# INFINITE SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS REVISITED

THOMAS KOSHY

Abstract. We explore two infinite sums involving gibonacci polynomials and their numeric versions, and then extract their Pell and Jacobsthal counterparts.

## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where $x$ is a positive integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the $n$th *Lucas polynomial*. They can also be defined by *Binet-like* formulas. So, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [1, 3, 4].

*Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* $P_n$ and *Pell-Lucas numbers* $Q_n$ are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [4].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th *Jacobsthal-Lucas polynomial* [2, 4]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. So, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Gibonacci and Jacobsthal polynomials are linked by the relationships
$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so $z_n$ will mean $z_n(x)$. In addition, we let $g_n = f_n$ or $l_n$, and $c_n = J_n(x)$ or $j_n(x)$, $\Delta = \sqrt{x^2+4}$, $D = \sqrt{4x+1}$, $E = \sqrt{x^2+1}$, $2\alpha(x) = x + \Delta$, $2\beta(x) = x - \Delta$, $\alpha = \alpha(1)$, $\beta(1) = \beta$, $\gamma(x) = x + E$, $\delta(x) = x - E$, and $\delta = \delta(1)$. It follows by the *Binet-like formulas* [4] that $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$, $\lim_{m\to\infty} \frac{f_{m+k}}{l_m} = \frac{\alpha^k(x)}{\Delta}$, and $\lim_{m\to\infty} \frac{l_{m+k}}{f_m} = \alpha^k(x)\Delta$.

### 1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following fundamental properties [4]:

a) $f_{2n} = f_n l_n$;  b) $l_{2n} = l_n^2 - 2(-1)^n$;
c) $l_{2n} = \Delta^2 f_n^2 + 2(-1)^n$;  d) $f_{n+k} + f_{n-k} = f_k l_n$, where $k$ is odd;
e) $l_{n+k} + l_{n-k} = l_k l_n$, where $k$ is even.

Properties (b) and (c) imply that $l_{2^n} = l_{2^{n-1}}^2 - 2$ and $l_{2^n} = \Delta^2 f_{2^{n-1}}^2 + 2$, respectively, where $n \geq 2$.

## 2. Infinite Gibonacci Sums

With this background, we begin our explorations with the counterpart of Lemma 1 in [5], where $k$ has even parity.

**Lemma 2.1.** *Let $k$ be an even positive integer and $\Delta = \sqrt{x^2+4}$. Then,*

$$\sum_{n=0}^{\infty} \frac{l_2}{f_{k\cdot 2^n}} = \frac{l_2}{f_k} + \frac{l_2}{f_{2k}} + \frac{l_2}{f_{4k}} + \frac{l_{4k-2}}{f_{4k}} - \Delta\beta^2(x). \tag{2.1}$$

*Proof.* With $m \geq 2$ and $k$ even, it follows by identity (e) that

$$\begin{aligned} l_{k\cdot 2^m-2} + l_2 &= l_{[k\cdot 2^{m-1}+(k\cdot 2^{m-1}-2)]} + l_{[k\cdot 2^{m-1}-(k\cdot 2^{m-1}-2)]} \\ &= l_{k\cdot 2^{m-1}-2} l_{k\cdot 2^{m-1}}. \end{aligned}$$

Using this result, identity (e), and recursion [4], we will first confirm that

$$\sum_{n=0}^{m} \frac{l_2}{f_{k\cdot 2^n}} = \frac{l_2}{f_k} + \frac{l_2}{f_{2k}} + \frac{l_2}{f_{4k}} + \frac{l_{4k-1}}{f_{4k}} - \frac{l_{k\cdot 2^m-2}}{f_{k\cdot 2^m}}, \tag{2.2}$$

where $m \geq 2$.

