INFINITE SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS

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Abstract. We explore four infinite sums involving gibonacci polynomials and their numeric cases, and then extract their Pell and Jacobsthal versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is a positive integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 7].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [7].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [4, 7]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [7].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, $\Delta = \sqrt{x^2+4}$, $D = \sqrt{4x+1}$, $E = \sqrt{x^2+1}$, $2\alpha(x) = x + \Delta$, $2\beta(x) = x - \Delta$, $\alpha = \alpha(1)$, $\beta(1) = \beta$, $\gamma(x) = x + E$, $\delta(x) = x - E$, and $\delta = \delta(1)$. It follows by the *Binet-like formulas* [7] that $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$, $\lim_{m\to\infty} \frac{f_{m+k}}{l_m} = \frac{\alpha^k(x)}{\Delta}$, and $\lim_{m\to\infty} \frac{l_{m+k}}{f_m} = \alpha^k(x)\Delta$.

1.1. Fundamental Identities.

Gibonacci polynomials satisfy the following fundamental properties [7]:

a) $f_{2n} = f_n l_n$; b) $l_{2n} = l_n^2 - 2(-1)^n$;
c) $l_{2n} = \Delta^2 f_n^2 + 2(-1)^n$; d) $f_{n+k} + f_{n-k} = f_k l_n$, where k is odd.

Properties (b) and (c) imply that $l_{2^n} = l_{2^{n-1}}^2 - 2$ and $l_{2^n} = \Delta^2 f_{2^{n-1}}^2 + 2$, respectively, where $n \geq 2$.

2. Infinite Gibonacci Sums

With this background, we begin our explorations with a sum involving the reciprocals of a special class of Fibonacci polynomials.

Lemma 2.1. *Let k be an odd positive integer. Then,*

$$\sum_{n=0}^{\infty} \frac{1}{f_{k\cdot 2^n}} = \frac{1}{f_k} + \frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}} + \beta(x). \tag{2.1}$$

Proof. With $m \geq 2$, it follows by property (d) that

$$\begin{aligned} f_{k\cdot 2^m-1} + 1 &= f_{k\cdot 2^m-1} + f_1 \\ &= f_{k\cdot 2^{m-1}+(k\cdot 2^{m-1}-1)} + f_{k\cdot 2^{m-1}-(k\cdot 2^{m-1}-1)} \\ &= f_{k\cdot 2^{m-1}-1} l_{k\cdot 2^{m-1}}. \end{aligned}$$

Using recursion [7], we will now establish that

$$\sum_{n=0}^{m} \frac{1}{f_{k\cdot 2^n}} = \frac{1}{f_k} + \frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}} - \frac{f_{k\cdot 2^m-1}}{f_{k\cdot 2^m}}. \tag{2.2}$$

To this end, let A_m denote the LHS of this equation and B_m its RHS. Using property (d), we then have

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{k\cdot 2^{m-1}-1}}{f_{k\cdot 2^{m-1}}} - \frac{f_{k\cdot 2^m-1}}{f_{k\cdot 2^m}} \\ &= \frac{f_{k\cdot 2^{m-1}-1} l_{k\cdot 2^{m-1}}}{f_{k\cdot 2^m}} - \frac{f_{k\cdot 2^m-1}}{f_{k\cdot 2^m}} \\ &= \frac{f_{k\cdot 2^m-1} + 1}{f_{k\cdot 2^m}} - \frac{f_{k\cdot 2^m-1}}{f_{k\cdot 2^m}} \\ &= \frac{1}{f_{k\cdot 2^m}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \left(\frac{1}{f_k} + \frac{1}{f_{2k}} + \frac{1}{f_{4k}}\right) - \left[\left(\frac{1}{f_k} + \frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}}\right) - \frac{f_{4k-1}}{f_{4k}}\right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, validating formula (2.2).

