# PRODUCTS OF MULTIPLE-INDEX FIBONACCI NUMBERS

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Abstract. Consider the generating function (gf) for the integer sequence $(F_{mi}F_{ni} : i \in \mathbb{N}_0)$, where $m$ and $n$ are positive integer parameters. We may compute this gf in terms of Fibonacci/Lucas numbers using an implementation of an algorithm by Zeilberger. However, for the case where the integers $m$ and $n$ have the same parity, we have experimentally discovered that there is a simpler way of expressing this gf compared with the corresponding expression obtained via Zeilberger's procedure. We prove this equivalence via Binet's formula, and then apply our simplified gf evaluation to generalize a classic Fibonacci sum identity given by Freitag, and some order-2 Fibonacci-type sums in the recent work of Melham. Our evaluations of finite sums over $F_{mi}F_{ni}$ are dramatically simpler compared with the corresponding output obtained via Zeilberger's `Cfinite` Maple package.

## 1. Introduction

A $C$-finite sequence is a sequence that satisfies a linear recurrence equation with constant coefficients [20]. As indicated in [20], the set of all $C$-finite sequences is closed under multiplication, i.e., under the Hadamard product operation on sequences. Given two sequences $C_1$ and $C_2$, the Hadamard product $C_1C_2$ refers to the sequence $(C_1(i)C_2(i) : i \in \mathbb{N}_0)$. Letting $(F_i : i \in \mathbb{N}_0)$ denote the Fibonacci sequence, if we set $C_1$ as the sequence $(F_{mi} : i \in \mathbb{N}_0)$ for a natural number $m$, and if we set $C_2$ as $(F_{ni} : i \in \mathbb{N}_0)$ for a parameter $n \in \mathbb{N}$, a Maple implementation of a procedure by Zeilberger [20] allows us to easily express the generating function (gf) for the Hadamard product

$$(F_{mi}F_{ni} : i \in \mathbb{N}_0) \tag{1.1}$$

in terms of Fibonacci/Lucas numbers. Using the Mathematica computer algebra system (CAS) and the On-line Encyclopedia of Integer Sequences (OEIS) [16], we found a strikingly simpler way of evaluating the gf for (1.1) in the case where $n$ and $m$ have the same parity. However, state-of-the-art symbolic computation software cannot directly confirm or verify that our simplified gf for (1.1) is equivalent to the output obtained via Zeilberger's procedure [20] for the gf for (1.1). In this article, we prove this equivalence using Binet's formula. We then apply our simplified gf to generalize a classic result on Fibonacci sums from [5], and to build on recently introduced results on order-2 Fibonacci-type sums given in [12].

### 1.1. **An Application of Zeilberger's Procedure.**

For a parameter $n$ in $\mathbb{N}$, it is easily seen that the gf for the sequence $(F_{ni} : i \in \mathbb{N}_0)$ is of the form

$$\frac{F_n x}{(-1)^n x^2 - L_n x + 1},$$

recalling that the sequence $(L_i : i \in \mathbb{N}_0)$ of Lucas numbers is defined as $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, as in sequence `A000032` in the OEIS [16]. Implementing Zeilberger's `Cfinite` package by inputting

```
KefelR(Fm*x/((-1)^m*x^2 - Lm*x + 1), Fn*x/((-1)^n*x^2 - Ln*x + 1), x)
```

into Maple and letting expressions such as `Fm` denote variables or parameters in Maple, we find that the gf for the Hadamard product of the sequences $(F_{mi} : i \in \mathbb{N}_0)$ and $(F_{ni} : i \in \mathbb{N}_0)$ may be expressed as:

$$\frac{-F_m\,F_n\,x\left(x^2\,(-1)^{m+n}-1\right)}{1-L_m\,L_n\,x+x^2L_m^2\,(-1)^n+x^2{L_n}^2\,(-1)^m-2\,x^2\,(-1)^{m+n}-L_m\,L_n\ (-1)^{m+n}\,x^3+x^4}.$$

Using a heavily experimental approach based on the use of Mathematica and the OEIS [16], we came to find, conjecturally, that: If $m$ and $n$ have the same parity, then

$$\frac{F_m\,F_n\,x\left(1-x^2\right)}{1-L_m\,L_n\,x+x^2L_m^2\,(-1)^n+x^2{L_n}^2\,(-1)^m-2\,x^2-L_m\,L_nx^3+x^4} \tag{1.2}$$

may be written as the following remarkably simpler expression:

$$\frac{1}{5}\left(1-x^2\right)\left(\frac{1}{1-xL_{n+m}+x^2}-\frac{1}{1+(-1)^{m+1}xL_{n-m}+x^2}\right). \tag{1.3}$$

