SUMS INVOLVING GIBONACCI POLYNOMIALS

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Abstract. We explore sums involving gibonacci polynomials, and deduce the Pell versions for two of them.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $E = \sqrt{x^2+1}$.

It follows by the Binet-like formulas that $\lim_{n\to\infty} \frac{g_{n+k}}{g_n} = \alpha^k(x)$, where $2\alpha(x) = x + \Delta$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [3]:

$$f_{n+k} - f_{n-k} = \begin{cases} f_n l_k, & \text{if } k \text{ is odd;} \\ f_k l_n, & \text{otherwise;} \end{cases} \tag{1}$$

$$l_{n+k} - l_{n-k} = \begin{cases} l_k l_n, & \text{if } k \text{ is odd;} \\ \Delta^2 f_k f_n, & \text{otherwise;} \end{cases} \tag{2}$$

$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n; \tag{3}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise.} \end{cases} \tag{4}$$

[*Note*: Identity (2) gives the correct version of Exercise 40 on page 57 in [3].]

Using the gibonacci recurrence and identity (4), we can establish that

$$g_{n+2}g_{n-3} - g_{n+1}g_{n-2} = \begin{cases} (-1)^n f_4, & \text{if } g_n = f_n; \\ (-1)^{n+1}\Delta^2 f_4, & \text{otherwise;} \end{cases} \tag{5}$$

$$g_{n+2}g_{n-2} - g_{n+1}g_{n-1} = \begin{cases} (-1)^{n+1} f_3, & \text{if } g_n = f_n; \\ (-1)^n \Delta^2 f_3, & \text{otherwise.} \end{cases} \tag{6}$$

Identities (5) and (6) with $g_n = f_n$ and $x = 1$ appear in [2, 10].

It also follows from identity (4) that [3, 4]

$$f_{n+2}f_{n+1}f_{n-1}f_{n-2} = f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2. \tag{7}$$

This is the polynomial version of the *Gelin-Cesàro identity* [4, 8]

$$F_{n+2}F_{n+1}F_{n-1}F_{n-2} = F_n^4 - 1.$$

The Lucas counterpart of identity (7) is [3, 4]

$$l_{n+2}l_{n+1}l_{n-1}l_{n-2} = l_n^4 + (-1)^n(x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2. \tag{8}$$

This implies

$$L_{n+2}L_{n+1}L_{n-1}L_{n-2} = L_n^4 - 25.$$

These properties play a pivotal role in our discourse.

2. Gibonacci Polynomial Sums

With the above background, we begin our explorations with three lemmas.

Lemma 1. *Let $g_n = f_n$ or l_n, and k be a positive integer. Then,*

$$\sum_{\substack{n=k+1\\k\geq 1}}^{\infty}\left(\frac{1}{g_n g_{n-k}} - \frac{1}{g_{n+k}g_n}\right) = \sum_{r=1}^{k}\frac{1}{g_{k+r}g_r}. \tag{9}$$

Proof. Using recursion [3], we will first establish that

$$\sum_{\substack{n=k+1\\k\geq 1}}^{m}\left(\frac{1}{g_n g_{n-k}} - \frac{1}{g_{n+k}g_n}\right) = \sum_{r=1}^{k}\frac{1}{g_{k+r}g_r} - \sum_{r=1}^{k}\frac{1}{g_{m+r}g_{m+r-k}}.$$

To this end, we let A_m = LHS and B_m = RHS. Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=1}^{k}\left(\frac{1}{g_{m-1+r}g_{m-1+r-k}} - \frac{1}{g_{m+r}g_{m+r-k}}\right)\\ &= \frac{1}{g_m g_{m-k}} - \frac{1}{g_{m+k}g_m}\\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{k+1} - B_{k+1}\\ &= \left(\frac{1}{g_{k+1}g_1} - \frac{1}{g_{2k+1}g_{k+1}}\right) - \left(\sum_{r=1}^{k}\frac{1}{g_{k+r}g_r} - \sum_{r=1}^{k}\frac{1}{g_{k+r+1}g_{r+1}}\right)\\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, as desired.

