INFINITE SUMS INVOLVING JACOBSTHAL POLYNOMIALS REVISITED

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Abstract. We explore the Jacobsthal-Lucas implications of four finite sums involving gibonacci polynomials and then extract their infinite gibonacci versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$ [3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} \cdot f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} \cdot l_n(1/\sqrt{x})$ ([2], and [3] on page 566).

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n(x)$ or $j_n(x)$, $\Delta = \sqrt{x^2+4}$, and $D = \sqrt{4x+1}$.

We have $\lim_{m\to\infty} \frac{x^m}{J_m^2} = 0 = \lim_{m\to\infty} \frac{x^m}{j_m^2}$ [5], where x is a positive integer.

1.1. Gibonacci Polynomial Sums.

In [4], we explored the following finite sums involving gibonacci polynomials g_n:

$$\sum_{n=1}^{m} \frac{\Delta^2 x f_{2(2n+1)}}{\left(l_{2n+1}^2 + \Delta^2\right)^2} = \frac{1}{(x^2+2)^2} - \frac{1}{l_{2m+2}^2}; \tag{1.1}$$

$$\sum_{n=1}^{m} \frac{(x^3+2x)\Delta^2 f_{2(2n+2)}}{\left(l_{2n+2}^2 + \Delta^2 x^2\right)^2} = \frac{1}{(x^2+2)^2} + \frac{1}{(x^4+4x^2+2)^2} - \frac{1}{l_{2m+2}^2} - \frac{1}{l_{2m+4}^2}; \tag{1.2}$$

$$\sum_{n=1}^{m} \frac{(x^3+2x)\Delta^2 f_{2(2n+1)}}{\left(l_{2n+1}^2 - \Delta^2 x^2\right)^2} = \frac{1}{x^2} + \frac{1}{(x^3+3x)^2} - \frac{1}{l_{2m+1}^2} - \frac{1}{l_{2m+3}^2}; \tag{1.3}$$

$$\sum_{n=1}^{m} \frac{\Delta^2 x f_{2(2n+2)}}{\left(l_{2n+2}^2 - \Delta^2\right)^2} = \frac{1}{(x^3+3x)^2} - \frac{1}{l_{2m+3}^2}. \tag{1.4}$$

Our goal is to find their Jacobsthal consequences and then extract their infinite versions.

2. Jacobsthal Polynomial Sums

2.1. **Jacobsthal Version of Formula** (1.1)**.** Let $A = \dfrac{\Delta^2 x f_{2(2n+1)}}{\left(l_{2n+1}^2 + \Delta^2\right)^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{4n}, we get

$$\begin{aligned} A &= \frac{D^2 x f_{2(2n+1)}}{\sqrt{x}\left(x l_{2n+1}^2 + D^2\right)^2} \\ &= \frac{D^2 x \cdot x^{(4n-1)/2}\left[x^{(4n+1)/2} f_{2(2n+1)}\right]}{\sqrt{x}\left(x^{2n+1} l_{2n+1}^2 + D^2 x^{2n}\right)^2} \\ &= \frac{D^2 x^{2n} J_{2(2n+1)}}{\left(j_{2n+1}^2 + D^2 x^{2n}\right)^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{D^2 x^{2n} J_{2(2n+1)}}{x\left(j_{2n+1}^2 + D^2 x^{2n}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Now, let $B = \dfrac{1}{(x^2+2)^2} - \dfrac{1}{l_{2m+2}^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{2m+2}. This yields

$$\begin{aligned} B &= \frac{x^2}{(2x+1)^2} - \frac{1}{l_{2m+2}^2} \\ &= \frac{x^2}{(2x+1)^2} - \frac{x^{2m+2}}{\left[x^{(2m+2)/2} l_{2m+2}\right]^2}; \\ \text{RHS} &= \frac{x^2}{(2x+1)^2} - \frac{x^{2m+2}}{j_{2m+2}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Equating the two sides, we get

$$\sum_{n=1}^{m} \frac{D^2 x^{2n} J_{2(2n+1)}}{\left(j_{2n+1}^2 + D^2 x^{2n}\right)^2} = \frac{x^2}{(2x+1)^2} - \frac{x^{2m+2}}{j_{2m+2}^2}. \tag{2.1}$$