To realize this goal, let $A_n$ denote the LHS of this equation and $B_n$ its RHS. Using property (e), we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{l_{k\cdot 2^{m-1}-2}}{f_{k\cdot 2^{m-1}}} - \frac{l_{k\cdot 2^m-2}}{f_{k\cdot 2^m}} \\ &= \frac{l_{k\cdot 2^{m-1}-2} l_{k\cdot 2^{m-1}}}{f_{k\cdot 2^m}} - \frac{l_{k\cdot 2^m-2}}{f_{k\cdot 2^m}} \\ &= \frac{l_{k\cdot 2^m-2} + l_2}{f_{k\cdot 2^m}} - \frac{l_{k\cdot 2^m-2}}{f_{k\cdot 2^m}} \\ &= \frac{l_2}{f_{k\cdot 2^m}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \left(\frac{l_2}{f_k} + \frac{l_2}{f_{2k}} + \frac{l_2}{f_{4k}}\right) - \left[\left(\frac{l_2}{f_k} + \frac{l_2}{f_{2k}} + \frac{l_2}{f_{4k}} + \frac{l_{4k-1}}{f_{4k}}\right) - \frac{l_{4k-1}}{f_{4k}}\right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, confirming formula (2.2). Because $\lim_{n\to\infty} \frac{l_n}{f_{n+2}} = \frac{\Delta}{\alpha^2(x)} = \Delta\beta^2(x)$, the given result follows from equation (2.2), as desired. □

It follows from equation (2.1) that

$$\sum_{n=0}^{\infty} \frac{l_2}{f_{2^{n+1}}} = \frac{l_2}{f_2} + \frac{l_2}{f_4} + \frac{l_2}{f_8} + \frac{l_6}{f_8} - \Delta\beta^2(x); \quad \sum_{n=0}^{\infty} \frac{l_2}{f_{2^{n+2}}} = \frac{l_2}{f_4} + \frac{l_2}{f_8} + \frac{l_2}{f_{16}} + \frac{l_{14}}{f_{16}} - \Delta\beta^2(x).$$

Consequently, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^{n+1}}} = \frac{5}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2^{n+2}}} = \frac{3}{2} - \frac{\sqrt{5}}{2}.$$

The following lemma is an alternate version of Lemma 2.1. Both give the same formula when $k = 2$.

**Lemma 2.2.** *Let $k$ be an even positive integer and $\Delta = \sqrt{x^2+4}$. Then,*

$$\sum_{n=0}^{\infty} \frac{l_k}{f_{k\cdot 2^n}} = \frac{l_k}{f_k} + \frac{l_k}{f_{2k}} + \frac{l_k}{f_{4k}} + \frac{l_{3k}}{f_{4k}} - \Delta\beta^k(x). \tag{2.3}$$

*Proof.* With $m \geq 2$ and $k$ even, it follows by identity (e) that

$$\begin{aligned} l_{k(2^m-1)} + l_k &= l_{[k\cdot 2^{m-1}+k(2^{m-1}-1)]} + l_{[k\cdot 2^{m-1}-k(2^{m-1}-1)]} \\ &= l_{k(2^{m-1}-1)} l_{k\cdot 2^{m-1}}. \end{aligned}$$

With this result, identity (a), and recursion [4], we will now establish that

$$\sum_{n=0}^{m} \frac{l_k}{f_{k\cdot 2^n}} = \frac{l_k}{f_k} + \frac{l_k}{f_{2k}} + \frac{l_k}{f_{4k}} + \frac{l_{3k}}{f_{4k}} - \frac{l_{k(2^m-1)}}{f_{k\cdot 2^m}}, \tag{2.4}$$

where $m \geq 2$. Again, we let $A_m$ be the LHS of this equation and $B_m$ its RHS. Then,