Because $\lim_{m\to\infty}\frac{1}{g_m} = 0$, the given result now follows. □

Consequences: Lemma 1 has interesting consequences, depending on the value of g_n and the parity of k. First, notice that

$$\begin{aligned}\sum_{\substack{n=k+1\\k\geq 1}}^{\infty} \frac{g_{n+k}-g_{n-k}}{g_{n+k}g_n g_{n-k}} &= \sum_{\substack{n=k+1\\k\geq 1}}^{\infty} \left(\frac{1}{g_n g_{n-k}} - \frac{1}{g_{n+k}g_n}\right)\\ &= \sum_{r=1}^{k} \frac{1}{g_{k+r}g_r}. \qquad (10)\end{aligned}$$

<u>Case 1</u>. Suppose $g_n = f_n$. If k is odd, then by equation (1), this yields

$$\sum_{\substack{n=k+1\\k\geq 1,\,\text{odd}}}^{\infty} \frac{l_k}{f_{n+k}f_{n-k}} = \sum_{r=1}^{k} \frac{1}{f_{k+r}f_r}. \qquad (11)$$

Consequently,

$$\sum_{n=3}^{\infty} \frac{1}{f_{n+1}f_{n-1}} = \frac{1}{f_2^2} - \frac{1}{f_3}.$$

On the other hand, if k is even, we get

$$\sum_{\substack{n=k+1\\k\geq 2,\,\text{even}}}^{\infty} \frac{f_k l_n}{f_{n+k}f_n f_{n-k}} = \sum_{r=1}^{k} \frac{1}{f_{k+r}f_r}. \qquad (12)$$

<u>Case 2</u>. Suppose $g_n = l_n$. If k is odd, then by equations (2) and (10), we get

$$\sum_{\substack{n=k+1\\k\geq 1,\,\text{odd}}}^{\infty} \frac{l_k}{l_{n+k}l_{n-k}} = \sum_{r=1}^{k} \frac{1}{l_{k+r}l_r}; \qquad (13)$$

otherwise, we get

$$\sum_{\substack{n=k+1\\k\geq 2,\,\text{even}}}^{\infty} \frac{\Delta^2 f_k f_n}{l_{n+k}l_n l_{n-k}} = \sum_{r=1}^{k} \frac{1}{l_{k+r}l_r}. \qquad (14)$$

With identity (4), equations (11) and (13) yield

$$\begin{aligned}\sum_{\substack{n=k+1\\k\geq 1,\,\text{odd}}}^{\infty} \frac{l_k}{f_n^2 + (-1)^n f_k^2} &= \sum_{r=1}^{k} \frac{1}{f_{k+r}f_r}; \qquad (15)\\ \sum_{\substack{n=k+1\\k\geq 1,\,\text{odd}}}^{\infty} \frac{l_k}{l_n^2 - (-1)^n \Delta^2 f_k^2} &= \sum_{r=1}^{k} \frac{1}{l_{k+r}l_r},\end{aligned}$$

respectively.

Consequently, we have

$$\sum_{n=2}^{\infty} \frac{1}{F_n^2 + (-1)^n} = 1; \qquad \sum_{n=2}^{\infty} \frac{1}{L_n^2 - 5(-1)^n} = \frac{1}{3}.$$

With the sum $\sum_{n=0}^{\infty} \frac{x}{f_{2n}^2+1} = \alpha(x)$ [4], formula (15) yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{x}{f_{2n}^2+1} + \sum_{n=2}^{\infty} \frac{x}{f_{2n-1}^2-1} &= \sum_{n=2}^{\infty} \frac{x}{f_n^2+(-1)^n}\\ &= \frac{1}{x};\\ \sum_{n=2}^{\infty} \frac{x}{f_{2n-1}^2-1} &= \frac{1}{x} - [\alpha(x)-x]\\ &= \frac{x^2 - x\alpha(x)+1}{x}.\end{aligned}$$

This implies

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n-1}^2-1} = \frac{3-\sqrt{5}}{2}.$$

It follows from equations (12) and (14) that

$$\sum_{n=3}^{\infty} \frac{L_n}{F_{n+2}F_nF_{n-2}} = \frac{5}{6}; \qquad \sum_{n=3}^{\infty} \frac{F_n}{L_{n+2}L_nL_{n-2}} = \frac{5}{84},$$

respectively.

Using identity (4), we can rewrite equations (12) and (14) in a different way:

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{xl_n}{f_n^3-(-1)^n x^2 f_n} &= \frac{1}{f_3f_1} + \frac{1}{f_4f_2};\\ \sum_{n=3}^{\infty} \frac{xf_n}{l_n^3+(-1)^n \Delta^2 x^2 l_n} &= \frac{1}{\Delta^2}\left(\frac{1}{l_3 1_1} + \frac{1}{l_4l_2}\right),\end{aligned}$$

respectively.