This gf equivalence is a main result in this article. Inputting the difference of (1.2) and (1.3) into our current CAS software, and using commands such as Maple's `simplify`, we find that such software cannot confirm or "detect" the equality of (1.2) and (1.3) with the required parity condition. Our simplified gf evaluation in (1.3) is of interest in its own right, as emphasized in Proposition 1.1, noting the symmetry or similarity between the denominators in (1.5), in contrast to the relatively unwieldy expression in (1.4). Apart from aesthetic considerations, we want to find concrete applications concerning our symbolic form in (1.3). Because (1.3) expands as a linear combination of rational functions with quadratic denominators with closed-form coefficients, we can construct identities for sums involving $F_{mi}F_{ni}$ and combinations of order-2 recurrences. We do this in Section 3.2. In Example 3.2, our evaluations for sums of the form

$$\sum_{i=0}^{j} F_{mi}F_{ni}$$

are extremely simple compared with the corresponding evaluations obtained via Zeilberger's `Cfinite` package [20].

**Proposition 1.1.** *If $m$ and $n$ are natural numbers with the same parity, then*

$$\frac{5\,F_m\,F_n\,x}{1-L_m\,L_n\,x+x^2L_m^2\,(-1)^n+x^2L_n^2\,(-1)^m-2\,x^2-L_m\,L_nx^3+x^4} \tag{1.4}$$

*equals*

$$\frac{1}{1-xL_{n+m}+x^2}-\frac{1}{1+(-1)^{m+1}xL_{n-m}+x^2} \tag{1.5}$$

*for suitably bounded $x$.*

## 2. A Simplified Generating Function Evaluation

Again, it is not clear why the identity in Proposition 1.1 is true. We prove this proposition in this section.

**Theorem 2.1.** *If $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ have the same parity, then the gf for $F_{ni}F_{mi}$ is $\frac{1}{5}\left(1-x^2\right)\left(\frac{1}{1-xL_{n+m}+x^2}-\frac{1}{1+(-1)^{m+1}xL_{n-m}+x^2}\right)$.*

*Proof.* Suppose that $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ have the same parity. By Binet's formula, the gf for the integer sequence $(F_{ni}F_{mi})_{i\in\mathbb{N}_0}$ is:

$$-\frac{x\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\left((-1)^{m+n+1}+x^2\right)\phi^{m+n}}{5\left((-\phi)^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}.$$

Because $m$ and $n$ have the same parity, the above expression may be rewritten as follows:

$$-\frac{x\left(x^2-1\right)\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\phi^{m+n}}{5\left(\phi^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}.$$

By Binet's formula for Lucas numbers, it remains to prove that the above expression is equal to:

$$\frac{x\left(x^2-1\right)\left(\left((-1)^m\phi^{-2m}-1\right)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}\right)}{5\left((-1)^{m+1}x\left(\phi^{m-n}+\phi^{n-m}\right)+x^2+1\right)\left(-x\left(\phi^{-m-n}+\phi^{m+n}\right)+x^2+1\right)}.$$

Equivalently, it remains to prove that

$$-\frac{\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\phi^{m+n}}{\left(\phi^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}$$

is equal to:

$$\frac{\left(\left((-1)^m\phi^{-2m}-1\right)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}\right)}{\left((-1)^{m+1}x\left(\phi^{m-n}+\phi^{n-m}\right)+x^2+1\right)\left(-x\left(\phi^{-m-n}+\phi^{m+n}\right)+x^2+1\right)}.$$

Now, consider the ratio of the former numerator to the latter numerator:

$$\frac{\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\phi^{m+n}}{\left(\left((-1)^m\phi^{-2m}-1\right)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}\right)}.$$

It is easily seen that the above quotient is equal to $\phi^{2m+2n}$. This can be verified by expanding the above numerator and expanding the expression $\phi^{2m+2n}((((-1)^m\phi^{-2m}-1)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}))$. So, it remains to prove that

$$\frac{1}{-\left(\phi^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}$$

is equal to:

$$\frac{1}{\phi^{2m+2n}\left((-1)^{m+1}x\left(\phi^{m-n}+\phi^{n-m}\right)+x^2+1\right)\left(-x\left(\phi^{-m-n}+\phi^{m+n}\right)+x^2+1\right)}.$$