$$\begin{aligned} B_m - B_{m-1} &= \frac{l_{k(2^{m-1}-1)}}{f_{k\cdot 2^{m-1}}} - \frac{l_{k(2^m-1)}}{f_{k\cdot 2^m}} \\ &= \frac{l_{k(2^{m-1}-1)} l_{k\cdot 2^{m-1}}}{f_{k\cdot 2^m}} - \frac{l_{k(2^m-1)}}{f_{k\cdot 2^m}} \\ &= \frac{l_{k(2^m-1)} + l_k}{f_{k\cdot 2^m}} - \frac{l_{k(2^m-1)}}{f_{k\cdot 2^m}} \\ &= \frac{l_k}{f_{k\cdot 2^m}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \left(\frac{l_k}{f_k} + \frac{l_k}{f_{2k}} + \frac{l_k}{f_{4k}}\right) - \left[\left(\frac{l_k}{f_k} + \frac{l_k}{f_{2k}} + \frac{l_k}{f_{4k}} + \frac{l_{3k}}{f_{4k}}\right) - \frac{l_{3k}}{f_{4k}}\right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, confirming formula (2.4).

Because $\lim_{m\to\infty} \frac{l_m}{f_{m+k}} = \frac{\Delta}{\alpha^k(x)} = \Delta\beta^k(x)$, equation (2.4) yields the given result, as desired. □

It follows from Lemma 2.2 that

$$\sum_{n=0}^{\infty} \frac{l_2}{f_{2^{n+1}}} = \frac{l_2}{f_2} + \frac{l_2}{f_4} + \frac{l_2}{f_8} + \frac{l_6}{f_8} - \Delta\beta^2(x); \quad \sum_{n=0}^{\infty} \frac{l_4}{f_{2^{n+2}}} = \frac{l_4}{f_4} + \frac{l_4}{f_8} + \frac{l_4}{f_{16}} + \frac{l_{12}}{f_{16}} - \Delta\beta^4(x).$$

They yield

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^{n+1}}} = \frac{5}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2^{n+2}}} = \frac{3}{2} - \frac{\sqrt{5}}{2},$$

respectively, as obtained earlier.

With Lemma 2.1 at our disposal, we now establish the counterpart of Theorem 1 in [5] for $k$ with even parity.

**Theorem 2.3.** *Let $k$ be an even positive integer and $\Delta = \sqrt{x^2+4}$. Then,*

$$\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} = \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-2}}{f_{4k}l_2} - \frac{\Delta\beta^2(x)}{l_2}\right]. \tag{2.5}$$

*Proof.* With property (c), Lemma 2.1, and $m \geq 3$, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} &= \frac{f_k}{l_{2k}-2} + \sum_{n=2}^{m} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} \\
&= \frac{f_k}{l_{2k}-2} + \sum_{n=2}^{m} \frac{f_{k\cdot 2^{n-1}}}{\Delta^2 f^2_{k\cdot 2^{n-1}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\sum_{n=2}^{m} \frac{1}{f_{k\cdot 2^{n-1}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\sum_{n=1}^{m-1} \frac{1}{f_{k\cdot 2^{n}}}; \\
\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} &= \frac{f_k}{l_{2k}-2} - \frac{1}{\Delta^2 f_k} + \frac{1}{\Delta^2}\sum_{n=0}^{\infty} \frac{1}{f_{k\cdot 2^{n}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{4k-2}}{f_{4k}l_2} - \frac{\Delta\beta^2(x)}{l_2}\right],
\end{aligned}$$

as desired. □

It follows from equation (2.5) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{f_{2^n}}{l_{2^{n+1}}-2} &= \frac{f_2}{l_4-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_4} + \frac{1}{f_8} + \frac{l_6}{f_8 l_2} - \frac{\Delta\beta^2(x)}{l_2}\right]; \\
\sum_{n=1}^{\infty} \frac{f_{2^{n+1}}}{l_{2^{n+2}}-2} &= \frac{f_4}{l_8-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_8} + \frac{1}{f_{16}} + \frac{l_{14}}{f_{16} l_2} - \frac{\Delta\beta^2(x)}{l_2}\right].
\end{aligned}$$

They yield

$$\sum_{n=1}^{\infty} \frac{F_{2^n}}{L_{2^{n+1}}-2} = \frac{1}{2} - \frac{\sqrt{5}}{10}; \qquad \frac{F_{2^{n+1}}}{L_{2^{n+2}}-2} = \frac{3}{10} - \frac{\sqrt{5}}{10},$$

respectively.