Consequently, we have

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{xl_n}{f_{n+2}f_nf_{n-2}} &= \frac{1}{f_3f_1} + \frac{1}{f_4f_2}; & \sum_{n=3}^{\infty} \frac{L_n}{F_n^3-(-1)^nF_n} &= \frac{5}{6};\\ \sum_{n=3}^{\infty} \frac{xf_n}{l_{n+2}l_nl_{n-2}} &= \frac{1}{\Delta^2}\left(\frac{1}{l_3l_1} + \frac{1}{l_4l_2}\right); & \sum_{n=3}^{\infty} \frac{F_n}{L_n^3+5(-1)^nL_n} &= \frac{5}{84}.\end{aligned}$$

The next lemma explores an application of identity (5).

Lemma 2. *Let $g_n = f_n$ or l_n. Then,*

$$\sum_{n=3}^{\infty} (-1)^n \left(\frac{g_{n-3}}{g_{n-2}} - \frac{g_{n+1}}{g_{n+2}}\right) = -\frac{g_0}{g_1} + \frac{g_1}{g_2} - \frac{g_2}{g_3} + \frac{g_3}{g_4}. \tag{16}$$

Proof. Let $R = \text{RHS}$. We will first establish that

$$\sum_{n=3}^{m} (-1)^n \left(\frac{g_{n-3}}{g_{n-2}} - \frac{g_{n+1}}{g_{n+2}}\right) = R + (-1)^m \left(\frac{g_{m-2}}{g_{m-1}} - \frac{g_{m-1}}{g_m} + \frac{g_m}{g_{m+1}} - \frac{g_{m+1}}{g_{m+2}}\right). \tag{17}$$

Clearly, the LHS is a telescoping sum. So, when m is odd, we get LHS $= R - S_m$, where

$$S_m = \frac{g_{m-2}}{g_{m-1}} - \frac{g_{m-1}}{g_m} + \frac{g_m}{g_{m+1}} - \frac{g_{m+1}}{g_{m+2}};$$

otherwise, we get RHS $= R + S_m$.

Combining the two cases, we get formula (17), as expected.

Because $\lim_{n\to\infty} \frac{g_n}{g_{n+1}} = \frac{1}{\alpha(x)}$, it follows that $\lim_{n\to\infty} S_m = 0$. Consequently, formula (17) yields the given result, as desired. □

In particular, we have

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{F_{n-3}}{F_{n-2}} - \frac{F_{n+1}}{F_{n+2}}\right) = \frac{7}{6}; \qquad \sum_{n=3}^{\infty}(-1)^n\left(\frac{L_{n-3}}{L_{n-2}} - \frac{L_{n+1}}{L_{n+2}}\right) = -\frac{155}{84}.$$

This lemma yields has a delightful byproduct.

Lemma 3. *Let $g_n = f_n$ or l_n, and $S_{g_3} = \frac{g_0}{g_1} - \frac{g_1}{g_2} + \frac{g_2}{g_3} - \frac{g_3}{g_4}$. Then,*

$$\sum_{n=3}^{\infty} \frac{1}{g_{n+2}g_{n-2}} = \begin{cases} -\frac{1}{f_4}S_{f_3}, & \text{if } g_n = f_n; \\ \frac{1}{\Delta^2 f_4}S_{l_3}, & \text{otherwise.} \end{cases} \tag{18}$$

Proof. We will establish this using identity (5) and Lemma 2.

Case 1. Let $g_n = f_n$. Then,

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{f_{n+2}f_{n-2}} &= \frac{1}{f_4}\sum_{n=3}^{\infty}(-1)^n \frac{f_{n+2}f_{n-3} - f_{n+1}f_{n-2}}{f_{n+2}f_{n-2}} \\ &= \frac{1}{f_4}\sum_{n=3}^{\infty}(-1)^n\left(\frac{f_{n-3}}{f_{n-2}} - \frac{f_{n+1}}{f_{n+2}}\right) \\ &= -\frac{1}{f_4}\left(\frac{f_0}{f_1} - \frac{f_1}{f_2} + \frac{f_2}{f_3} - \frac{f_3}{f_4}\right) \\ &= -\frac{1}{f_4}S_{f_3}. \end{aligned}$$

Case 2. Let $g_n = l_n$. Then,

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{l_{n+2}l_{n-2}} &= -\frac{1}{\Delta^2 f_4}\sum_{n=3}^{\infty}(-1)^n \frac{l_{n+2}l_{n-3} - l_{n+1}l_{n-2}}{l_{n+2}l_{n-2}} \\ &= -\frac{1}{\Delta^2 f_4}\sum_{n=3}^{\infty}(-1)^n\left(\frac{l_{n-3}}{l_{n-2}} - \frac{l_{n+1}}{l_{n+2}}\right) \\ &= \frac{1}{\Delta^2 f_4}\left(\frac{l_0}{l_1} - \frac{l_1}{l_2} + \frac{l_2}{l_3} - \frac{l_3}{l_4}\right) \\ &= \frac{1}{\Delta^2 f_4}S_{l_3}. \end{aligned}$$

