_{F_m} = F_{F_{m-2}}F_{F_{m-1-1}} + F_{F_{m-2+1}}F_{F_{m-1}}$$

and

$$F_{F_{m+1}} = F_{F_{m-2}}F_{F_{m-1}} + F_{F_{m-2+1}}F_{F_{m-1+1}},$$

and both of these remaining identities may be proved similarly, compared with our proofs for (2.2) and (2.4).  $\square$

### 3. CONCLUSION

We are broadly interested in the subject of Fibonacci-like recursions, inspired, in part, by past research on such recursions as in [1, 2, 9, 10, 12]. We note that Bakir Farhi has obtained a number of interesting infinite series evaluations involving  $F_{F_n}$  [4], as in the formula

$$\sum_{n=1}^{\infty} (-1)^{F_n} \frac{F_{F_{n+1}}}{F_{F_n} F_{F_{n+1}}} = 1 - \sqrt{5},$$

and we encourage the consideration as to how identities as in Theorem 2.1 relate to such series evaluations. As in [4], we note that closed-form evaluations such as

$$\sum_{n=1}^{\infty} (-1)^{F_n} \frac{F_{F_{n-1}}}{F_{F_n} F_{F_{n+1}}} = \frac{1 - \sqrt{5}}{2}$$

were also given by Bruckman and Good in [3]; see also [11].

We greatly encourage the development of generalizations of the Fibonacci-like recurrence given in Theorem 2.1. In this direction, it seems that there is a great amount to explore. For example, setting  $M_0 = I_3$  and

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and again enforcing the recursion  $M_n = M_{n-2}M_{n-1}$ , we have experimentally discovered that the (1, 3)-, (2, 3)-, (3, 1)-, and (3, 2)-entries of  $M_n$  agree with entry A179823 in the OEIS [6], which concerns denominators in the approximation of  $\sqrt{2}$ . For the sake of brevity, we save this topic for a separate research endeavour.

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