*An Alternate Version.* Using Lemma 2.2, we can rewrite equation (2.5) in an alternate way:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2} &= \frac{f_k}{l_{2k}-2} - \frac{1}{\Delta^2}\sum_{n=0}^{\infty} \frac{1}{f_{k\cdot 2^{n}}} \\
&= \frac{f_k}{l_{2k}-2} + \frac{1}{\Delta^2}\left[\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{f_{3k}}{f_{4k}l_k} - \frac{\beta^k(x)}{\Delta l_k}\right]. \qquad (2.6)
\end{aligned}$$

This implies

$$\sum_{n=1}^{\infty} \frac{F_{k\cdot 2^{n-1}}}{L_{k\cdot 2^n}-2} = \frac{F_k}{L_{2k}-2} + \frac{1}{5}\left(\frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{L_{3k}}{F_{4k}L_k} - \frac{\sqrt{5}\beta^k}{L_k}\right).$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{F_{2^n}}{L_{2^{n+1}}-2} \;=\; \frac{1}{2}-\frac{\sqrt{5}}{10}; \qquad \frac{F_{2^{n+1}}}{L_{2^{n+2}}-2} \;=\; \frac{3}{10}-\frac{\sqrt{5}}{10},$$

as we found before.

Finally, we explore the counterpart of Theorem 2 in [5] with $k$ even.

**Theorem 2.4.** *Let $k$ be an even positive integer and $\Delta = \sqrt{x^2+4}$. Then,*

$$\sum_{n=1}^{\infty} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}+1} = \frac{f_k - f_{2k}}{l_{2k}+1} + \frac{1}{\Delta}. \tag{2.7}$$

*Proof.* With $m \geq 1$ and recursion [4], we will first confirm that

$$\sum_{n=1}^{m} \frac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}+1} = \frac{f_k - f_{2k}}{l_{2k}+1} + \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1}. \tag{2.8}$$

As before, let $A_m$ = LHS and $B_m$ = RHS. Using properties (a) and (b), we get

$$\begin{aligned}
B_m - B_{m-1} &= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^{m-1}}}{l_{k\cdot 2^{m-1}}+1}\\
&= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^{m-1}}\left(l_{k\cdot 2^{m-1}}-1\right)}{\left(l_{k\cdot 2^{m-1}}+1\right)\left(l_{k\cdot 2^{m-1}}-1\right)}\\
&= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^m} - f_{k\cdot 2^{m-1}}}{l^2_{k\cdot 2^{m-1}}-1}\\
&= \frac{f_{k\cdot 2^m}}{l_{k\cdot 2^m}+1} - \frac{f_{k\cdot 2^m} - f_{k\cdot 2^{m-1}}}{l_{k\cdot 2^m}+1}\\
&= \frac{f_{k\cdot 2^{m-1}}}{l_{k\cdot 2^m}+1}\\
&= A_m - A_{m-1}.
\end{aligned}$$

Recursively, this yields

$$\begin{aligned}
A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1\\
&= \frac{f_k}{l_{2k}+1} - \left(\frac{f_k - f_{2k}}{l_{2k}+1} + \frac{f_{2k}}{l_{2k}+1}\right)\\
&= 0.
\end{aligned}$$

Thus, $A_m = B_m$, as predicted.

