FIBONACCI IN SOMOS-5 BY COMPLEXIFICATION

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Abstract. We use a complex seed for the Somos-5 sequence to get Gaussian integers with real and imaginary parts related by the Fibonacci sequence. For example, if $\{\mathbf{s}_n\}$ is Somos-5 with seed $(1, 1, \mathbf{i}, 1, 1)$, then $\{\mathbf{s}_{2n}\}_{n\geq 1}$ is the sequence

$$1,\ 1,\ 1+\mathbf{i},\ 2-\mathbf{i},\ 2+3\,\mathbf{i},\ 5-3\,\mathbf{i},\ 5+8\,\mathbf{i},\ 13-8\,\mathbf{i},\ 13+21\,\mathbf{i},\ 34-21\,\mathbf{i},\ldots,$$

and the sequence of L_1-norms is the Fibonacci sequence starting at 1.

The **Somos-5** sequence begins with $s_1 = \cdots = s_5 = 1$, that is, with the *seed* $(1, 1, 1, 1, 1)$, and evolves using Michael Somos' five-term recursion. For $n \geq 6$,

$$s_n := S(s_{n-1}, s_{n-2}, \ldots, s_{n-5}) := \frac{s_{n-1}\, s_{n-4} + s_{n-2}\, s_{n-3}}{s_{n-5}}.$$

Surprisingly, this sequence [4, A006721] has been proven to stay integer [5]. Somos sequences [6] have been called nonlinear versions of Fibonacci [1].

Let $\{F_n\}$ denote the sequence $F_{-1} = 1$, $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, etc. Two complex numbers $\mathbf{z}$ and $\mathbf{w}$ are **F-dependent** if there is a nonnegative r such that $F_{2r-1}\,\mathbf{z}+F_{2r}\,\mathbf{w}\,\mathbf{i} = 0$ or $F_{2r+1}\,\mathbf{w} - F_{2r}\,\mathbf{z}\,\mathbf{i} = 0$ ($\mathbf{i} := \sqrt{-1}$) and **F-independent** otherwise. Note that F-independent complex numbers are both nonzero. The following are proved below.

Theorem 1. *With seed* $(1, \mathbf{z}, \mathbf{i}, \mathbf{w}, 1)$, $\mathbf{z}, \mathbf{w}$ *F-independent, we have for* $r \geq 0$,

(1) $\mathbf{s}_{4r+1} = 1$;
(2) $\mathbf{s}_{4r+2} = F_{2r-1}\,\mathbf{z} + F_{2r}\,\mathbf{w}\,\mathbf{i}$;
(3) $\mathbf{s}_{4r+3} = \mathbf{i}$;
(4) $\mathbf{s}_{4r+4} = F_{2r+1}\,\mathbf{w} - F_{2r}\,\mathbf{z}\,\mathbf{i}$.

Corollary 1. *With seed* $(1, \mathbf{z}, \mathbf{i}, \mathbf{w}, 1)$, $\mathbf{z}, \mathbf{w}$ *F-independent Gaussian integers, Somos-5 is a sequence* $\{\mathbf{s}_n\}_{n=1}^{\infty}$ *of* nonzero Gaussian integers.

Let $\ell_n := \|\mathbf{s}_n\|_1 := |\mathrm{Re}(\mathbf{s}_n)| + |\mathrm{Im}(\mathbf{s}_n)|$; put $x_n := \ell_{n-4} + \ell_{n-2} - \ell_n$, $n \geq 6$ even.

Theorem 2. *Sequence* $\{x_{2n}\}$ *is* (i) *nonnegative,* (ii) *even,* (iii) *non-increasing, and* (iv) *if* $x_{2n} = x_{2n+2}$*, then* $x_{2n+4} = 0$.

By Theorem 2, once $\{x_{2n}\}$ reaches zero, it stays there, and this means that the sequence $\{\ell_{2n}\}$ satisfies the Fibonacci recursion.

Corollary 2. *With seed* $(1, \mathbf{z}, \mathbf{i}, \mathbf{w}, 1)$, $\mathbf{z}, \mathbf{w}$ *F-independent Gaussian integers, there is a unique smallest* $N := N(\mathbf{z}, \mathbf{w}) \geq 3$ *such that* $x_{2n} = 0$ *for all* $n \geq N$.

Although the sequence $\{\ell_{2n}\}$ eventually satisfies the recursion, it appears that delay can be arbitrarily long for suitably chosen $\mathbf{z}, \mathbf{w}$. Indeed, if $m \geq 1$, and $\mathbf{z} := F_{m+1}+\mathbf{i}$ and $\mathbf{w} := 1+F_m\,\mathbf{i}$, then calculations shows $N(\mathbf{z}, \mathbf{w}) = m + 3$ for $1 \leq m \leq 30$.

Proof of Theorem 1. We induct on r, as the result holds for $r = 0$ by definition. There is an asymmetry because the behavior of $\mathbf{s}_n$ depends on n mod 4 whereas there are five elements in the seed, so (1) holds for $r = 0$ and $r = 1$.