Because $\lim\limits_{n\to\infty} \dfrac{f_{n+k}}{f_n} = \alpha^k(x)$, equation (2.2) yields the given result, as desired. □

It follows from equation (2.1) that

$$\sum_{n=0}^{\infty} \frac{1}{f_{2^n}} = 1 + \frac{1}{f_2} + \frac{1}{f_4} + \frac{f_3}{f_4} + \beta(x); \qquad \sum_{n=0}^{\infty} \frac{1}{f_{3\cdot 2^n}} = \frac{1}{f_3} + \frac{1}{f_6} + \frac{1}{f_{12}} + \frac{f_{11}}{f_{12}} + \beta(x).$$

They yield [2, 6, 9]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=0}^{\infty} \frac{1}{F_{3\cdot 2^n}} = \frac{7}{4} - \frac{\sqrt{5}}{2}.$$

Next, we present an alternate version of Lemma 2.1.

Lemma 2.2. *Let k be an odd positive integer. Then,*

$$\sum_{n=0}^{\infty} \frac{f_k}{f_{k\cdot 2^n}} = 1 + \frac{f_k}{f_{2k}} + \frac{f_k}{f_{4k}} + \frac{f_{3k}}{f_{4k}} + \beta^k(x). \tag{2.3}$$

Proof. With $m \geq 2$ and k odd, it follows by property (d) that

$$\begin{aligned} f_{k(2^m-1)} + f_k &= f_{k\cdot 2^{m-1}+k(2^{m-1}-1)} + f_{k\cdot 2^{m-1}-k(2^{m-1}-1)} \\ &= f_{k(2^{m-1}-1)} l_{k\cdot 2^{m-1}}. \end{aligned}$$

Using recursion [7], we will now establish that

$$\sum_{n=0}^{m} \frac{f_k}{f_{k\cdot 2^n}} = 1 + \frac{f_k}{f_{2k}} + \frac{f_k}{f_{4k}} + \frac{f_{3k}}{f_{4k}} - \frac{f_{k(2^m-1)}}{f_{k\cdot 2^m}}. \tag{2.4}$$

We let A_m denote the LHS of this equation and B_m its RHS. Using property (d), we then have

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{k(2^{m-1}-1)}}{f_{k\cdot 2^{m-1}}} - \frac{f_{k(2^m-1)}}{f_{k\cdot 2^m}} \\ &= \frac{f_{k(2^{m-1}-1)} l_{k\cdot 2^{m-1}}}{f_{k\cdot 2^m}} - \frac{f_{k(2^m-1)}}{f_{k\cdot 2^m}} \\ &= \frac{f_{k(2^m-1)} + f_k}{f_{k\cdot 2^m}} - \frac{f_{k(2^m-1)}}{f_{k\cdot 2^m}} \\ &= \frac{f_k}{f_{k\cdot 2^m}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \left(1 + \frac{f_k}{f_{2k}} + \frac{f_k}{f_{4k}}\right) - \left[\left(1 + \frac{f_k}{f_{2k}} + \frac{f_k}{f_{4k}} + \frac{f_{3k}}{f_{4k}}\right) - \frac{f_{3k}}{f_{4k}}\right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, validating formula (2.4).

Because $\lim_{n\to\infty} \frac{f_n}{f_{n+k}} = \frac{1}{\alpha^k(x)}$ and k is odd, the given result follows from equation (2.4), as desired. □

In particular, we have

$$\sum_{n=0}^{\infty} \frac{1}{f_{2^n}} = 1 + \frac{1}{f_2} + \frac{1}{f_4} + \frac{f_3}{f_4} + \beta(x); \qquad \sum_{n=0}^{\infty} \frac{f_3}{f_{3\cdot 2^n}} = 1 + \frac{f_3}{f_6} + \frac{f_3}{f_{12}} + \frac{f_9}{f_{12}} + \beta^3(x).$$

It then follows that [2, 6, 9]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=0}^{\infty} \frac{1}{F_{3\cdot 2^n}} = \frac{7}{4} - \frac{\sqrt{5}}{2},$$

respectively, as found earlier.

With Lemma 2.1, we now establish the next result.