Because $\lim_{n\to\infty} \frac{f_n}{l_n} = \Delta$, equation (2.7) follows from equation (2.8), as desired. □

In particular, Theorem 2.4 yields

$$\sum_{n=1}^{\infty} \frac{f_{2^n}}{l_{2^{n+1}}+1} \;=\; \frac{f_2 - f_4}{l_4+1} + \frac{1}{\Delta}; \qquad \sum_{n=1}^{\infty} \frac{f_{2^{n+1}}}{l_{2^{n+2}}+1} \;=\; \frac{f_4 - f_8}{l_8+1} + \frac{1}{\Delta}.$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{F_{2^n}}{L_{2^{n+1}}+1} \;=\; -\frac{1}{4}+\frac{\sqrt{5}}{5}; \qquad \sum_{n=1}^{\infty} \frac{F_{2^{n+1}}}{L_{2^{n+2}}+1} \;=\; -\frac{3}{8}+\frac{\sqrt{5}}{5},$$

respectively.

Next, we explore the Pell implications of formulas (2.1), (2.3), (2.5), (2.6), and (2.7).

## 3. Pell Versions

Because $b_n(x) = g_n(2x)$, it follows from equations (2.1), (2.3), (2.5), (2.6), and (2.7) that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q_2}{p_{k\cdot 2^n}} &= \frac{q_2}{p_k} + \frac{q_2}{p_{2k}} + \frac{q_2}{p_{4k}} + \frac{q_{4k-2}}{p_{4k}} - 2E\delta^2(x);\\
\sum_{n=0}^{\infty} \frac{q_k}{p_{k\cdot 2^n}} &= \frac{q_k}{p_k} + \frac{q_k}{p_{2k}} + \frac{q_k}{p_{4k}} + \frac{q_{3k}}{p_{4k}} - 2E\delta^k(x);\\
\sum_{n=1}^{\infty} \frac{p_{k\cdot 2^{n-1}}}{q_{k\cdot 2^n} - 2} &= \frac{p_k}{q_{2k} - 2} + \frac{1}{4E^2}\left(\frac{1}{p_{2k}} + \frac{1}{p_{4k}} + \frac{q_{4k-2}}{p_{4k}q_2}\right) - \frac{\delta^2(x)}{2Eq_2};\\
\sum_{n=0}^{\infty} \frac{p_{k\cdot 2^{n-1}}}{q_{k\cdot 2^n} - 2} &= \frac{p_k}{q_{2k} - 2} + \frac{1}{4E^2}\left(\frac{1}{p_{2k}} + \frac{1}{p_{4k}} + \frac{q_{3k}}{p_{4k}q_k}\right) - \frac{\delta^k(x)}{2Eq_k};\\
\sum_{n=1}^{\infty} \frac{p_{k\cdot 2^{n-1}}}{q_{k\cdot 2^n} + 1} &= \frac{p_k - p_{2k}}{q_{2k} + 1} + \frac{1}{2E},
\end{aligned}$$

respectively, where $k$ is an even positive integer. Consequently, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{P_{k\cdot 2^n}} &= \frac{1}{P_k} + \frac{1}{P_{2k}} + \frac{1}{P_{4k}} + \frac{Q_{4k-2}}{3P_{4k}} - \frac{\sqrt{2}\delta^2}{3};\\
\sum_{n=0}^{\infty} \frac{Q_k}{P_{k\cdot 2^n}} &= \frac{Q_k}{P_k} + \frac{Q_k}{P_{2k}} + \frac{Q_k}{P_{4k}} + \frac{Q_{3k}}{P_{4k}} - \sqrt{2}\delta^k;\\
\sum_{n=1}^{\infty} \frac{P_{k\cdot 2^{n-1}}}{Q_{k\cdot 2^n} - 1} &= \frac{P_k}{Q_{2k} - 1} + \frac{1}{4}\left(\frac{1}{P_{2k}} + \frac{1}{P_{4k}} + \frac{Q_{4k-2}}{3P_{4k}} - \frac{\sqrt{2}\delta^2}{6}\right);\\
\sum_{n=1}^{\infty} \frac{P_{k\cdot 2^{n-1}}}{Q_{k\cdot 2^n} - 1} &= \frac{P_k}{Q_{2k} - 1} + \frac{1}{4}\left(\frac{1}{P_{2k}} + \frac{1}{P_{4k}} + \frac{Q_{3k}}{P_{4k}Q_k} - \frac{\sqrt{2}\delta^k}{2Q_k}\right);\\
\sum_{n=1}^{\infty} \frac{P_{k\cdot 2^{n-1}}}{2Q_{k\cdot 2^n} + 1} &= \frac{P_k - P_{2k}}{2Q_{2k} + 1} + \frac{\sqrt{2}}{4},
\end{aligned}$$

respectively, where $k$ is an even positive integer.