Expand the former denominator as follows:

$$x^4\phi^{2m+2n}-x^3\phi^{m+n}+(-1)^{m+1}x^3\phi^{3m+n}+(-1)^{n+1}x^3\phi^{m+3n}-x^3\phi^{3m+3n}+x^2(-1)^{m+n+2}\phi^{2m+2n}+x^2\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{4m+2n}+(-1)^{n+2}x^2\phi^{2m+4n}+x(-1)^{m+n+3}\phi^{m+n}+(-1)^{m+3}x\phi^{3m+n}+(-1)^{n+3}x\phi^{m+3n}+x(-1)^{m+n+3}\phi^{3m+3n}+(-1)^{m+n+4}\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{2m}+(-1)^{n+2}x^2\phi^{2n}.$$

Expand the latter denominator as follows:

$$x^4\phi^{2m+2n}-x^3\phi^{m+n}+(-1)^{m+1}x^3\phi^{3m+n}+(-1)^{m+1}x^3\phi^{m+3n}-x^3\phi^{3m+3n}+(-1)^{m+2}x^2\phi^{2n}+2x^2\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{4m+2n}+(-1)^{m+2}x^2\phi^{2m+4n}-x\phi^{m+n}+(-1)^{m+1}x\phi^{3m+n}+(-1)^{m+1}x\phi^{m+3n}-x\phi^{3m+3n}+\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{2m}.$$

Because $m$ and $n$ have the same parity, we can simplify the former denominator and see that the above two expressions are equal. □

Zeilberger's implementation of an algorithm for computing Hadamard products of gfs [20] gives us a computer proof that the gf for (1.1) is given by the `KefelR` function output shown in Section 1.1. However, using the given parity constraints, we have simplified this gf in Theorem 2.1.

## 3. Order-2 Fibonacci Sums

Multiplying our simplified gf in Theorem 2.1 by expressions such as $\frac{1}{1-x}$ and using the Cauchy product for gfs, we obtain interesting results such as Theorems 3.1–3.4. We find it worthwhile to first describe how these theorems build on and relate to relevant background material.

3.1. **Background.** In 1973, Freitag [5] proved the summation identity

$$\sum_{i=1}^{j} F_{ni} = \frac{(-1)^n F_{jn} + F_n - F_{(j+1)n}}{(-1)^n + 1 - L_n}.$$

As described in [5], it is a natural mathematical problem to consider finite sums involving entries in the Fibonacci sequence that "skip" by a fixed period. Referring to the terms *double-index harmonic number* [18] and *Euler sum with multiple argument* [17], we consider sums over products of *multiple-index* Fibonacci numbers. After studying [5] and [3, 14], it seems that our results on sums involving $F_{ni}F_{mi}$ are not known. Letting $u$ and $v$ denote Fibonacci-type or Lucas-type sequences, the evaluation of sums of the form

$$\sum_{i=1}^{j} u_{a+bi} v_{c+di}$$

is a topic that has been explored in many references, e.g., [10] and [8, 9, 13].

For integer values $a_\ell$ and $b_\ell$, an algorithm was given by Greene and Wilf [6] (cf. [20]) that may be used to express

$$\sum_{i=0}^{j} F_{a_1 j + b_1 i + c_1} F_{a_2 j + b_2 i + c_2} \cdots F_{a_k j + b_k i + c_k} \tag{3.1}$$

in closed form with Fibonacci numbers. With regard to Zeilberger's Maple implementation of an algorithm for evaluating finite sums as in (3.1), it is unclear how this can be used to obtain our parity-dependent results on sums involving $F_{mi}F_{ni}$ in Section 3, with specific reference to the `FindCHvTN` program from Zeilberger's `Cfinite` Maple package [20]. In particular, it seems that `FindCHvTN` does not apply to summands with additional parameters apart from the upper limit of a given finite sum under consideration. For example, by applying Zeilberger's `Cfinite` package and inputting

```
FindCHvTN(Fn(), 1, 5, [2*j, 3*j], n, j, 1)
```

we obtain an evaluation for $\sum_{j=0}^{n-1} F_{2j}F_{3j}$; but inputting an expression such as

```
FindCHvTN(Fn(), 1, 5, [a*j, b*j], n, j, 1)
```

results in an error message. It is unclear how the algorithms in [6] may be applied to obtain our results in Theorem 2.1, relative to the corresponding `Cfinite` output under consideration in Section 1.1.