Theorem 2.3. *Let k be an odd positive integer and $\Delta = \sqrt{x^2+4}$. Then,*

$$\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n} - 2} = \frac{f_k}{l_{2k} - 2} + \frac{1}{\Delta^2}\left[\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}} + \beta(x)\right]. \tag{2.5}$$

Proof. Using property (c) and Lemma 2.1, and with $m \geq 3$, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} &= \frac{f_k}{l_{2k}-2} + \sum_{n=2}^{m} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} \\
&= \frac{f_k}{l_{2k}-2} + \sum_{n=2}^{m} \frac{f_{k\cdot 2^{n-1}}}{\Delta^2 f^2_{k\cdot 2^{n-1}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\sum_{n=2}^{m} \frac{1}{f_{k\cdot 2^{n-1}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\sum_{n=1}^{m-1} \frac{1}{f_{k\cdot 2^{n}}}; \\
\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} &= \frac{f_k}{l_{2k}-2} - \frac{1}{\Delta^2 f_k} + \frac{1}{\Delta^2}\sum_{n=0}^{\infty} \frac{1}{f_{k\cdot 2^{n}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}} + \beta(x)\right],
\end{aligned}$$

as desired. □

It follows from equation (2.5) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{f_{2^{n-1}}}{l_{2^n}-2} &= \frac{1}{x^2} + \frac{1}{\Delta^2}\left[\frac{1}{f_2} + \frac{1}{f_4} + \frac{f_3}{f_4} + \beta(x)\right]; \\
\sum_{n=1}^{\infty} \frac{f_{3\cdot 2^{n-1}}}{l_{3\cdot 2^n}-2} &= \frac{f_3}{l_6-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_6} + \frac{1}{f_{12}} + \frac{f_{11}}{f_{12}} + \beta(x)\right].
\end{aligned}$$

Consequently, we have [9, 10]

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n}-2} = \frac{3}{2} - \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{F_{3\cdot 2^{n-1}}}{L_{3\cdot 2^n}-2} = \frac{3}{8} - \frac{\sqrt{5}}{10},$$

respectively.

An Interesting Observation: With Lemma 2.2, we can rewrite equation (2.5) in a different way:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} &= \frac{f_k}{l_{2k}-2} - \frac{1}{\Delta^2 f_k} + \frac{1}{\Delta^2}\sum_{n=0}^{\infty} \frac{1}{f_{k\cdot 2^{n}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2 f_k}\left[\frac{f_k}{f_{2k}} + \frac{f_k}{f_{4k}} + \frac{f_{3k}}{f_{4k}} + \beta^k(x)\right]. \qquad (2.6)
\end{aligned}$$

In particular, we then have

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n}-2} = \frac{3}{2} - \frac{\sqrt{5}}{10}; \qquad \sum_{n=1}^{\infty} \frac{F_{3\cdot 2^{n-1}}}{L_{3\cdot 2^n}-2} = \frac{3}{8} - \frac{\sqrt{5}}{10},$$

as we just found.

Next, we investigate another gibonacci sum.

Theorem 2.4. *Let k be an odd positive integer and $\Delta = \sqrt{x^2+4}$. Then,*

$$\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}+1} = \frac{f_k}{l_{2k}+1} + \frac{f_{2k}-f_{4k}}{l_{4k}+1} + \frac{1}{\Delta}. \tag{2.7}$$

Proof. With $m \geq 2$ and recursion [7], we will first confirm that

$$\sum_{n=2}^{m} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}+1} = \frac{f_{2k}-f_{4k}}{l_{4k}+1} + \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1}. \tag{2.8}$$

Let A_m = LHS and B_m = RHS. Using properties (a) and (b), we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^{m-1}}}{l_{k\cdot 2^{m-1}}+1} \\ &= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^{m-1}}\left(l_{k\cdot 2^{m-1}}-1\right)}{\left(l_{k\cdot 2^{m-1}}+1\right)\left(l_{k\cdot 2^{m-1}}-1\right)} \\ &= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^m}-f_{k\cdot 2^{m-1}}}{l^2_{k\cdot 2^{m-1}}-1} \\ &= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^m}-f_{k\cdot 2^{m-1}}}{l_{k\cdot 2^m}+1} \\ &= \frac{f_{k\cdot 2^{m-1}}}{l_{k\cdot 2^m}+1} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \frac{f_{2k}}{l_{4k}+1} - \left(\frac{f_{4k}}{l_{4k}+1} + \frac{f_{2k}-f_{4k}}{l_{4k}+1}\right) \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, as expected.