## 4. Jacobsthal Consequences

Using the gibonacci-Jacobsthal relationships, we now explore the Jacobsthal implications of formulas (2.1), (2.5), and (2.7). In each case, for clarity and convenience, we let $A$ denote the fractional expression on the left side and $B$ the corresponding expression on the right side, and LHS and RHS those of the Jacobsthal formula to be found. In the interest of brevity, we omit the Jacobsthal implications of formulas (2.3) and (2.6).

### 4.1. **Jacobsthal Version of Formula** (2.1).

*Proof.* We have $A = \dfrac{l_2}{f_{k\cdot 2^n}}$ and $B = \dfrac{l_2}{f_k} + \dfrac{l_2}{f_{2k}} + \dfrac{l_2}{f_{4k}} + \dfrac{l_{4k-2}}{f_{4k}} - \Delta\beta^2(x)$.

Replacing $x$ with $1/\sqrt{x}$ in $A$, and then multiplying the numerator and denominator with $x^{(k\cdot 2^n-1)/2}$, we get

$$\begin{aligned} A &= \frac{x^{(k\cdot 2^n-3)/2}(x^{2/2}l_2)}{x^{(k\cdot 2^n-1)/2}f_{k\cdot 2^n}} \\ &= \frac{x^{(k\cdot 2^n-3)/2}j_2}{J_{k\cdot 2^n}}; \\ \text{LHS} &= \sum_{n=0}^{\infty}\frac{x^{(k\cdot 2^n-3)/2}j_2}{J_{k\cdot 2^n}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we replace $x$ with $1/\sqrt{x}$ in $B$, and then multiply the numerator and denominator with $x^{(4k-1)/2}$. This gives

$$\begin{aligned} B &= \frac{l_2}{f_k}+\frac{l_2}{f_{2k}}+\frac{l_2}{f_{4k}}+\frac{l_{4k-2}}{f_{4k}}-\frac{D(1-D)^2\sqrt{x}}{4x^2} \\ &= \frac{x^{(k-3)/2}(x^{2/2}l_2)}{x^{(k-1)/2}f_k}+\frac{x^{2k-3)/2}(x^{2/2}l_2)}{x^{(2k-1)/2}f_{2k}}+\frac{x^{(4k-3)/2}(x^{2/2}l_2)}{x^{(4k-1)/2}f_{4k}} \\ &\quad + \frac{\sqrt{x}\left[x^{(4k-2)/2}l_{4k-2}\right]}{x^{(4k-1)/2}f_{4k}}-\frac{D(1-D)^2\sqrt{x}}{4x^2}; \\ \text{RHS} &= \frac{x^{(k-3)/2}j_2}{J_k}+\frac{x^{(2k-3)/2}j_2}{J_{2k}}+\frac{x^{(4k-3)/2}j_2)}{J_{4k}}+\frac{\sqrt{x}J_{4k-1}}{J_{4k}}-\frac{D(1-D)^2\sqrt{x}}{4x^2}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides gives the desired Jacobsthal version:

$$\sum_{n=0}^{\infty}\frac{x^{k\cdot 2^{n-1}-1}j_2}{J_{k\cdot 2^n}} = \frac{x^{(k-2)/2}j_2}{J_k}+\frac{x^{k-1}j_2}{J_{2k}}+\frac{x^{2k-1}j_2}{J_{4k}}+\frac{xj_{4k-1}}{J_{4k}}-\frac{D(1-D)^2}{4x}.$$

