GENERALIZED SCHREIER SETS, LINEAR RECURRENCE RELATION, AND TURÁN GRAPHS

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Abstract. We prove a linear recurrence relation for a large family of generalized Schreier sets, which generalizes the Fibonacci recurrence proved by Bird and higher order Fibonacci recurrence proved by the second author, et al. Furthermore, we show a relationship between Schreier-type sets and Turán graphs.

1. Introduction

A finite set $F \subset \mathbb{N}$ is said to be *Schreier* if $\min F \geq |F|$, where $|F|$ is the cardinality of F. The namesake of Schreier sets is Jósef Schreier who introduced these sets in the construction of a Banach space solving a problem of Banach and Saks [7]. In a blog post [1], Alistair Bird showed that the Fibonacci sequence appears if we count Schreier sets under certain conditions. In particular, if we set $S_n := \{F \subset \mathbb{N} : \min F \geq |F| \text{ and } \max F = n\}$, then $|S_1| = 1$, $|S_2| = 1$, and $|S_{n+2}| = |S_{n+1}| + |S_n|$ for all $n \geq 1$. There has been research on generalizing Bird's result to higher order recurrences (see [2, Theorems 4, 5, 6] and [3, Theorems 1.1, 1.3]) and on investigating the relationship between Schreier-type sets and partial sums of the Fibonacci and Gibonacci sequences [4, 5].

The first main result of this paper proves a recurrence relation from a large family of generalized Schreier sets. For $(p, q, n) \in \mathbb{N}^3$, we define

$$S_n^{p/q} \ = \ \{F \subset \mathbb{N} : q \min F \geq p|F| \text{ and } \max F = n\}.$$

Observe that $S_n^{p/1}$ is a special case considered in [3, Theorem 1.1].

Theorem 1.1. *Let $(p, q) \in \mathbb{N}^2$. For $n \in \mathbb{N}$ with $n \geq p + q$, we have*

$$|S_n^{p/q}| \ = \ \sum_{k=1}^{q} (-1)^{k+1} \binom{q}{k} |S_{n-k}^{p/q}| + |S_{n-(p+q)}^{p/q}|. \tag{1.1}$$

If $p = q = 1$, we have the Fibonacci recurrence stated above and proved by Bird. If $q = 1$ and $p \in \mathbb{N}$, we have [3, Theorem 1.1]

$$|S_n^p| \ = \ |S_{n-1}^p| + |S_{n-(p+1)}^p|.$$

The cases $q \in \mathbb{N}$ and $p = 1$ are new and, in the authors' opinion, unexpected and elegant. We also note that p/q need not be in simplified form. For different forms p/q, (1.1) gives equivalent recurrences.[1]

Our second result of this short note connects Schreier-type sets with Turán graphs. A Turán graph, denoted by $T(n, p)$, is the n-vertex complete p-partite graph whose parts differ in size by at most 1. That is, $T(n, p)$ has n vertices separated into p subsets, with sizes as equal as possible, and two vertices are connected by an edge if and only if they belong to different

[1] This independence of the recurrence relation has the same spirit as [3, Remark 1.5], where the depth of the recurrence is independent of one of the parameters.

subsets [8]. With an abuse of notation, we also write $T(n,p)$ to mean the number of edges of the corresponding graph. For each fixed $p \geq 2$, the sequence $(T(n,p))_{n=1}^{\infty}$ is available in OEIS [6] (for example, see `https://oeis.org/A002620`, `https://oeis.org/A000212`, `https://oeis.org/A033436`, and `https://oeis.org/A033437`.)

Define

$$Sr(n,p) \ = \ |\{F \subset [n] \ : \ p \min F \geq |F| \text{ and } F \text{ is an interval}\}|,$$

where $[n] = \{1, 2, \ldots, n\}$. Notice that in contrast to the definition of $S_n^{p/q}$, the definition of $Sr(n,p)$ does not require $\max F = n$.

Theorem 1.2. *For all $(n,p) \in \mathbb{N}^2$ with $n \geq p$, we have*

$$Sr(n,p) \ = \ T(n+1, p+1).$$

2. Proof of Theorem 1.1

The main tool is the following lemma.

Lemma 2.1. *Fix $(p,q) \in \mathbb{N}^2$, $n \geq p+q$, and let $G \subset \{n-q, \ldots n-1\}$ be nonempty. Define*

$$A_G \ = \ \{F \in S_n^{p/q} : G \cap F = \emptyset\}.$$

Then, $|A_G| \ = \ |S_{n-|G|}^{p/q}|$.