Combining the two cases, we get the desired result. □

It follows from equation (18) that

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{f_n^2 - (-1)^n x^2} &= -\frac{1}{f_4} S_{f_3}; & \sum_{n=3}^{\infty} \frac{1}{F_n^2 - (-1)^n} &= \frac{7}{18}. \\
\sum_{n=2}^{\infty} \frac{1}{F_{2n}^2 - 1} + \sum_{n=2}^{\infty} \frac{1}{F_{2n-1}^2 + 1} &= \frac{7}{18}; & \sum_{n=3}^{\infty} \frac{1}{l_n^2 + (-1)^n \Delta^2 x^2} &= \frac{1}{\Delta^2 f_4} S_{l_3} \\
\sum_{n=3}^{\infty} \frac{1}{L_n^2 + 5(-1)^n} &= \frac{31}{252}; & \sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 + 5} + \sum_{n=2}^{\infty} \frac{1}{L_{2n-1}^2 - 5} &= \frac{31}{252}.
\end{aligned}$$

The lemmas, coupled with identities (6) through (8), yield the next result.

Theorem 1.

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{f_n^4 - (-1)^n (x^2 - 1) f_n^2 - x^2} = -\frac{1}{f_3 f_4^2}. \tag{19}$$

Proof. With identities (6) through (8) and the lemmas, we get

$$\begin{aligned}
\text{LHS} &= \sum_{n=3}^{\infty} \frac{(-1)^n}{f_{n+2} f_{n+1} f_{n-1} f_{n-2}} \\
&= -\frac{1}{f_3} \sum_{n=3}^{\infty} \frac{f_{n+2} f_{n-2} - f_{n+1} f_{n-1}}{f_{n+2} f_{n+1} f_{n-1} f_{n-2}} \\
&= -\frac{1}{f_3} \sum_{n=3}^{\infty} \left(\frac{1}{f_{n+1} f_{n-1}} - \frac{1}{f_{n+2} f_{n-2}} \right) \\
&= -\frac{1}{f_3} \left[\left(\frac{1}{f_2^2} - \frac{1}{f_3} \right) + \frac{1}{f_4} \left(\frac{f_0}{f_1} - \frac{f_1}{f_2} + \frac{f_2}{f_3} - \frac{f_3}{f_4} \right) \right] \\
&= -\frac{1}{f_3 f_4^2},
\end{aligned}$$

as desired. □

Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [3], we can rewrite equation (17) in a different way:

$$\sum_{n=3}^{\infty} \frac{(-1)^n \Delta^4}{l_n^4 - (-1)^n [(x^2 - 1)\Delta^2 + 8] l_n^2 - (x^4 + 4)\Delta^2 + 16} = -\frac{1}{f_3 f_4^2}. \tag{20}$$

It follows from equations (19) and (20) that [2, 5]

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{F_n^4 - 1} = -\frac{1}{18}; \tag{21}$$

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{L_n^4 - 8(-1)^n L_n^2 - 9} = -\frac{1}{450}, \tag{22}$$

respectively.

Equation (21), coupled with the equation [4, 6, 8]

$$\sum_{n=3}^{\infty} \frac{1}{F_n^4 - 1} = \frac{35}{18} - \frac{5\sqrt{5}}{6},$$

yields

$$-\sum_{n=2}^{\infty}\frac{1}{F_{2n-1}^4-1}+\sum_{n=2}^{\infty}\frac{1}{F_{2n}^4-1} = -\frac{1}{18};$$
$$\sum_{n=2}^{\infty}\frac{1}{F_{2n-1}^4-1}+\sum_{n=2}^{\infty}\frac{1}{F_{2n}^4-1} = \frac{35}{18}-\frac{5\sqrt{5}}{6},$$

respectively. It follows from these two equations that

$$\sum_{n=2}^{\infty}\frac{1}{F_{2n}^4-1} = \frac{17}{18}-\frac{5\sqrt{5}}{12}; \qquad \sum_{n=2}^{\infty}\frac{1}{F_{2n-1}^4-1} = 1-\frac{5\sqrt{5}}{12}.$$

Equation (22) implies

$$\sum_{n=2}^{\infty}\frac{1}{L_{2n-1}^4+8L_{2n-1}^2-9}-\sum_{n=2}^{\infty}\frac{1}{L_{2n}^4-8L_{2n}^2-9}=\frac{1}{450}.$$

2.1. **Lucas Versions.** We now explore the Lucas version of Theorem 1 and its consequences.

Theorem 2.

