FIBONACCI SUMMATION IDENTITIES ARISING FROM CATALAN'S IDENTITY

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Abstract. We show how Catalan's identity for the Fibonacci numbers can be leveraged to construct a large family (with some eccentric members) of summation identities involving the Fibonacci numbers. In the process, we provide a solution for a few problems that were posed in the problem sections of *The Fibonacci Quarterly*.

1. Introduction

Many problems for solution in *The Fibonacci Quarterly*, where it is asked to evaluate a sum, boil down to finding a way of rewriting the sum as a telescoping sum that, by its nature, is easily evaluated. These type of problems are nothing new. Today's versions just have more complex summands than those of yesteryear; the simpler summands already having been exhausted. To set the scene and showcase the principle, we start with a problem by Lucas from about 150 years ago and provide some historical comments on Catalan's identity. We then take Catalan's identity and leverage it to derive a theorem that allows one to generate a host of Fibonacci summation identities.

2. Question 494

Édouard Lucas (1842–1891), in the June 1879 issue of the Belgian mathematical educational journal *Nouvelle Correspondance Mathématique* (NCM), posed what might well be one of the first problems on Fibonacci numbers and telescoping sums [7]. The NCM was founded in 1874 by Eugène Catalan (1814–1894) and Paul Mansion (1844–1919), with Catalan being the editor-in-chief.

In "Question 494", Lucas asked to find the sum of the infinite series

$$\frac{1}{1^2} - \frac{1}{1^2+1^2} + \frac{1}{1^2+1^2+2^2} - \frac{1}{1^2+1^2+2^2+3^2} + \frac{1}{1^2+1^2+2^2+3^2+5^2} - \frac{1}{1^2+1^2+2^2+3^2+5^2+8^2} + \cdots . \tag{2.1}$$

The September 1880 issue of the NCM contained a solution by Mangon [8], a sub-lieutenant in the artillery corps of the Belgian army. Mangon noted that the sum of squares in consecutive denominators evaluates to 1, 1×2, 2×3, 3×5, and so on. He recognized that these are the product of the terms u_{n-1} and u_n in Lamé's sequence: 1, 2, 3, 5, 8, 13, ..., that obey the law of formation $u_n = u_{n-1} + u_{n-2}$, starting with $u_1 = 1$ and $u_2 = 2$. Integrating the difference equation, Mangon derives the closed-form expression

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \tag{2.2}$$

and also points out that

$$u_n u_{n-3} - u_{n-1}u_{n-2} = (-1)^{n-1}. \tag{2.3}$$

Thus, the sum of the first n terms in the infinite series is equal to

$$S_n = 1 - \frac{1}{1\times 2} + \frac{1}{2\times 3} - \frac{1}{3\times 5} + \frac{1}{5\times 8} - \cdots + (-1)^{n-1}\frac{1}{u_{n-1}u_n}. \tag{2.4}$$

He then writes the fractions as differences, using (2.3) for the general term, and obtains the telescoping sum

$$S_n = 1 - \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{3}{8}\right) + \cdots + \left(\frac{u_{n-3}}{u_{n-1}} - \frac{u_{n-2}}{u_n}\right). \tag{2.5}$$

Intermediate terms cancel, so that one is left with only the first and last term, and the sum simplifies to

$$S_n = 1 - \frac{u_{n-2}}{u_n} = \frac{u_{n-1}}{u_n}. \tag{2.6}$$

An application of (2.2) gives the limiting value and the answer to Lucas' question as $2/(1+\sqrt{5})$.[1]

In modern notation, recognizing that $u_n = F_{n+1}$, this can be condensed to

$$\sum_{i=1}^{n} \frac{(-1)^{i-1}}{F_1^2 + \cdots + F_i^2} = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{F_i F_{i+1}} = \sum_{i=1}^{n} \left(\frac{F_{i-2}}{F_i} - \frac{F_{i-1}}{F_{i+1}}\right) = \frac{F_{-1}}{F_1} - \frac{F_{n-1}}{F_{n+1}} = \frac{F_n}{F_{n+1}},$$

with the desired limit, as $F_n \sim \varphi^n/\sqrt{5}$, for large positive n, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