Because $\lim_{n\to\infty} \frac{l_n}{f_n} = \Delta$, equation (2.8) yields

$$\sum_{n=2}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}+1} = \frac{f_{2k}-f_{4k}}{l_{4k}+1} + \frac{1}{\Delta}.$$

This yields the given result, as desired. □

In particular, Theorem 2.4 implies

$$\sum_{n=1}^{\infty} \frac{f_{2^{n-1}}}{l_{2^n}+1} = \frac{1}{l_2+1} + \frac{f_2-f_4}{l_4+1} + \frac{1}{\Delta}; \qquad \sum_{n=1}^{\infty} \frac{f_{3\cdot 2^{n-1}}}{l_{3\cdot 2^n}+1} = \frac{f_3}{l_6+1} + \frac{f_6-f_{12}}{l_{12}+1} + \frac{1}{\Delta}.$$

It then follows that [3, 8]

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n}+1} = \frac{\sqrt{5}}{5}; \qquad \sum_{n=1}^{\infty} \frac{F_{3\cdot 2^{n-1}}}{L_{3\cdot 2^n}+1} = -\frac{6}{19} + \frac{\sqrt{5}}{5},$$

respectively.

Next, we explore the Pell implications of formulas (2.1), (2.3), (2.5), (2.6), and (2.7).

3. Pell Versions

Because $b_n(x) = g_n(2x)$, it follows from equations (2.1), (2.3), (2.5), (2.6), and (2.7) that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{p_{k\cdot 2^n}} &= \frac{1}{p_k} + \frac{1}{p_{2k}} + \frac{1}{p_{4k}} + \frac{p_{4k-1}}{p_{4k}} + \delta(x);\\
\sum_{n=0}^{\infty} \frac{p_k}{p_{k\cdot 2^n}} &= 1 + \frac{p_k}{p_{2k}} + \frac{p_k}{p_{4k}} + \frac{p_{3k}}{p_{4k}} + \delta^k(x);\\
\sum_{n=1}^{\infty} \frac{p_{k\cdot 2^{n-1}}}{q_{k\cdot 2^n} - 2} &= \frac{p_k}{q_{2k}-2} + \frac{1}{4E^2}\left[\frac{1}{p_{2k}} + \frac{1}{p_{4k}} + \frac{p_{4k-1}}{p_{4k}} + \delta(x)\right];\\
\sum_{n=1}^{\infty} \frac{p_{k\cdot 2^{n-1}}}{q_{k\cdot 2^n} - 2} &= \frac{p_k}{q_{2k}-2} + \frac{1}{4E^2 p_k}\left[\frac{p_k}{p_{2k}} + \frac{p_k}{p_{4k}} + \frac{p_{3k}}{p_{4k}} + \delta^k(x)\right];\\
\sum_{n=1}^{\infty} \frac{p_{k\cdot 2^{n-1}}}{q_{k\cdot 2^n} + 1} &= \frac{1}{2E} + \frac{p_k}{q_{2k}+1} + \frac{p_{2k}-p_{4k}}{q_{4k}+1},
\end{aligned}$$

respectively, where k is an odd positive integer. Consequently, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{P_{k\cdot 2^n}} &= \frac{1}{P_k} + \frac{1}{P_{2k}} + \frac{1}{P_{4k}} + \frac{P_{4k-1}}{P_{4k}} + \delta;\\
\sum_{n=0}^{\infty} \frac{P_k}{P_{k\cdot 2^n}} &= 1 + \frac{P_k}{P_{2k}} + \frac{P_k}{P_{4k}} + \frac{P_{3k}}{P_{4k}} + \delta^k;\\
\sum_{n=1}^{\infty} \frac{P_{k\cdot 2^{n-1}}}{Q_{k\cdot 2^n} - 1} &= \frac{P_k}{Q_{2k}-1} + \frac{1}{4}\left(\frac{1}{P_{2k}} + \frac{1}{P_{4k}} + \frac{P_{4k-1}}{P_{4k}} + \delta\right);\\
\sum_{n=1}^{\infty} \frac{P_{k\cdot 2^{n-1}}}{Q_{k\cdot 2^n} - 1} &= \frac{P_k}{Q_{2k}-1} + \frac{1}{4P_k}\left(\frac{P_k}{P_{2k}} + \frac{P_k}{P_{4k}} + \frac{P_{3k}}{P_{4k}} + \delta^k\right);\\
\sum_{n=1}^{\infty} \frac{P_{k\cdot 2^{n-1}}}{2Q_{k\cdot 2^n} + 1} &= \frac{\sqrt{2}}{4} + \frac{P_k}{2Q_{2k}+1} + \frac{P_{2k}-P_{4k}}{2Q_{4k}+1},
\end{aligned}$$

