CATALAN NUMBERS AND NON-INTERSECTING LATTICE PATHS

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Abstract. The convolution relation for Catalan numbers is used to show that they count certain non-intersecting lattice paths. Related uses and interpretations of the Catalan numbers in this context are also noted.

1. Introduction

Although less well-known than the Fibonacci numbers to non-mathematical audiences, the Catalan numbers $\{1, 1, 2, 5, 14, 42, 132, \ldots\}$ (OEIS [2] A000108) have a similarly wide range of applications. Named after the Belgian mathematician Eugène Charles Catalan (1814–1894), they arise as solutions to such counting problems as Catalan's binary bracketing problem, Euler's polygon division problem, and the ballot problem, and are associated with binary trees, flexagons, frieze paths, and Dyck paths [1, 4, 5]. Catalan numbers may be defined for integers $n \geq 0$ by $C_n = \frac{1}{n+1}\binom{2n}{n}$, so they are scaled versions of the central binomial coefficients. They also satisfy the convolution relation

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \quad \text{with} \quad C_0 = 1. \tag{1.1}$$

In this paper, we show that Catalan numbers count certain non-intersecting pairs of paths on lattices consisting of n diagonals. These "2-paths" are (weakly) monotone increasing and begin and end on pairs of consecutive points on diagonals of a lattice, as illustrated in Figure 1. Shapiro [3] analyzed an alternative formulation of these paths by use of a partial recurrence relation. This note presents a proof that these 2-paths are Catalan, based on the ordinary recurrence relation (1.1). Figure 1 is almost a "visual" proof itself. Conversely, these 2-paths provide a combinatorial interpretation of (1.1). Additional uses and interpretations of Catalan numbers in this context are also described.

2. 2-paths Are Catalan

We define a 2-path as two (weakly) monotone increasing, non-intersecting lattice paths (of unit steps either up or to the right) passing from consecutive left-hand points to consecutive right-hand points on diagonals of a lattice, as illustrated in Figure 1. Let c_n count the number of 2-paths passing through $n \geq 1$ diagonals (so each individual path making up the 2-path connects n lattice points and has $n-1$ edges). Each 2-path counted by c_n then starts on diagonal 1 and ends on diagonal n, so c_n counts non-intersecting pairs of paths to any lattice points (a, b) and $(a-1, b+1)$, where $a + b = n$ (for integers $a \geq 1$, $b \geq 0$). For example, $c_1 = 1$ counts the trivial 2-path with zero edges on a lattice with just one diagonal (e.g., just the initial diagonal at the lower left in Figure 1), whereas $c_2 = 2$ by inspection (consider non-intersecting pairs of paths, each with one edge, on the first two diagonals of Figure 1). Our aim is to show that c_n satisfies (1.1).