3. Armchair Comments

Enjoying a coffee in one's easy chair, having the benefit of today's knowledge of things Fibonacci, it is tempting to go through Mangon's proof and critique the finer points. However, I find his proof is well laid out, progresses methodologically, meets with success, and reflects the state of general Fibonacci knowledge at the time. I would say, *bien fait!* Around the time that Lucas posed "Question 494", the connection to Fibonacci, the Binet formula, and other formulae we now take for granted were not common knowledge. In 1876, Lucas [5] published his first article in the NCM on Lamé's sequence, as the Fibonacci sequence was known then. The notation u_i, to denote terms of the sequence, is derived from a textbook (1857) by Catalan [2, p. 86], used at the prestigious École Polytechnique. In 1877, Lucas [6] published a lengthy, two-part article (totalling 120 pages) on the works of Leonard de Pise, but that was in the Italian journal *Bullettino di Bibliography.* This would have only been known in select academic circles. In the article, Lucas gives some properties of the Fibonacci sequence, one being

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}, \tag{3.1}$$

that he subsequently used as a cloak in "Question 494".[2]

Catalan's identity is named after Eugène Catalan, whom we have already mentioned as cofounder of the NCM, and states that

$$F_n^2 - F_{n-d}F_{n+d} = (-1)^{n-d}F_d^2, \tag{3.2}$$

where n and d are arbitrary integers. We give Catalan's identity in this particular form to reflect that it relates a particular Fibonacci number to the Fibonacci numbers at a distance of d apart. The identity is frequently referenced in the literature with the year 1879. This is

[1] The published solution contains a typographic error and gives $2/(1-\sqrt{5})$ as the limiting value. An erratum in the October 1880 issue of the NCM corrects this [8].

[2] It seems a time-honored practice for academics to take a new result that they have just discovered to dress up an old problem and pose the outcome as a "new problem" for solution.

only partly correct. Catalan's identity first appeared in print in December 1886 as part of the second instalment of Catalan's collected works, entitled *Mélanges Mathématiques*, published over three volumes of the *Mémoires de la Société Royale des Sciences de Liège*. Catalan retired from the University of Liège in 1884 and was officially promoted to professor emeritus; he regarded the *Mélanges Mathématiques* as his mathematical testament. The *Mélanges* consists of 299 individual sections with a list of Catalan's publications and a few errata at the end. Most sections contain annotated reprints of articles that were previously published. However, some of the later sections seem to be based on his private and unpublished research notes. These notes are dated and annotated, but contain no reference to the material having been published before. Section 189, entitled *Sur la série de Lamé* [3], is one of the latter. In this section, one can find a theorem that is now known as Catalan's identity, with a date, presumably when Catalan wrote the note, given as October 1879.

Leonard Dickson (1874–1954), in his influential textbook *History of the Theory of Numbers*, cites the above mentioned *Mémoires* as the source and a publication date of 1886 for Catalan's identity [4, p. 402]. Raymond Archibald (1875–1955), in a historic review of the Fibonacci sequence in the MONTHLY, mentions that Catalan found the identity in 1879, also citing the *Mémoires* as the source, but with a publication date for the latter as 1887 [1, pp. 236–237]. Perhaps, the tagline

"Discovered in October 1879, and not published until December 1886"

would be a more accurate description when mentioning the origins of Catalan's identity.

It is clear that Mangon discovered identity (2.3) under his own steam, and could not have been aware of Catalan's identity. Could it have been that Mangon [or Lucas] inspired Catalan, who as editor-in-chief of the NCM saw all correspondence, to write up his own research note with a more general identity, that now bears his name? The timeline of events does not preclude the possibility. Regardless, it shows that clearing one's desk before retirement can be beneficial to one's legacy.