respectively.

4. Jacobsthal Consequences

Using the gibonacci-Jacobsthal relationships, we now explore the Jacobsthal implications of formulas (2.1), (2.5), and (2.7). In each case, for clarity and convenience, we let A denote the fractional expression on the left side and B the corresponding expression on the right side, and LHS and RHS those of the Jacobsthal formula to be found.

4.1. **Jacobsthal Version of Formula** (2.1)**.**

Proof. We have $A = \dfrac{1}{f_{k\cdot 2^n}}$ and $B = \dfrac{1}{f_k} + \dfrac{1}{f_{2k}} + \dfrac{1}{f_{4k}} + \dfrac{f_{4k-1}}{f_{4k}} + \beta(x)$.

Replacing x with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator with $x^{(k\cdot 2^n-1)/2}$, we get

$$\begin{aligned} A &= \frac{x^{(k\cdot 2^n-1)/2}}{x^{(k\cdot 2^n-1)/2} f_{k\cdot 2^n}} \\ &= \frac{x^{(k\cdot 2^n-1)/2}}{J_{k\cdot 2^n}}; \\ \text{LHS} &= \sum_{n=0}^{\infty} \frac{x^{(k\cdot 2^n-1)/2}}{J_{k\cdot 2^n}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, replace x with $1/\sqrt{x}$ in B, and then multiply the numerator and denominator with $x^{(4k-1)/2}$. This yields

$$\begin{aligned} B &= \frac{1}{f_k} + \frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}} + \frac{1-D}{2\sqrt{x}} \\ &= \frac{x^{3k/2}}{x^{(k-1)/2} f_k} + \frac{x^k}{x^{(2k-1)/2} f_{2k}} + \frac{x^{(4k-1)/2}}{x^{(4k-1)/2} f_{4k}} + \frac{\sqrt{x}\left[x^{(4k-2)/2} f_{4k-1}\right]}{x^{(4k-1)/2} f_{4k}} + \frac{1-D}{2\sqrt{x}}; \\ \text{RHS} &= \frac{x^{3k/2}}{J_k} + \frac{x^k}{J_{2k}} + \frac{x^{(4k-1)/2}}{J_{4k}} + \frac{\sqrt{x} J_{4k-1}}{J_{4k}} + \frac{1-D}{2\sqrt{x}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides gives the desired Jacobsthal version:

$$\sum_{n=0}^{\infty} \frac{x^{k\cdot 2^{n-1}}}{J_{k\cdot 2^n}} = \frac{x^{(3k+1)/2}}{J_k} + \frac{x^{(2k+1)/2}}{J_{2k}} + \frac{x^{2k}}{J_{4k}} + \frac{x J_{4k-1}}{J_{4k}} + \frac{1-D}{2}. \tag{4.1}$$

□

In particular, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{k\cdot 2^n}} = \frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{F_{4k-1}}{F_{4k}} + \beta.$$

This yields [2, 6]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=0}^{\infty} \frac{1}{F_{3\cdot 2^n}} = \frac{7}{4} - \frac{\sqrt{5}}{2},$$

as found earlier.

4.2. **Jacobsthal Version of Formula** (2.5)**.**

Proof. We have $A = \dfrac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n} - 2}$ and $B = \dfrac{f_k}{l_{2k} - 2} + \dfrac{1}{\Delta^2}\left[\dfrac{1}{f_{2k}} + \dfrac{1}{f_{4k}} + \dfrac{f_{4k-1}}{f_{4k}} + \beta(x)\right]$.