□

This implies

$$\sum_{n=0}^{\infty}\frac{L_2}{F_{k\cdot 2^n}} = \frac{L_2}{F_k}+\frac{L_2}{F_{2k}}+\frac{L_2}{F_{4k}}+\frac{L_{4k-2}}{F_{4k}}-\sqrt{5}\beta^2.$$

It then follows that

$$\sum_{n=0}^{\infty}\frac{1}{F_{2^{n+1}}} = \frac{5}{2}-\frac{\sqrt{5}}{2}; \qquad \sum_{n=0}^{\infty}\frac{1}{F_{2^{n+2}}} = \frac{3}{2}-\frac{\sqrt{5}}{2},$$

as obtained earlier.

4.2. **Jacobsthal Version of Formula** (2.5)**.**

*Proof.* We have $A = \dfrac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n}-2}$ and $B = \dfrac{f_k}{l_{2k}-2}+\dfrac{1}{\Delta^2}\left[\dfrac{1}{f_{2k}}+\dfrac{1}{f_{4k}}+\dfrac{f_{4k-2}}{f_{4k}l_2}-\dfrac{\Delta\beta^2(x)}{l_2}\right]$.

Replacing $x$ with $1/\sqrt{x}$ in $A$, and then multiplying the numerator and denominator with $x^{(k\cdot 2^n)/2}$, we get

$$\begin{aligned} A &= \frac{\sqrt{x}\left[x^{(k\cdot 2^{n-1}-1)/2} f_{k\cdot 2^{n-1}}\right]}{\left[x^{(k\cdot 2^n)/2} l_{k\cdot 2^n}\right] - 2x^{k\cdot 2^{n-1}}} \\ &= \frac{\sqrt{x} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{\sqrt{x} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, replace $x$ with $1/\sqrt{x}$ in $B$, and then multiply the numerator and denominator with $x^{(4k+1)/2}$. This gives

$$\begin{aligned} B &= \frac{f_k}{l_{2k}-2} + \frac{x}{D^2}\left[\frac{1}{f_{2k}} + \frac{1}{f_{4k}} + \frac{l_{4k-2}}{f_{4k} l_2} - \frac{D(1-D)^2}{4x\sqrt{x} l_2}\right] \\ &= \frac{x^{(k+1)/2}\left[x^{(k-1)/2} f_k\right]}{x^{2k/2} l_{2k} - 2x^k} + \frac{x}{D^2}\left[\frac{x^{(2k-1)/2}}{x^{(2k-1)/2} f_{2k}} + \frac{x^{(4k-1)/2}}{x^{(4k-1)/2} f_{4k}}\right] \\ &\quad + \frac{x}{D^2}\left[\frac{x\sqrt{x}\left[x^{(4k-2)/2} l_{(4k-2)/2}\right]}{x^{(4k-1)/2} f_{4k}\left(x^{2/2} l_2\right)} - \frac{D(1-D)^2}{4\sqrt{x}\left(x^{2/2} l_2\right)}\right]; \\ \text{RHS} &= \frac{x^{(k+1)/2} J_k}{j_{2k} - 2x^k} + \frac{x}{D^2}\left[\frac{x^{(2k-1)/2}}{J_{2k}} + \frac{x^{(4k-1)/2}}{J_{4k}} + \frac{x\sqrt{x} j_{4k-2}}{J_{4k} j_2} + \frac{D(1-D)^2}{4\sqrt{x} j_2}\right], \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

By combining the two sides, we get the Jacobsthal version of the formula:

$$\sum_{n=1}^{\infty} \frac{J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} - 2x^{k\cdot 2^{n-1}}} = \frac{x^{k/2} J_k}{j_{2k} - 2x^k} + \frac{x}{D^2}\left[\frac{x^{k-1}}{J_{2k}} + \frac{x^{2k-1}}{J_{4k}} + \frac{x j_{4k-2}}{J_{4k} j_2} - \frac{D(1-D)^2}{4x j_2}\right].$$