4. Problem H-832

The motivation to look closer into sums, whose evaluation boil down to telescoping sums, comes from a well-crafted and visually pleasing problem for solution that Hideyuki Ohtsuka posed in the Advanced Problem Section of the *Fibonacci Quarterly* [11]. In the November 2018 issue, he proposed Problem H-832 and asked to find closed-form expressions for the sums

$$\sum_{k=0}^{n} F_{rk}^3 L_{rk} \quad \text{and} \quad \sum_{k=0}^{n} F_{2F_k}^3 F_{2L_k}, \tag{4.1}$$

where n and r are nonnegative integers. A solution by Ohtsuka was published in the November 2020 issue, where he used Catalan's identity to rewrite the summations as telescoping sums [12]. This made me wonder what type of problems can be constructed if one takes Catalan's identity as a point of departure.

5. Parametric Fibonacci Summation Identities

Let a and b be integers with the same parity, then we can write Catalan's identity in the form

$$F_{b+a} F_{b-a} = F_b^2 - F_a^2, \tag{5.1}$$

in which one can recognize the "Ansatz" to a telescoping sum. As a first illustration, take $a = 2k$ and $b = 2k+2$, so that the parity condition is satisfied, and sum both sides of (5.1) over k

from zero to n to give

$$\sum_{k=0}^{n} F_{4k+2}F_2 = \sum_{k=0}^{n}\left(F_{2(k+1)}^2 - F_{2k}^2\right) = \sum_{k=1}^{n+1} F_{2k}^2 - \sum_{k=0}^{n} F_{2k}^2 = F_{2n+2}^2 - F_0^2. \tag{5.2}$$

This shows a telescoping sum in action, and one sees that all elements cancel out, except the last and first. Tidying up this example, using $F_2 = 1$ and $F_0 = 0$, and considering its companion sum, obtained by taking $a = 2k - 1$ and $b = 2k + 1$, gives the twin set

$$\sum_{k=0}^{n} F_{4k+2} = F_{2n+2}^2 \quad \text{and} \quad \sum_{k=0}^{n} F_{4k} = F_{2n+1}^2 - 1. \tag{5.3}$$

One can easily extend this procedure, introduce parameters, and derive a closed-form expression for a more general sum. In this case,

$$F_{2r} \times \sum_{k=0}^{n} F_{4rk+2(r+c)} = F_{2r(n+1)+c}^2 - F_c^2, \tag{5.4}$$

where r and c are arbitrary integers. The utility of introducing parameters is that, once the final identity has been obtained, one can then select parameter values to obtain specific cases that are visually more appealing or intriguing, such as

$$F_{2n} \times \sum_{k=0}^{n} F_{4nk} = F_{(2n+1)n}^2 - F_n^2, \tag{5.5}$$

where n, the upper limit of the summation, is also part of the summand's index. Of course, when one proposes to find a closed-form expression for a Fibonacci summation as a "problem for solution", one can also select the parameters to obfuscate the origins of the problem and provide less handles for its solution. In particular, one should choose the parameter values such that the corresponding sequence does not have an entry in the On-Line Encyclopedia of Integer Sequences (OEIS) [10], where one can find references for a particular number sequence and often closed-form expressions. This rules out the identities in (5.3); one can find the corresponding number sequences as sequences A049684 and A058038 in the OEIS. The summation in identity (5.5) does not (yet) have an entry in the OEIS.

A more versatile generalization can be obtained by letting a and b be consecutive elements in a general sequence of integers, as we have done in the following theorem.

Theorem 5.1. *Let $\{\mu_n\}$ be a sequence of integers that have the same parity. Then,*

$$\sum_{k=0}^{n} F_{\mu_{k+1}+\mu_k} F_{\mu_{k+1}-\mu_k} = F_{\mu_{n+1}}^2 - F_{\mu_0}^2. \tag{5.6}$$

Proof. Take $a = \mu_k$ and $b = \mu_{k+1}$ in (5.1) and sum both sides over the index k, where the right-hand side simplifies as it is a telescoping sum. □

The identities in (5.3) can be obtained from Theorem 5.1 by taking $\mu_k = 2k$ and $\mu_k = 2k-1$, respectively. The general identity (5.4) by taking $\mu_k = 2rk + c$, and the special case (5.5) by taking $\mu_k = n(2k - 1)$.