Suppose the result holds for $r \geq 0$ fixed and also for $\mathbf{s}_{4r+5}$. We successively calculate $\mathbf{s}_{4r+6}, \ldots, \mathbf{s}_{4r+9}$ to prove (2), (3), (4) for $r+1$ and (1) for $r+2$.

Using the induction hypotheses and the Fibonacci and Somos-5 recursions,

$$\mathbf{s}_{4r+6} = \Big(1 \cdot F_{2r-1}\,\mathbf{z} + F_{2r}\,\mathbf{w}\,\mathbf{i} + (F_{2r+1}\,\mathbf{w} - F_{2r}\,\mathbf{z}\,\mathbf{i}) \cdot \mathbf{i}\Big)/1 = F_{2r+1}\,\mathbf{z} + F_{2r+2}\,\mathbf{w}\,\mathbf{i},$$

and

$$\mathbf{s}_{4r+7} = \frac{(F_{2r+1}\,\mathbf{z} + F_{2r+2}\,\mathbf{w}\,\mathbf{i}) \cdot \mathbf{i} \;+\; 1 \cdot F_{2r+1}\,\mathbf{w} - F_{2r}\,\mathbf{z}\,\mathbf{i}}{F_{2r-1}\,\mathbf{z} + F_{2r}\,\mathbf{w}\,\mathbf{i}} = \mathbf{i},$$

so (2) and (3) hold for $r+1$; (4) for $r+1$ and (1) for $r+2$ are similar. $\square$

The proof of Theorem 1 shows the following, which implies Corollary 1.

$$\mathbf{s}_{4r+6} = \mathbf{s}_{4r+2} + \mathbf{i} \cdot \mathbf{s}_{4r+4} \quad \text{and} \quad \mathbf{s}_{4r+8} = \mathbf{s}_{4r+4} - \mathbf{i} \cdot \mathbf{s}_{4r+6}. \tag{1}$$

Suppose $\phi(\mathbf{z}, \mathbf{w}) := \mathbf{z} + \mathbf{i}\,\mathbf{w}$ and $\psi(\mathbf{z}, \mathbf{w}) := \mathbf{z} - \mathbf{i}\,\mathbf{w}$; now define $\Lambda : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\Lambda(\mathbf{z}, \mathbf{w}) := (\phi(\mathbf{z}, \mathbf{w}), \psi(\mathbf{w}, \phi(\mathbf{z}, \mathbf{w})))$. We have $\Lambda(\mathbf{s}_{4r+2}, \mathbf{s}_{4r+4}) = (\mathbf{s}_{4r+6}, \mathbf{s}_{4r+8})$.

Proof of Theorem 2. Let $n \geq 2$ be even and let $\mathbf{s}_n = a + b\,\mathbf{i}$, $\mathbf{s}_{n+2} = c + d\,\mathbf{i}$, a, b, c, d real. By equation (1), $\mathbf{s}_{n+4} = \mathbf{s}_n \pm \mathbf{i}\,\mathbf{s}_{n+2} = a \mp d + (b \pm c)\,\mathbf{i}$ if $n \equiv 2$ or $0 \pmod 4$. We consider the first case, $n \equiv 2$, and leave the other to the reader. Then, $x_{n+4} = \|\mathbf{s}_n\| + \|\mathbf{s}_{n+2}\| - \|\mathbf{s}_{n+4}\| = |a| + |b| + |c| + |d| - |a - d| - |b + c|$.

But, $|a - d| = |a| + |d|$ unless $a \neq 0 \neq d$ and $a/|a| = d/|d|$, in which case $|a - d| = |a| + |d| - 2\,\min(|a|, |d|)$ and we say that a and d **conflict**. Similarly, $|b + c| = |b| + |c|$ unless $b \neq 0 \neq c$ and $b/|b| = -c/|c|$, in which case $|b + c| = |b| + |c| - 2\,\min(|b|, |c|)$ and b and c are said to conflict. It follows that $x_{n+4} \geq 0$ is even, so (i) and (ii) hold.

Also, $x_{n+4} = 0$ unless conflict occurs. By (1), $\mathbf{s}_{n+6} = \mathbf{s}_{n+2} + \mathbf{i}\,\mathbf{s}_{n+4} = b + 2c + (2d - a)\,\mathbf{i}$ so if $x_{n+4} = 0$, then $x_{n+6} = 0$ because u and v conflict if and only if u and κv conflict, $\kappa > 0$. The effects of (a, d) and (b, c) conflicts are additive, so we consider only (a, d), setting $b = 0 = c$. If $x_{n+4} > 0$, then $x_{n+4} = 2\min(|a|, |d|)$. By a corresponding calculation, one has

$$x_{n+6} = \ell_{n+2} + \ell_{n+4} - \ell_{n+6} = 2\Big(\min(|2d|, |a|) - \min(|a|, |d|)\Big).$$

If $|a| \leq |d|$, then $x_{n+6} = 0$. If $|d| < |a|$ and $|2d| < |a|$, then $x_{n+6} = 2|d| = x_{n+4}$.
If $|d| < |a|$ and $|a| \leq |2d|$, then $x_{n+6} = 2(|a| - |d|) \leq 2|d| = x_{n+4}$, and equality holds if and only if $|a| = 2|d|$. So (iii) holds.