Proof. Fix G and let $\psi_G : \{1, \ldots, n\} \setminus G \to \{1, \ldots, n - |G|\}$ be the unique increasing bijection. Define $\phi_G : A_G \to S_{n-|G|}^{p/q}$ by

$$\phi_G(F) \ = \ \{\psi_G(i) : i \in F\}.$$

Showing that ϕ_G is a bijection is straightforward but technical.

First, we show that ϕ_G is well-defined; that is, the range of ϕ_G is $S_{n-|G|}^{p/q}$. Let $F \in A_G$. By definition, $n = \max F$ and so, $n - |G| = \max \phi_G(F)$. Note that $p|\phi_G(F)| = p|F|$. If $\min F < \min G$, then $\min \phi_G(F) = \min F$, and in this case,

$$p|\phi_G(F)| \ = \ p|F| \ \leq \ q \min F \ = \ q \min \phi_G(F),$$

as desired. Otherwise, $\min F > \min G$. In this case, $\min \phi_G(F) \geq n - q$ and $|F| \leq q$ because $G \subset \{n-1, \ldots, n-q\}$, $F \cap G = \emptyset$, and G is nonempty. Because $n \geq p + q$, we have

$$p|\phi_G(F)| \ = \ p|F| \ \leq \ pq \ \leq \ q(n-q) \ \leq \ q \min \phi_G(F).$$

This is the desired result.

The injectivity of ϕ_G follows immediately from the definition of ψ_G and ϕ_G. It remains to show that ϕ_G is surjective. Fix $H \in S_{n-|G|}^{p/q}$. Define

$$F \ = \ \{\psi_G^{-1}(i) : i \in H\}.$$

By definition, $\phi_G(F) = H$ and $F \cap G = \emptyset$. Note that $\max F = n$ and

$$p|F| \ = \ p|H| \ \leq \ q \min H \ = \ q\psi_G(\min F) \ \leq \ q \min F.$$

This finishes the proof of the lemma. □

Proof of Theorem 1.1. Using notation from Lemma 2.1, the set $S_n^{p/q} \setminus \cup_{i=1}^{q} A_{\{n-i\}}$ is

$$A \ := \ \{F \in S_n^{p/q} : \{n-q, \ldots, n-1\} \subset F\}.$$

We claim that $|A| = |S_{n-(p+q)}^{p/q}|$. The bijection $\phi : A \to S_{n-(p+q)}^{p/q}$ is defined by

$$\phi(F) \ = \ (F \setminus \{n-q+1, \ldots, n\}) - p.$$

Note first that $n-q = \max(F \setminus \{n-q+1,\dots,n\})$ and so $n-(p+q) = \max\phi(F)$. In addition we have

$$p|\phi(F)| \;=\; p(|F|-q) \;\le\; q\min F - pq \;=\; q(\min F - p) \;=\; q\min\phi(F).$$

Therefore, $\phi(F) \in S^{p/q}_{n-(p+q)}$. To see that ϕ is injective is trivial. We show that ϕ is surjective. Let $H \in S^{p/q}_{n-(p+q)}$ and define $F = (H+p)\cup\{n-q+1,\dots,n\}$. Then, $\phi(F) = H$ and $F \in A$, because

$$p|F| \;=\; p(|H|+q) \;\le\; q(\min H + p) \;=\; q\min F.$$

Let $\mathcal{G}_i = \{G \subset \{n-q,\dots,n-1\} : |G| = i\}$. By the inclusion-exclusion principle and Lemma 2.1, we obtain

$$\begin{aligned} |S^{p/q}_n| \;&=\; |A| + \sum_{G\in\mathcal{G}_1}|A_G| - \sum_{G\in\mathcal{G}_2}|A_G| + \sum_{G\in\mathcal{G}_3}|A_G| - \cdots + (-1)^{q+1}\sum_{G\in\mathcal{G}_q}|A_G| \\ &=\; |S^{p/q}_{n-(p+q)}| + \binom{q}{1}|S^{p/q}_{n-1}| - \binom{q}{2}|S^{p/q}_{n-2}| + \cdots + (-1)^{q+1}\binom{q}{q}|S^{p/q}_{n-q}|. \end{aligned} \tag{2.1}$$

This is the desired result. □

3. Proof of Theorem 1.2

The following is well-known and can, for example, be found on the Wikipedia page for Turán graphs.

Lemma 3.1. *For all $(n,p)\in\mathbb{N}^2$ with $n>p$, it holds that*

$$T(n,p) \;=\; \frac{p-1}{2p}(n^2-q^2) + \binom{q}{2}, \tag{3.1}$$

where $q := n - p\lfloor n/p\rfloor$.