Finite sums of the form indicated in (3.1) for the case where $k = 2$ are often referred to as order-2 Fibonacci sums [11, 12]. Our proofs for the results given in Section 3.2 significantly build on past results on order-2 Fibonacci sums, as in the identity

$$\sum_{i=0}^{j} F_{2i}^2 = \frac{F_{4j+2} - 2j - 1}{5}$$

given in [19, p. 70] (cf. [11]).

In 2017 [12], Melham introduced a method for evaluating certain order-2 Fibonacci-type sums. More specifically, writing

$$W_n(a, b, p) = W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b$$

as in [12], a method for evaluating

$$\sum_{i=1}^{j} W_{ai+b_1} W_{ai+b_2}$$

is given, noting that the coefficient of $i$ in the above summand is the same for both of the indices of the above $W$-expressions. So, it is unclear how the material from [12] could be reformulated to be applicable to sums as in Theorem 3.1 below.

3.2. **Finite Sums of Products of Multiple-index Fibonacci Numbers.** A remarkable aspect about our identity, highlighted in Theorem 3.1, is how the right side of this identity is simple compared with the corresponding output from the above referenced `FindCHvTN` program [20], as in Example 3.2.

**Theorem 3.1.** *If $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ are distinct and of the same parity, then*

$$\sum_{i=0}^{j} F_{mi} F_{ni} = \frac{1}{5} \left( \frac{(-1)^{jm+1} \left( (-1)^m F_{j(n-m)} + F_{(j+1)(n-m)} \right)}{F_{n-m}} + \frac{F_{j(m+n)} + F_{(j+1)(m+n)}}{F_{m+n}} \right).$$

*Proof.* This can be shown in a direct way by applying the partial sum operator $\frac{1}{1-x}$ to the gf in Theorem 2.1, under the given assumptions on $m$ and $n$. □

**Example 3.2.** *We input*

```
FindCHvTN(Fn(), 1, 6, [2*j, 4*j], n, j, 1)
```

*into Maple, using Zeilberger's* `Cfinite` *package* [20]*. This gives us that the sum*

$$\sum_{i=0}^{j} F_{2i} F_{4i} \tag{3.2}$$

*equals the following for all $j \in \mathbb{N}_0$:*

$$\frac{275\, F_{j+1}^6}{4} - \frac{315\, F_j F_{j+1}^5}{2} + \frac{41\, F_j F_{j+1}}{2} - 30\, F_j^2 F_{j+1}^4 + 170\, F_j^3 F_{j+1}^3 - \frac{275\, F_{j+1}^2}{4} - 3\, F_j^2 + F_{2j} F_{4j}.$$

*In stark contrast, by setting $m = 2$ and $n = 4$ in Theorem 3.1, we obtain that the sum in* (3.2) *also equals*

$$\frac{F_{6j} + F_{6j+6}}{40} - \frac{F_{2j} + F_{2j+2}}{5} \tag{3.3}$$

*for all $j \in \mathbb{N}_0$. It does not seem to be possible to evaluate* (3.2) *with less terms using* `FindCHvTN` *compared with the above* `FindCHvTN` *output. Without knowing that the second-to-last displayed expression and the dramatically simpler expression in* (3.3) *are related via the sum in* (3.2)*, it is unclear why these two expressions should be equal for all $j \in \mathbb{N}_0$. This is indicative of the computational usefulness of our identities in Theorem 3.1.*

**Theorem 3.3.** *If $m \in \mathbb{N}_0$ is even, then the identity*

$$\sum_{i=0}^{j} F_{mi}^2 = \frac{1}{5}\left(\frac{F_{2jm}}{F_{2m}} + \frac{F_{2(j+1)m}}{F_{2m}} - (2j+1)\right)$$

*is true (cf. Theorem 2.3 in* [12]*).*

*Proof.* Again, this follows directly by applying the partial sum operator to the gf in Theorem 2.1. □

**Theorem 3.4.** *If $m \in \mathbb{N}_0$ is odd, then the identity*

$$\sum_{i=0}^{j} F_{mi}^2 = \frac{1}{5}\left(\frac{F_{2jm}}{F_{2m}} + \frac{F_{2(j+1)m}}{F_{2m}} - (-1)^j\right)$$

*holds (cf. Theorem 2.3 in* [12]*).*

*Proof.* Again, this follows directly from Theorem 2.1. □

We may obtain many similarly elegant results by applying operators such as $\frac{1}{1+x}$ and $\frac{1}{1-x^2}$ to our gf evaluation in Theorem 2.1. For the sake of brevity, we leave a full exploration of this kind of topic to a separate research endeavor. Also we can consider the repeated application of the partial sum operator to our gf, to build on John Ivie's work on *multiple Fibonacci sums* in [7], with Chu's recent work in [2]. The material in [4] motivates extending our results to $k$-Fibonacci number sequences.