From equation (1), $x_{n+8} = |a - d| + |2d - a| - |2a - 3d|$; after some canceling,

$$x_{n+8} = 2\Big(\min(2|a|, 3|d|) - \min(|a|, |d|) - \min(|a|, 2|d|)\Big).$$

If $|2d| < |a|$, then $|d| < |a|$, so $|3d| < |2a|$; hence, $x_{n+8} = 0$. If $|d| < |a| = |2d|$, then $x_{n+8} = 2(3|d| - |d| - 2|d|) = 0$. Hence, (iv) also holds. $\square$

Example. If $z = 60 - 40\,\mathbf{i}$ and $w = 24 + 37\,\mathbf{i}$, then for $n = 1, \ldots, 14$,

$$\mathbf{s}_{2n} = 60 - 40\,\mathbf{i},\ 24 + 37\,\mathbf{i},\ 23 - 16\,\mathbf{i},\ 8 + 14\,\mathbf{i},\ 9 - 8\,\mathbf{i},\ 5\,\mathbf{i},\ 4 - 8\,\mathbf{i},\ -8 + \mathbf{i},$$

$$3 - 16\,\mathbf{i}, -24 - 2\,\mathbf{i}, 5 - 40\,\mathbf{i}, -64 - 7\,\mathbf{i}, 12 - 104\,\mathbf{i}, -168 - 19\,\mathbf{i}.$$

The corresponding sequence of L_1 norms $\ell_{2n} := \|\mathbf{s}_{2n}\|_1$, $1 \leq n \leq 14$, is

100, 61, 39, 22, 17, 5, 12, 9, 19, 26, 45, 71, 116, 187.

Both types of conflict occur, and the values of $x_3, \ldots, x_{14}$ are accordingly

122, 78, 44, 34, 10, 8, 2, 2, 0, 0, 0, 0.

Remarks. There are three other Somos recursions (with seeds of length $k = 4, 6, 7$) that, with all "1"s in their seed, give only integers [2, Chap. 1]. These real Somos sequences are interesting because of their unexpected integrality. They produce integer sequences when properly begun, but seeds that lead to such good properties are rare. For Somos-5, although seed (1,2,1,1,1) gives only integers, the seed (1,3,1,1,1) does not produce an integral sequence. However, $\sigma = (1, \mathbf{z}, \mathbf{i}, \mathbf{w}, 1)$ gives rise to an infinite sequence of *Gaussian* integers if $\mathbf{z}$ and $\mathbf{w}$ are F-independent Gaussian integers. *Complexification improves integrality.*

The real Somos sequence grows at a quadratically exponential rate (e.g., [3]), but the L_1-norm of the complexified version grows with the Fibonacci recursion, and so is exponential. *The complexified process has slower growth.*

A possible reason for introducing complex numbers into real calculations is to improve computational efficiency. Integrality would seem to avoid floating-point issues, whereas smaller numbers require less storage and CPU-time. Is the instance given here a one-off or can it be more generally applied?

Another way to understand the Somos-5 recursion is in terms of a kind of twisted dot-product of triples. Let $\alpha := (a, b, c)$ and $\beta := (d, e, f)$ be any two triples of real or complex numbers. Define $\alpha \star \beta := be + cd - af$ and call α and β **$\star$-orthogonal** if $\alpha \star \beta = 0$. Then s_n is a Somos sequence precisely when each triple (s_n, s_{n+1}, s_{n+2}) is $\star$-orthogonal to the consecutive triple $(s_{n+3}, s_{n+4}, s_{n+5})$. In the real-integer case, only rapid growth can achieve such an orthogonal-turn "spiral" in the geometry given by $(\mathbb{R}^3, \star)$, but a much more compact trajectory is possible in $(\mathbb{C}^3, \star)$.

References

[1] Y. N. Federov and A. N. W. Hone, *Sigma-function solution to the general Somos-6 recurrence via hyperelliptic Prym varieties*, J. of Integrable Systems, **1** (2016), 1–34.

[2] D. Gale, *Tracking the Automatic Ant*, Springer, 1998; Chap. 1 is a reprint of *The strange and surprising saga of the Somos sequences*, Math. Intelligencer, **13.1** (1991), 40–42.

[3] R. Wm. Gosper and R. Schroeppel, *Somos sequence near-addition formulas and modular theta functions*, IACR Cryptol. ePrint Arch., **96** (2007). https://arxiv.org/pdf/math/0703470.pdf

[4] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[5] R. M. Robinson, *Periodicity of Somos sequences*, Proc. AMS, **116.3** (1992), 613–619.

[6] M. Somos, *Brief history of the Somos sequence problem*, 07 May 1990 (update 30 Oct. 1994). https://faculty.uml.edu//jpropp/somos/history.txt

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