Now, replace x with $1/\sqrt{x}$ in A, and then multiply the numerator and denominator with $x^{(k\cdot 2^n)/2}$. We then get

$$\begin{aligned} A &= \frac{x^{(k\cdot 2^{n-1}+1)/2}\left[x^{(k\cdot 2^{n-1}-1)/2} f_{k\cdot 2^{n-1}}\right]}{x^{(k\cdot 2^n)/2} l_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}} \\ &= \frac{x^{(k\cdot 2^{n-1}+1)/2} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{x^{(k\cdot 2^{n-1}+1)/2} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Replacing x with $1/\sqrt{x}$ in B, and then multiplying the numerator and denominator with $x^{(4k-1)/2}$, yield

$$\begin{aligned} B &= \frac{f_k}{l_{2k}-2} + \frac{x}{D^2}\left(\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-1}}{f_{4k}} + \frac{1-D}{2\sqrt{x}}\right) \\ &= \frac{x^k\left[x^{(k-1)/2} f_k\right]}{x^{2k/2} l_{2k} - 2x^k} + \frac{x}{D^2}\left[\frac{x^k}{x^{(2k-1)/2} f_{2k}} + \frac{x^{(4k-1)/2}}{x^{(4k-1)/2} f_{4k}} + \frac{\sqrt{x}\left[x^{(4k-2)/2} f_{4k-1}\right]}{x^{(4k-1)/2} f_{4k}} + \frac{1-D}{2\sqrt{x}}\right]; \\ \text{RHS} &= \frac{x^k J_k}{j_{2k} - 2x^k} + \frac{1}{D^2}\left[\frac{x^{k+1}}{J_{2k}} + \frac{x^{(4k+1)/2}}{J_{4k}} + \frac{x^{3/2} J_{4k-1}}{J_{4k}} + \frac{(1-D)\sqrt{x}}{2}\right], \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

By combining the two sides, we get the Jacobsthal version of the formula:

$$\sum_{n=1}^{\infty} \frac{x^{k\cdot 2^{n-2}} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}} = \frac{x^{(2k-1)/2} J_k}{j_{2k} - 2x^k} + \frac{1}{D^2}\left[\frac{x^{(2k+1)/2}}{J_{2k}} + \frac{x^{2k}}{J_{4k}} + \frac{x J_{4k-1}}{J_{4k}} + \frac{1-D}{2}\right]. \tag{4.2}$$

□

This implies

$$\sum_{n=1}^{\infty} \frac{F_{k\cdot 2^{n-1}}}{L_{k\cdot 2^n} - 2} = \frac{F_k}{L_{2k} - 2} + \frac{1}{5}\left(\frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{F_{4k-1}}{F_{4k}} + \beta\right).$$

It then follows that [9, 10]

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n} - 2} = \frac{3}{2} - \frac{\sqrt{5}}{10}; \qquad \frac{F_{3\cdot 2^{n-1}}}{L_{3\cdot 2^n} - 2} = \frac{3}{8} - \frac{\sqrt{5}}{10},$$

as obtained earlier.

4.3. **Jacobsthal Version of Formula** (2.7).

Proof. We have $A = \dfrac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n} + 1}$ and $B = \dfrac{f_k}{l_{2k} + 1} + \dfrac{f_{2k} - f_{4k}}{l_{4k} + 1} + \dfrac{1}{\Delta}$.

Replacing x with $1/\sqrt{x}$ in A, and then multiplying the numerator and denominator with $x^{k\cdot 2^{n-1}}$, we get

$$\begin{aligned} A &= \frac{x^{(k\cdot 2^{n-1}+1)/2}\left[x^{(k\cdot 2^{n-1}-1)/2} f_{k\cdot 2^{n-1}}\right]}{x^{(k\cdot 2^n)/2} l_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}} \\ &= \frac{x^{(k\cdot 2^{n-1}+1)/2} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{x^{(k\cdot 2^{n-1}+1)/2} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, replace x with $1/\sqrt{x}$ in B, and then multiply the numerator and denominator with x^{2k}. This yields