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{l_n^4+(-1)^n(x^2-1)\Delta^2 l_n^2-\Delta^4x^2}=-\frac{1}{f_4l_4l_3l_2}. \tag{23}$$

Proof. By equation (13) and Lemma 3, we have

$$\sum_{n=3}^{\infty}\frac{1}{l_{n+1}l_{n-1}} = \frac{1}{l_2l_1^2}-\frac{1}{l_3l_1};$$
$$\sum_{n=3}^{\infty}\frac{1}{l_{n+2}l_{n-2}} = \frac{1}{\Delta^2 f_4}\left(\frac{l_0}{l_1}-\frac{l_1}{l_2}+\frac{l_2}{l_3}-\frac{l_3}{l_4}\right),$$

respectively.

Using identities (6) and (8), we then get

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{(-1)^n}{l_n^4-(-1)^n(x^2-1)\Delta^2 l_n^2-\Delta^4x^2} &= \frac{1}{\Delta^2 f_3}\sum_{n=3}^{\infty}\frac{l_{n+2}l_{n-2}-l_{n+1}l_{n-1}}{l_{n+2}l_{n+1}l_{n-1}l_{n-2}}\\ &= \frac{1}{\Delta^2 f_3}\sum_{n=3}^{\infty}\left(\frac{1}{l_{n+1}l_{n-1}}-\frac{1}{l_{n+2}l_{n-2}}\right)\\ &= \frac{1}{\Delta^2 f_3}\left[\left(\frac{1}{l_2l_1^2}-\frac{1}{l_3l_1}\right)-\frac{1}{\Delta^2 f_4}\left(\frac{l_0}{l_1}-\frac{l_1}{l_2}+\frac{l_2}{l_3}-\frac{l_3}{l_4}\right)\right]\\ &= -\frac{1}{f_4l_4l_3l_2},\end{aligned}$$

as desired. □

In particular, we have

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{L_n^4-25} = -\frac{1}{252};$$
$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}^4-25}-\sum_{n=2}^{\infty}\frac{1}{L_{2n-1}^4-25} = \frac{7}{504}. \tag{24}$$

Equation (24), coupled with the result [4, 7, 9]

$$\sum_{n=3}^{\infty} \frac{1}{L_n^4 - 25} = \frac{5}{63} - \frac{\sqrt{5}}{30}$$

yields

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^4 - 25} = \frac{1}{18} - \frac{\sqrt{5}}{60}; \qquad \sum_{n=2}^{\infty} \frac{1}{L_{2n-1}^4 - 25} = \frac{1}{24} - \frac{\sqrt{5}}{60}.$$

With identity (3), we can rewrite equation (23) as

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\Delta^4 f_n^4 + (-1)^n \Delta^2[(x^2-1)\Delta^2 + 2] f_n^2 - \Delta^2(x^4 + 3x^2 + 1)} = -\frac{1}{f_4 l_4 l_3 l_2}.$$

This yields

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{5F_n^4 + 2(-1)^n F_n^2 - 5} = -\frac{5}{252}.$$

3. Pell Implications

The Pell versions of sums involving gibonacci polynomials can be obtained using the relationship $b_n(x) = g_n(2x)$. For example, those of equations (19) and (23) are:

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{p_n^4 - (-1)^n(4x^2-1)p_n^2 - 4x^2} = -\frac{1}{p_3 p_4^2};$$

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{q_n^4 + 4(-1)^n(4x^2-1)E^2 q_n^2 - 64x^2E^4} = -\frac{1}{p_4 q_4 q_3 q_2},$$

respectively. In the interest of brevity, we omit the others.

Acknowledgment

The author thanks the reviewer for the careful reading of the article and encouraging words.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] S. Edwards, *Solution to Problem B-1174*, The Fibonacci Quarterly, **54.3** (2016), 277–278.
[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[4] T. Koshy, *Infinite sums involving Gibonacci polynomial products*, The Fibonacci Quarterly, **59.3** (2021), 237–245.
[5] H. Ohtsuka, *Problem B-1174*, The Fibonacci Quarterly, **53.3** (2015), 273.
[6] H. Ohtsuka, *Problem H-783*, The Fibonacci Quarterly, **54.1** (2016), 87.
[7] À. Plaza, *Problem H-810*, The Fibonacci Quarterly, **55.3** (2017), 282.
[8] À. Plaza, *Solution to Problem H-783*, The Fibonacci Quarterly, **56.1** (2018), 90–91.
[9] À. Plaza, *Solution to Problem H-810*, The Fibonacci Quarterly, **57.3** (2019), 284.
[10] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Halstead Press, 1989.

MSC2020: Primary 11B37, 11B39, 11C08

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