This implies

$$\sum_{n=1}^{\infty} \frac{D^2 x^{2n} J_{2(2n+1)}}{\left(j_{2n+1}^2 + D^2 x^{2n}\right)^2} = \frac{x^2}{(2x+1)^2}. \tag{2.2}$$

Consequently, we have [4]

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 + 5\right)^2} = \frac{1}{45}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(j_{2n+1}^2 + 9 \cdot 4^n\right)^2} = \frac{4}{225}.$$

2.2. **Jacobsthal Version of Formula** (1.2)**.** Let $A = \frac{(x^3+2x)\Delta^2 f_{2(2n+2)}}{\left(l_{2n+2}^2+\Delta^2x^2\right)^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{4n}, we get

$$\begin{aligned} A &= \frac{(2x+1)D^2x^2 f_{2(2n+2)}}{\sqrt{x}\left(x^2 l_{2n+2}^2+D^2\right)^2} \\ &= \frac{(2x+1)D^2x^{2n}\left[x^{(4n+3)/2}f_{2(2n+2)}\right]}{\left(x^{2n+2}l_{2n+2}^2+D^2x^{2n}\right)^2} \\ &= \frac{(2x+1)D^2x^{2n}J_{2(2n+2)}}{\left(j_{2n+2}^2+D^2x^{2n}\right)^2}; \\ \text{LHS} &= \sum_{n=1}^{m}\frac{(2x+1)D^2x^{2n}J_{2(2n+2)}}{\left(j_{2n+2}^2+D^2x^{2n}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

We now let $B = \frac{1}{(x^2+2)^2} + \frac{1}{(x^4+4x^2+2)^2} - \frac{1}{l_{2m+2}^2} - \frac{1}{l_{2m+4}^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{2m+4}. This yields

$$\begin{aligned} B &= \frac{x^2}{(2x+1)^2} + \frac{x^4}{(2x^2+4x+1)^2} - \frac{1}{l_{2m+2}^2} - \frac{1}{l_{2m+4}^2} \\ &= \frac{x^2}{(2x+1)^2} + \frac{x^4}{(2x^2+4x+1)^2} - \frac{x^{2m+2}}{x^{m+1}l_{2m+2}^2} - \frac{x^{2m+4}}{x^{m+2}l_{2m+4}^2} \\ \text{RHS} &= \frac{x^2}{(2x+1)^2} + \frac{x^4}{(2x^2+4x+1)^2} - \frac{x^{2m+2}}{j_{2m+2}^2} - \frac{x^{2m+4}}{j_{2m+4}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Equating the two sides, we get

$$\sum_{n=1}^{m}\frac{(2x+1)D^2x^{2n}J_{2(2n+2)}}{\left(j_{2n+2}^2+D^2x^{2n}\right)^2} = \frac{x^2}{(2x+1)^2} + \frac{x^4}{(2x^2+4x+1)^2} - \frac{x^{2m+2}}{j_{2m+2}^2} - \frac{x^{2m+4}}{j_{2m+4}^2}. \tag{2.3}$$

This implies

$$\sum_{n=1}^{\infty}\frac{(2x+1)D^2x^{2n}J_{2(2n+2)}}{\left(j_{2n+2}^2+D^2x^{2n}\right)^2} = \frac{x^2}{(2x+1)^2} + \frac{x^4}{(2x^2+4x+1)^2}. \tag{2.4}$$