Now onto some other interesting applications of Theorem 5.1. Taking $\mu_k = 2F_k$ and $\mu_k = 2L_k$ gives identities where the Fibonacci and Lucas numbers appear as indices of the Fibonacci numbers:

$$\sum_{k=0}^{n} F_{2F_{k+2}} F_{2F_{k-1}} = F_{2F_{n+1}}^2 \quad \text{and} \quad \sum_{k=0}^{n} F_{2L_{k+2}} F_{2L_{k-1}} = F_{2L_{n+1}}^2 - 9. \tag{5.7}$$

Many more such identities can be derived. Noting that the parity of the Fibonacci numbers, starting with F_0, is $0, 1, 1, 0, 1, 1$, and so on, one sees that the Fibonacci numbers that are a distance of three apart have the same parity. The same goes for the Lucas numbers. Taking $\mu_k = F_{3k-1}$ and $\mu_k = L_{3k-1}$, and an application of Theorem 5.1 gives

$$\sum_{k=0}^{n} F_{2F_{3k+1}} F_{2F_{3k}} = F^2_{F_{3n+2}} - 1 \quad \text{and} \quad \sum_{k=0}^{n} F_{2L_{3k+1}} F_{2L_{3k}} = F^2_{L_{3n+2}} - 1, \tag{5.8}$$

where we have used that $F_{n-1} + F_{n+2} = 2F_{n+1}$ and $F_{n+2} - F_{n-1} = 2F_n$, for all n, in particular $n = 3k$, and the same holding for the Lucas numbers. This is still some way away from solving Problem H-832, but we have some success in that the first Fibonacci identity in (5.8) answers one of the questions in Problem B-1282, also posed by Ohtsuka [13].

Another play on parity along the same lines is obtained by taking $\mu_k = F_{k-1}F_kF_{k+1}$ and $\mu_k = L_{k-1}L_kL_{k+1}$, creating two sequences where all numbers are even. This gives

$$\sum_{k=0}^{n} F_{2F_kF^2_{k+1}} F_{2F^2_kF_{k+1}} = F^2_{F_nF_{n+1}F_{n+2}} \quad \text{and} \quad \sum_{k=0}^{n} F_{2L_kL^2_{k+1}} F_{2L^2_kL_{k+1}} = F^2_{L_nL_{n+1}L_{n+2}} - 1. \tag{5.9}$$

Of course, one can go completely overboard, take $\mu_k = 2F_{F_{F_k}}$, or have even more levels of depth, and derive the corresponding summation identities. However, that will show more clearly the structure, which might give a clue as to how the identity is constructed, and is thus less effective at bamboozling one's audience.

Some of the more unusual summation identities that were promised in the abstract are derived from Theorem 5.1 by the choice of $\mu_k = k(k-1)$, $\mu_k = 2^k$, and $\mu_k = k!$. These choices give

$$\sum_{k=0}^{n} F_{2k^2} F_{2k} = F^2_{n(n+1)}, \tag{5.10}$$

$$\sum_{k=0}^{n-1} F_{3\times 2^k} F_{2^k} = F^2_{2^n} + 1, \tag{5.11}$$

and

$$\sum_{k=1}^{n-1} F_{(k+2)\times k!} F_{k\times k!} = F^2_{n!} + 1, \tag{5.12}$$

respectively. Note that we derived the second and third summation, starting at the indices $k = 1$ and $k = 2$, respectively, to satisfy the parity condition, and then manually extended the range to start at $k = 0$ and $k = 1$, to improve their visual appearance. This manipulation also converts the right-hand side from the difference of two factors, which is a pointer to a telescopic sum, to the sum of two positive factors, obfuscating the underlying structure.