## 4. Conclusion

Generating function-based methods and results continue to be frequently involved in research on Fibonacci/Fibonacci-type sequences; in this regard, see the publications [1, 15, 22]. We encourage the use of the concepts and methods involved in our article to past research efforts on gf-based results on Fibonacci/Fibonacci-type sequences.

Combinatorial proofs for evaluating gfs for products of second-order recurrences are the subject of the Ph.D. thesis [21]. We encourage the development of combinatorial methods for proving and generalizing Proposition 1.1.

## References


[1] A. Bulawa and W. K. Lee, *Integer values of generating functions for the Fibonacci and related sequences*, The Fibonacci Quarterly, **55.1** (2017), 74–81.

[2] H. V. Chu, *Partial sums of the Fibonacci sequence*, The Fibonacci Quarterly, **59.2** (2021), 132–135.

[3] K. Dilcher, *Hypergeometric functions and Fibonacci numbers*, The Fibonacci Quarterly, **38.4** (2000), 342–363.

[4] S. Falcon and A. Plaza, *On k-Fibonacci numbers of arithmetic indexes*, Applied Mathematics and Computation, **208.1** (2009), 180–185.

[5] H. T. Freitag, *On summations and expansions of Fibonacci numbers*, The Fibonacci Quarterly, **11.1** (1973), 63–71, 84, 112.

[6] C. Greene and H. S. Wilf, *Closed form summation of C-finite sequences*, Transactions of the American Mathematical Society, **359.3** (2007), 1161–1189.

[7] J. Ivie, *Multiple Fibonacci sums*, The Fibonacci Quarterly, **7.3** (1969), 303–309.

[8] E. Kiliç, *Sums of the squares of terms of sequence $\{u_n\}$*, Indian Academy of Sciences. Proceedings. Mathematical Sciences, **118.1** (2008), 27–41.

[9] E. Kiliç, N. Ömür, and Y. T. Ulutaş, *Binomial sums whose coefficients are products of terms of binary sequences*, Utilitas Mathematica, **84** (2011), 45–52.

[10] E. Kiliç and H. Prodinger, *Sums of products of generalized Fibonacci and Lucas numbers*, Acta Mathematica Hungarica, **145.1** (2015), 17–25.

[11] R. Knott, *Fibonacci and golden ratio formulae*, 2012, http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibFormulae.html.

[12] R. S. Melham, *Closed forms for certain Fibonacci type sums that involve second order products*, The Fibonacci Quarterly, **55.3** (2017), 195–200.

[13] R. S. Melham, *On sums of powers of terms in a linear recurrence*, Portugaliae Mathematica, **56.4** (1999), 501–508.

[14] C. Panraksa and A. Tangboonduangjit, *On some arithmetic properties of a sequence related to the quotient of Fibonacci numbers*, The Fibonacci Quarterly, **55.1** (2017), 21–28.

[15] H. Prodinger and S. J. Selkirk, *Sums of squares of Tetranacci numbers: a generating function approach*, The Fibonacci Quarterly, **57.4** (2019), 313–317.

[16] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[17] A. Sofo, *General order Euler sums with multiple argument*, Journal of Number Theory, **189** (2018), 255–271.

[18] S. M. Stewart, *Explicit evaluation of some quadratic Euler-type sums containing double-index harmonic numbers*, Tatra Mountains Mathematical Publications, **77** (2020), 73–98.

[19] S. Vajda, *Fibonacci & Lucas numbers, and the golden section*, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1989.

[20] D. Zeilberger, *The C-finite ansatz*, Ramanujan Journal, **31.1–2** (2013), 23–32.

[21] Y. Zhang, *A combinatorial proof for the generating function of products of second-order recurrence sequences*, Ph.D. thesis, Central Michigan University, 2018.

[22] Y. Zhang and G. Grossman, *A combinatorial proof for the generating function of powers of the Fibonacci sequence*, The Fibonacci Quarterly, **55.3** (2017), 235–242.

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