Consequently, we have [4]

$$\sum_{n=1}^{\infty}\frac{F_{2(2n+2)}}{\left(L_{2n+2}^2+5\right)^2} = \frac{58}{6,615}; \qquad \sum_{n=1}^{\infty}\frac{4^nJ_{2(2n+2)}}{\left(j_{2n+2}^2+9\cdot 4^n\right)^2} = \frac{1,556}{325,125}.$$

2.3. **Jacobsthal Version of Formula** (1.3)**.** Let $A = \dfrac{(x^3+2x)\Delta^2 f_{2(2n+1)}}{\left(l_{2n+1}^2 - \Delta^2 x^2\right)^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{4n-2}, we get

$$\begin{aligned} A &= \frac{(2x+1)D^2x^2 f_{2(2n+1)}}{\sqrt{x}\left(x^2 l_{2n+1}^2 - D^2\right)^2} \\ &= \frac{(2x+1)D^2x^{2n}\left[x^{(4n+1)/2} f_{2(2n+1)}\right]}{x\left(x^{2n+1} l_{2n+1}^2 - D^2x^{2n-1}\right)^2} \\ &= \frac{(2x+1)D^2x^{2n} J_{2(2n+1)}}{x\left(j_{2n+1}^2 - D^2x^{2n-1}\right)^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{(2x+1)D^2x^{2n} J_{2(2n+1)}}{x\left(j_{2n+1}^2 - D^2x^{2n-1}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Now, let $B = \dfrac{1}{x^2} + \dfrac{1}{(x^3+3x)^2} - \dfrac{1}{l_{2m+1}^2} - \dfrac{1}{l_{2m+3}^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{2m+3}. This yields

$$\begin{aligned} B &= x + \frac{x^3}{(3x+1)^2} - \frac{1}{l_{2m+1}^2} - \frac{1}{l_{2m+3}^2} \\ &= x + \frac{x^3}{(3x+1)^2} - \frac{x^{2m+3}}{x^2\left[x^{(2m+1)/2} l_{2m+1}\right]^2} - \frac{x^{2m+3}}{\left[x^{(2m+3)/2} l_{2m+3}\right]^2}; \\ \text{RHS} &= x + \frac{x^3}{(3x+1)^2} - \frac{x^{2m+1}}{j_{2m+1}^2} - \frac{x^{2m+3}}{l_{2m+3}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Equating the two sides then yields

$$\sum_{n=1}^{m} \frac{(2x+1)D^2x^{2n} J_{2(2n+1)}}{x\left(j_{2n+1}^2 - D^2x^{2n-1}\right)^2} = x + \frac{x^3}{(3x+1)^2} - \frac{x^{2m+1}}{j_{2m+1}^2} - \frac{x^{2m+3}}{l_{2m+3}^2}. \tag{2.5}$$

This implies

$$\sum_{n=1}^{\infty} \frac{(2x+1)D^2x^{2n} J_{2(2n+1)}}{\left(j_{2n+1}^2 - D^2x^{2n-1}\right)^2} = x^2 + \frac{x^4}{(3x+1)^2}. \tag{2.6}$$

This yields [4]

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 - 5\right)^2} = \frac{17}{240}; \qquad \sum_{n=1}^{\infty} \frac{J_{2(2n+1)}}{\left(j_{2n+1}^2 - 9\cdot 2^{2n-1}\right)^2} = \frac{212}{2,205}.$$

2.4. **Jacobsthal Version of Sum** (1.4)**.** Let $A = \dfrac{\Delta^2 x f_{2(2n+2)}}{\left(l_{2n+2}^2 - \Delta^2\right)^2}$. Now, replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{4n+2}. This yields

$$\begin{aligned} A &= \frac{D^2 x f_{2(2n+2)}}{\sqrt{x}\left(x l_{2n+2}^2 - D^2\right)^2} \\ &= \frac{D^2 x^{2n+1}\left[x^{(4n+3)/2} f_{2(2n+2)}\right]}{\left(x^{2n+2} l_{2n+2}^2 - D^2 x^{2n+1}\right)^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{D^2 x^{2n+1} J_{2(2n+2)}}{\left(j_{2n+2}^2 - D^2 x^{2n+1}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