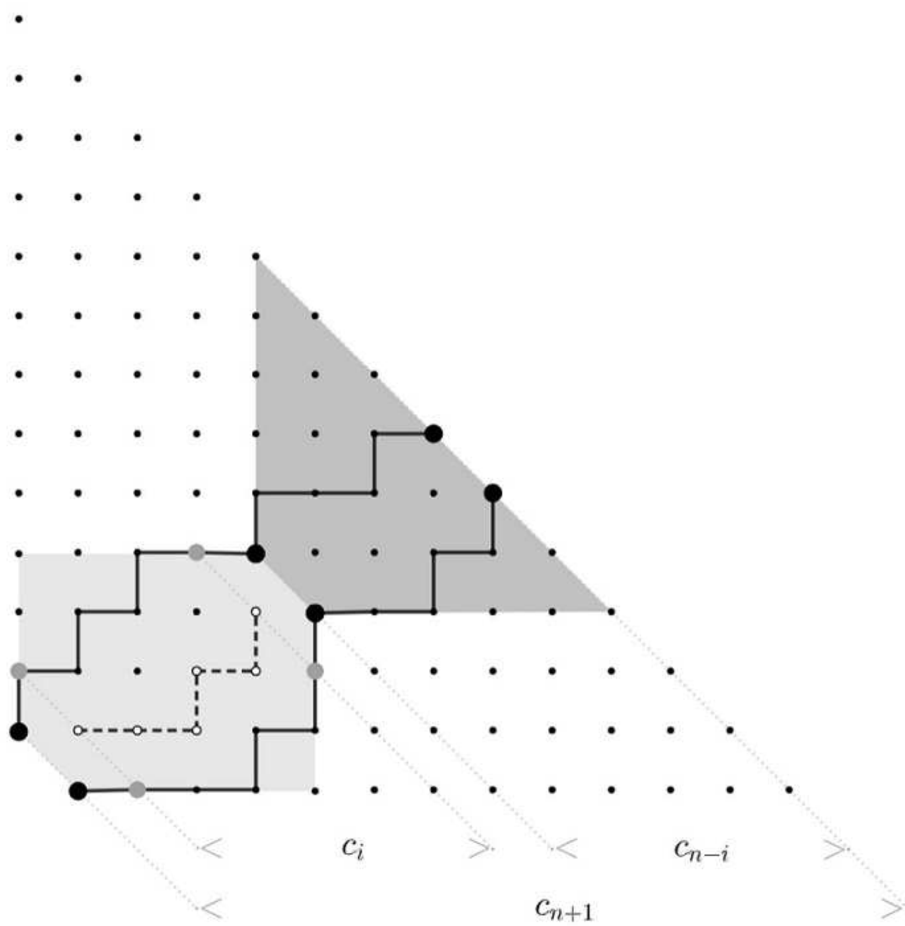

Figure 1. An example 2-path (solid lines) on a lattice of $n + 1$ diagonals showing where it *first* contains a sub-2-path (on $i + 2$ diagonals, $1 \leq i \leq n - 1$). Other possible sub-2-paths ending on the same two points will be contained in the light shaded region and each of these can be continued by each 2-path in the dark shaded region (of which there are c_{n-i}). The dark shaded region is the same for all pairs of consecutive points on the given diagonal.

Within the light shaded region, the individual paths making up a sub-2-path are separated by at least one point on each diagonal (otherwise a valid sub-path would have occurred earlier than shown). Removal of one of these points on each diagonal, such as those connected by the dashed line, collapses the paths between the light shaded points to produce a valid 2-path on i diagonals (of which there are c_i).

For $n \geq 2$, a 2-path counted by c_{n+1} traverses $n+1 \geq 3$ diagonals and may begin in one of three ways:

- with parallel vertical steps, followed by c_n 2-paths,
- with one vertical and one horizontal step, as in Figure 1, or
- with parallel horizontal steps, also followed by c_n 2-paths.

For the middle case above, consider Figure 1. Each 2-path will *first* have a complete sub-2-path ending on some diagonal from 3 to $n+1$, and suppose this is diagonal $i+2$, where $1 \leq i \leq n-1$, as illustrated within the light shaded region of Figure 1. Each of these sub-paths has at least one internal lattice point on diagonals 2 to $i+1$ and is counted by c_i for $1 \leq i \leq n-1$ because all valid 2-paths traversing $i+2-2=i$ diagonals could be created (such as between the light shaded points in Figure 1) by removal of an internal lattice point from each diagonal (such as on the dashed line in Figure 1). Further, no matter on which lattice diagonal the sub-path ends, there will always be $c_{n+1-(i+1)} = c_{n-i}$ possible ways to complete the 2-path (such as within the dark shaded region of Figure 1). In total, this gives $\sum_{i=1}^{n-1} c_i c_{n-i}$ possible paths for this case. Note that working with the *first* valid sub-path ensures that each possible 2-path is counted only once.

Combining the three cases above gives

$$c_{n+1} = c_n + \sum_{i=1}^{n-1} c_i c_{n-i} + c_n = \sum_{i=0}^{n} c_i c_{n-i}$$

upon defining $c_0 = 1$. Thus, $c_n = C_n$ and the Catalan numbers count 2-paths on a lattice of n diagonals, as we set out to show.

For comparison, Shapiro [3] defines a path as a sequence of lattice points beginning at $(0,0)$ obtained by unit steps either up or to the right. The number of pairs of these paths, each of length $n \geq 1$, that intersect only at the origin and end with x-coordinates differing by one, is denoted by B_{n1}, which is then shown to be C_n. As these pairs of paths intersect only at the origin, B_{n1} also counts pairs of monotone increasing, non-intersecting lattice paths with one path starting at $(1,0)$ and the other at $(0,1)$ and ending at lattice points (a,b) and $(a-1,b+1)$ respectively, where $a+b=n$. These are 2-paths with $n-1$ edges as defined in this paper, so $c_n = B_{n1}$.