Lemma 3.2. *We use $a\wedge b$ to indicate $\min\{a,b\}$. For all $(n,p)\in\mathbb{N}^2$, it holds that*

$$Sr(n,p) \;=\; \sum_{m=1}^{n} pm\wedge(n+1-m) \tag{3.2}$$

$$= \begin{cases} 1, & \text{if } n=1; \\ \binom{n+1}{2}, & \text{if } p>n\ge 2; \\ \frac{1}{2}\left(p(\Delta+1)\Delta + (n-\Delta+1)(n-\Delta)\right), & \text{if } p\le n; \end{cases} \tag{3.3}$$

where $\Delta = \lfloor (n+1)/(p+1)\rfloor$.

Proof. We build a set $F\subset[n]$ that satisfies: 1) $p\min F \ge |F|$ and 2) F is an interval. We denote the smallest element of F by m, which can be chosen from 1 to n. Once m is fixed, we choose $c := |F|$, which must satisfy $c\le pm$ and $c+m-1\le n$. The latter condition is to guarantee that $\max F\le n$. Once m and c are chosen, then F is unique because it is an interval. We obtain the formula for $Sr(n,p)$.

$$Sr(n,p) \;=\; \sum_{m=1}^{n}\sum_{c=1}^{pm\wedge(n+1-m)} 1 \;=\; \sum_{m=1}^{n} pm\wedge(n+1-m),$$

which is (3.2).

We now derive (3.3):

(1) By (3.2), $Sr(1,p) = p\wedge 1 = 1$.

(2) When $p > n \geq 2$, we have $pm \wedge (n+1-m) = n+1-m$ and so,

$$Sr(n,p) \ = \ \sum_{m=1}^{n}(n+1-m) \ = \ \binom{n+1}{2}.$$

(3) When $p \leq n$, we have

$$\begin{aligned}\sum_{m=1}^{n} pm \wedge (n+1-m) \ &= \ \sum_{m=1}^{\Delta} pm + \sum_{m=\Delta+1}^{n+1}(n+1-m) \\ &= \ \frac{1}{2}p(1+\Delta)\Delta + \frac{1}{2}(n-\Delta)(n-\Delta+1).\end{aligned}$$

This proves (3.3). □

Proof of Theorem 1.2. We prove $Sr(n,p) \ = \ T(n+1,p+1)$ for all $(n,p) \in \mathbb{N}^2$ with $n \geq p$. If $n = p$, then by definitions, $T(n+1,p+1) = \binom{n+1}{2}$, and $Sr(n,p) = n+(n-1)+\cdots+1 = \binom{n+1}{2}$. If $n > p$, by Lemmas 3.1 and 3.2, we want to show that

$$\frac{p((n+1)^2-q^2)}{2(p+1)} + \binom{q}{2} \ = \ \frac{1}{2}\left(p(\Delta+1)\Delta + (n-\Delta+1)(n-\Delta)\right),$$

which is equivalent to

$$\begin{aligned}&p\Delta(2n+2-(p+1)\Delta) + (n+1-(p+1)\Delta)(n-(p+1)\Delta) \\ &\qquad = \ p\Delta(\Delta+1) + (n-\Delta+1)(n-\Delta). \qquad (3.4)\end{aligned}$$

Simple algebraic manipulation of the three variables p, n, and Δ confirms that two sides of (3.4) are equal. This completes our proof. □

It would be interesting to see a proof of Theorem 1.2 that gives an explicit bijection between the edges of $T(n+1,p+1)$ and the elements of the set

$$\{F \subset [n] \ : \ p \min F \geq |F| \text{ and } F \text{ is an interval}\}.$$

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References

[1] A. Bird, *Schreier sets and the Fibonacci sequence*, `https://outofthenormmaths.wordpress.com/2012/05/13/jozef-schreier-schreier-sets-and-the-fibonacci-sequence/`.
[2] H. V. Chu, *The Fibonacci sequence and Schreier-Zeckendorf sets*, J. Integer Seq., **22** (2019).
[3] H. V. Chu, S. J. Miller, and Z. Xiang, *Higher order Fibonacci sequences from generalized Schreier sets*, The Fibonacci Quarterly, **58.3** (2020), 249–253.
[4] H. V. Chu, *Partial sums of the Fibonacci sequence*, The Fibonacci Quarterly, **59.2** (2021), 132–135.
[5] P. J. Mahanta, *Partial sums of the Gibonacci sequence*, preprint. Available at: `https://arxiv.org/pdf/2109.03534.pdf`.
[6] OEIS Foundation Inc. (2021), *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.
[7] J. Schreier, *Ein gegenbeispiel zur theorie der schwachen konvergentz*, Studia Math., **2** (1962), 58–62.
[8] P. Turán, *Eine extremalaufgabe aus der graphentheorie*, Mat. Fiz Lapook, **48** (1941), 436–452.

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