□

This implies

$$\sum_{n=1}^{\infty} \frac{F_{k\cdot 2^{n-1}}}{L_{k\cdot 2^n} - 2} = \frac{F_k}{L_{2k} - 2} + \frac{1}{5}\left(\frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{L_{4k-2}}{3F_{4k}} - \frac{\sqrt{5}\beta^2}{3}\right).$$

In particular, we then have

$$\sum_{n=1}^{\infty} \frac{F_{2^n}}{L_{2^{n+1}} - 2} = \frac{1}{2} - \frac{\sqrt{5}}{10}; \qquad \frac{F_{2^{n+1}}}{L_{2^{n+2}} - 2} = \frac{3}{10} - \frac{\sqrt{5}}{10},$$

as obtained earlier.

4.3. **Jacobsthal Version of Formula** (2.7)**.**

*Proof.* We have $A = \dfrac{f_{k\cdot 2^{n-1}}}{l_{k\cdot 2^n} + 1}$ and $B = \dfrac{f_k - f_{2k}}{l_{2k} + 1} + \dfrac{1}{\Delta}$.

First, replace $x$ with $1/\sqrt{x}$ in $A$, and then multiply the numerator and denominator with $x^{k\cdot 2^{n-1}}$. We then get

$$\begin{aligned} A &= \frac{x^{(k\cdot 2^{n-1}+1)/2}\left[x^{(k\cdot 2^{n-1}-1)/2} f_{k\cdot 2^{n-1}}\right]}{\left[x^{(k\cdot 2^n)/2} l_{k\cdot 2^n}\right] + x^{k\cdot 2^{n-1}}} \\ &= \frac{x^{(k\cdot 2^{n-1}+1)/2} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{x^{(k\cdot 2^{n-1}+1)/2} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, replace $x$ with $1/\sqrt{x}$ in $B$, and then multiply the numerator and denominator with $x^k$. This yields

$$\begin{aligned} B &= \frac{f_k - f_{2k}}{l_{2k}+1} + \frac{\sqrt{x}}{D} \\ &= \frac{x^{(k+1)/2}\left[x^{(k-1)/2} f_k\right] - \sqrt{x}\left[x^{(2k-1)/2} f_{2k}\right]}{\left(x^{2k/2} l_{2k}\right) + x^k} + \frac{\sqrt{x}}{D}; \\ \text{RHS} &= \frac{x^{(k+1)/2} J_k - \sqrt{x} J_{2k}}{j_{2k} + x^k} + \frac{\sqrt{x}}{D}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

By equating the two sides, we get the Jacobsthal version of the formula:

$$\sum_{n=1}^{\infty} \frac{x^{k\cdot 2^{n-2}} J_{k\cdot 2^{n-1}}}{j_{k\cdot 2^n} + x^{k\cdot 2^{n-1}}} = \frac{x^{k/2} J_k - J_{2k}}{j_{2k} + x^k} + \frac{1}{D}.$$

□

In particular, this yields

$$\sum_{n=1}^{\infty} \frac{F_{k\cdot 2^{n-1}}}{L_{k\cdot 2^n} + 1} = \frac{F_k - F_{2k}}{L_{2k}+1} + \frac{\sqrt{5}}{5}.$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{F_{2^n}}{L_{2^{n+1}} + 1} = -\frac{1}{4} + \frac{\sqrt{5}}{5}; \qquad \sum_{n=1}^{\infty} \frac{F_{2^{n+1}}}{L_{2^{n+2}} + 1} = -\frac{3}{8} + \frac{\sqrt{5}}{5},$$

as found earlier.

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Department of Mathematics, Framingham State University, Framingham, MA 01701, USA
*Email address*: tkoshy@emeriti.framingham.edu