$$\begin{aligned} B &= \frac{x^{(k+1)/2}\left[x^{(k-1)/2} f_k\right]}{x^{2k/2} l_{2k} + x^k} + \frac{x^{(2k+1)/2}\left[x^{(2k-1)/2} f_{2k}\right] - x^{1/2}\left[x^{(4k-1)/2} f_{4k}\right]}{x^{4k/2} l_{4k} + x^{2k}} + \frac{\sqrt{x}}{D}; \\ \text{RHS} &= \frac{\sqrt{x}}{D} + \frac{x^{(k+1)/2} J_k}{j_{2k} + x^k} + \frac{x^{(2k+1)/2} J_{2k} - \sqrt{x} J_{4k}}{j_{4k} + x^{2k}} + \frac{\sqrt{x}}{D}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the Jacobsthal version of the formula:

$$\sum_{n=1}^{\infty} \frac{x^{k\cdot 2^{n-2}} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}} = \frac{x^{k/2} J_k}{j_{2k} + x^k} + \frac{x^k J_{2k} - J_{4k}}{j_{4k} + x^{2k}} + \frac{1}{D}. \tag{4.3}$$

□

It follows from equation (4.3) that

$$\sum_{n=1}^{\infty} \frac{F_{k\cdot 2^{n-1}}}{L_{k\cdot 2^n} + 1} = \frac{F_k}{L_{2k} + 1} + \frac{F_{2k} - F_{4k}}{L_{4k} + 1} + \frac{\sqrt{5}}{5}.$$

In particular, we then have [3, 8]

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n} + 1} = \frac{\sqrt{5}}{5}; \qquad \sum_{n=1}^{\infty} \frac{F_{3\cdot 2^{n-1}}}{L_{3\cdot 2^n} + 1} = -\frac{6}{19} + \frac{\sqrt{5}}{5},$$

as found earlier.

Finally, we present the Jacobsthal versions of formulas (2.3) and (2.6); in the interest of brevity, we omit their proofs:

$$\sum_{n=1}^{\infty} \frac{x^{k\cdot 2^{n-1}} J_k}{J_{k\cdot 2^n}} = x^{k/2} + \frac{x^k J_k}{J_{2k}} + \frac{x^k J_k}{J_{4k}} + \frac{x^{2k} J_{3k}}{J_{4k}} + \frac{(1-D)^k}{2^k}; \tag{4.4}$$

$$\sum_{n=1}^{\infty} \frac{x^{k\cdot 2^{n-2}} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}} = \frac{J_k}{j_{2k} - 2x^k} + \frac{1}{D^2 J_k}\left[\frac{x^k J_k}{J_{2k}} + \frac{x^{2k} J_k}{J_{4k}} + \frac{x^k J_{3k}}{J_{4k}} + \frac{(1-D)^k}{2^k}\right], \tag{4.5}$$

respectively, where k is odd.

They yield

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{F_k}{F_{k\cdot 2^n}} &= 1 + \frac{F_k}{F_{2k}} + \frac{F_k}{F_{4k}} + \frac{F_{3k}}{F_{4k}} + \beta^k;\\ \sum_{n=1}^{\infty} \frac{F_{k\cdot 2^{n-1}}}{L_{k\cdot 2^n} - 2} &= \frac{F_k}{L_{2k} - 2} + \frac{1}{5F_k}\left(\frac{F_k}{F_{2k}} + \frac{F_k}{F_{4k}} + \frac{F_{3k}}{F_{4k}} + \beta^k\right),\end{aligned}$$

respectively, where k is odd.

It then follows that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} &= \frac{7}{2} - \frac{\sqrt{5}}{2}\ [2,\ 6]; & \sum_{n=0}^{\infty} \frac{1}{F_{3\cdot 2^n}} &= \frac{7}{4} - \frac{\sqrt{5}}{2};\\ \sum_{n=1}^{\infty} \frac{F_{2^{n-1}}}{L_{2^n} - 2} &= \frac{3}{2} - \frac{\sqrt{5}}{10}\ [9,\ 10]; & \sum_{n=1}^{\infty} \frac{F_{3\cdot 2^{n-1}}}{L_{3\cdot 2^n} - 2} &= \frac{3}{8} - \frac{\sqrt{5}}{10},\end{aligned}$$

as found earlier.

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