3. Additional Results

As a bonus, we have found that Catalan numbers also count "sub-2-path free" 2-paths; i.e., there are C_{n-2} 2-paths with no valid sub-2-paths on a lattice of $n \geq 2$ diagonals. Further, it is well-known that the number of single, monotone paths traversing a lattice from its lower left point to the point $0 \leq k \leq n-1$ places along the nth diagonal is $\binom{n-1}{k}$. Therefore, the total number of pairs of monotone increasing paths (intersecting and non-intersecting) beginning on consecutive points of diagonal 1 and ending on consecutive points (maintaining the order of the initial points) on diagonal $n \geq 1$ of a lattice is also related to the Catalan numbers:

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 = \binom{2(n-1)}{n-1} = nC_{n-1}.$$

Thus, i) C_{n-1} may be interpreted as the average number of pairs of these paths per diagonal on a lattice of $n \geq 1$ diagonals, and ii) there are $nC_{n-1} - C_n$ *intersecting* pairs of these paths on a lattice of $n \geq 1$ diagonals.

The results derived above are illustrated in Figure 2 and Table 1 for the case of $n = 4$.

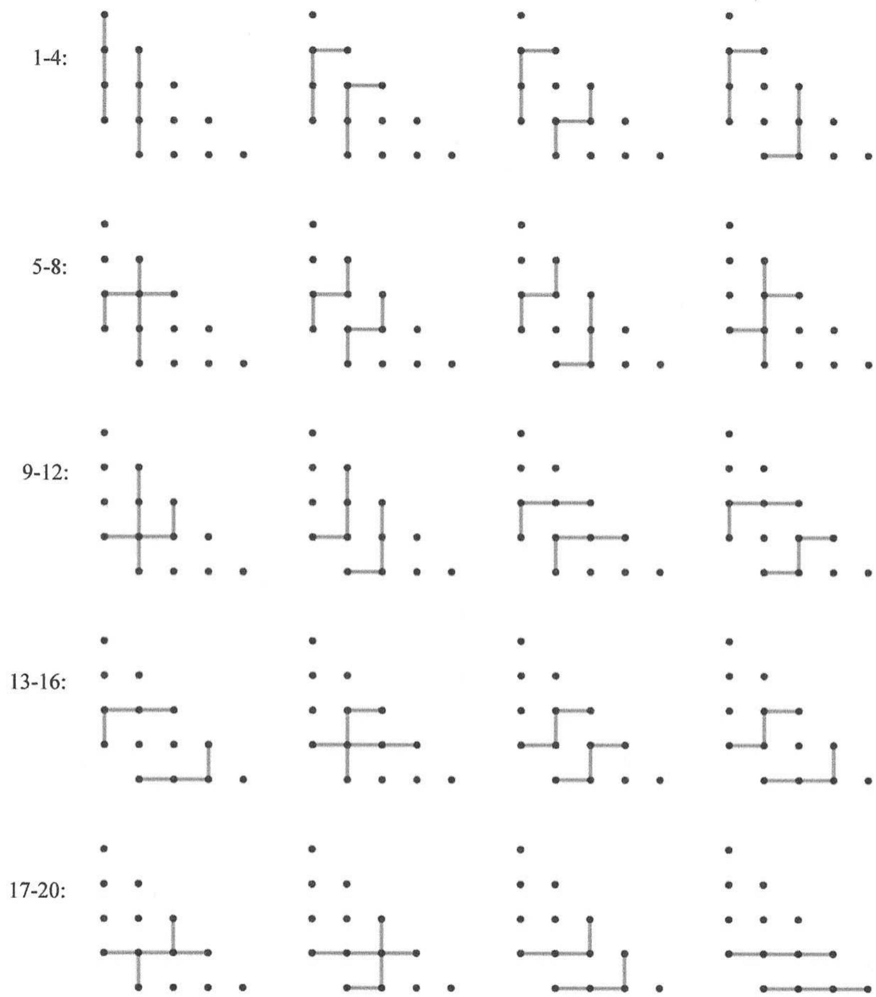

Figure 2. All pairs of monotone increasing paths (intersecting and non-intersecting) beginning on consecutive left-hand points and ending on consecutive (ordered) right-hand points on diagonal n of a lattice, where $n = 4$.

Description	**Formula**	**n = 4**	**Cases in Figure 2**
All paired paths	nC_{n-1}	$4C_3 = 4 \times 5 = 20$	(20 cases)
2-paths non-intersecting	C_n	$C_4 = 14$	1–4,6,7,10–13, 15,16,19,20 (14 cases)
Sub-2-path free 2-paths	C_{n-2}	$C_2 = 2$	4,13 (2 cases)
Average number of paired paths per diagonal	C_{n-1}	$C_3 = 5$	$\frac{20}{4} = 5$
Intersecting paired paths	$nC_{n-1} - C_n$	$20 - 14 = 6$	5,8,9,14,17,18 (6 cases)

Table 1. Application of formulas derived in the text to the $n = 4$ lattice example illustrated in Figure 2.

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References

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