We now let $B = \dfrac{1}{(x^3+3x)^2} - \dfrac{1}{l_{2m+3}^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{2m+3}, we get

$$\begin{aligned} B &= \frac{x^3}{(3x+1)^2} - \frac{1}{l_{2m+3}^2} \\ &= \frac{x^3}{(3x+1)^2} - \frac{x^{2m+3}}{\left[x^{(2m+3)/2} l_{2m+3}\right]^2}; \\ \text{RHS} &= \frac{x^3}{(3x+1)^2} - \frac{x^{2m+3}}{j_{2m+3}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides, we get

$$\sum_{n=1}^{m} \frac{D^2 x^{2n+1} J_{2(2n+2)}}{\left(j_{2n+2}^2 - D^2 x^{2n+1}\right)^2} = \frac{x^3}{(3x+1)^2} - \frac{x^{2m+3}}{j_{2m+3}^2}. \tag{2.7}$$

This implies

$$\sum_{n=1}^{\infty} \frac{D^2 x^{2n} J_{2(2n+2)}}{\left(j_{2n+2}^2 - D^2 x^{2n+1}\right)^2} = \frac{x^2}{(3x+1)^2}. \tag{2.8}$$

Consequently, we have [4]

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 - 5\right)^2} = \frac{1}{80}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(j_{2n+2}^2 - 18 \cdot 4^n\right)^2} = \frac{4}{441}.$$

3. Alternate Forms

Using the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$ ([3], page 446), we can rewrite the sums (2.2), (2.4), (2.6), and (2.8) in a different way:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{D^2 x^{2n} J_{2(2n+1)}}{\left[D^2(J_{2n+1}^2 + x^{2n}) - 4x^{2n+1}\right]^2} &= \frac{x^2}{(2x+1)^2};\\
\sum_{n=1}^{\infty} \frac{D^2 (2x+1) x^{2n} J_{2(2n+2)}}{\left[D^2(J_{2n+2}^2 + x^{2n}) + 4x^{2n+2}\right]^2} &= \frac{x^2}{(2x+1)^2} + \frac{x^4}{(2x^2+4x+1)^2};\\
\sum_{n=1}^{m} \frac{D^2 (2x+1) x^{2n} J_{2(2n+1)}}{\left[D^2(J_{2n+1}^2 - x^{2n-1}) - 4x^{2n+1}\right]^2} &= x^2 + \frac{x^4}{(3x+1)^2};\\
\sum_{n=1}^{\infty} \frac{D^2 x^{2n} J_{2(2n+2)}}{\left[D^2(J_{2n+2}^2 - x^{2n+1}) + 4x^{2n+2}\right]^2} &= \frac{x^2}{(3x+1)^2},
\end{aligned}$$

respectively.

Clearly, they yield the following Fibonacci results [4]:

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(5F_{2n+1}^2 + 1\right)^2} = \frac{1}{45}; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(5F_{2n+2}^2 + 9\right)^2} = \frac{58}{6,615};$$

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(5F_{2n+1}^2 - 9\right)^2} = \frac{17}{240}; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(5F_{2n+2}^2 - 1\right)^2} = \frac{1}{80},$$

respectively.

In addition, we have

$$\sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(9J_{2n+2}^2 + 4^n\right)^2} = \frac{4}{225}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(9J_{2n+2}^2 + 25 \cdot 4^n\right)^2} = \frac{1,556}{325,125};$$

$$\sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(9J_{2n+1}^2 - 25 \cdot 2^{2n-1}\right)^2} = \frac{212}{2,205}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(9J_{2n+2}^2 - 2 \cdot 4^n\right)^2} = \frac{4}{441},$$

respectively.

4. Acknowledgment

The author thanks the reviewer for the prompt and careful reading of the article, and for encouraging words.

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MSC2020: 11B39, 11B83, 11C08

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