One can even throw other recurrent sequences into the mix. Consider the Tribonacci sequence, defined by the recursion $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, with initial conditions $T_0 = T_1 = 0$ and $T_2 = 1$. Taking $\mu_k = T_{4k}$, using the property that Tribonacci numbers, a distance of four apart, have the same parity, gives

$$\sum_{k=0}^{n} F_{2T_{4k+3}} F_{2(T_{4k+2}+T_{4k+1})} = F^2_{T_{4n+4}}, \tag{5.13}$$

where we used $T_{n+4} + T_n = 2T_{n+3}$ and $T_{n+4} - T_n = 2T_{n+2} + 2T_{n+1}$.

The fun does not end here. Catalan's identity (3.2) also holds for the Fibonacci polynomials $F_n(x)$, defined by the recursion $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. This implies that Theorem 5.1 can be generalized and remains valid when we replace the Fibonacci numbers in (5.6) by the Fibonacci polynomials. This, in one stroke, generalizes all identities that we derived. One could thus ask to determine closed-form expressions for

$$\sum_{k=0}^{n} F_{4nk}(x) \quad \text{and} \quad \sum_{k=0}^{n} P_{2F_{k+2}} P_{2F_{k-1}}, \tag{5.14}$$

where P_n is the nth Pell number, defined by the recursion $P_{n+2} = 2P_{n+1} + P_n$, with initial conditions $P_0 = 0$ and $P_1 = 1$. The answers, mutatis mutandis, can be found in (5.5) and (5.7).

6. Complicating Things

Normally, one does not earn brownie points in mathematics, or any other discipline, for that matter, when one takes a simple identity and turns it into a more complex one. Constructing and posing "problems for solution" in mathematical journals might just be the only exception. So, let's go for some brownie points. Taking $r = 2$ in Theorem A.1 in the Appendix gives

$$\sum_{k=0}^{n} F_{\mu_{k+1}}^2 F_{\mu_{k+2}+\mu_k} F_{\mu_{k+2}-\mu_k} = F_{\mu_{n+1}}^2 F_{\mu_{n+2}}^2 - F_{\mu_0}^2 F_{\mu_1}^2, \tag{6.1}$$

where $\{\mu_n\}$ is a sequence of integers, with the property that elements, a distance of two apart, have the same parity. Taking $\mu_k = r(k-1)$ in (6.1) satisfies the parity condition for all integers r and gives the summand as $F_{rk}^2 F_{2rk} F_{2r}$. Now use $F_{2n} = F_n L_n$ to give

$$F_{2r} \sum_{k=0}^{n} F_{rk}^3 L_{rk} = F_{rn}^2 F_{r(n+1)}^2, \tag{6.2}$$

and provides a closed-form expression for the first sum by Ohtsuka in (4.1). Taking $\mu_k = 2F_{k-1}$ and $\mu_k = 2L_{k-1}$, both obviously satisfying the parity condition for (6.1), gives

$$\sum_{k=0}^{n} F_{2F_k}^3 F_{2L_k} = F_{2F_n}^2 F_{2F_{n+1}}^2 \quad \text{and} \quad \sum_{k=0}^{n} F_{2L_k}^3 F_{10F_k} = F_{2L_n}^2 F_{2L_{n+1}}^2 - 9, \tag{6.3}$$

respectively, where we use $L_n = F_{n-1} + F_{n+1}$ and $5F_n = L_{n+1} + L_{n-1}$. This provides a closed-form expression for the second sum by Ohtsuka in (4.1) and gives the Lucas equivalent as a bonus. To collect the last few brownie points, we take $r = 3$ in Theorem A.1. This gives

$$\sum_{k=0}^{n} F_{\mu_{k+1}}^2 F_{\mu_{k+2}}^2 F_{\mu_{k+r}+\mu_k} F_{\mu_{k+r}-\mu_k} = F_{\mu_{n+1}}^2 F_{\mu_{n+2}}^2 F_{\mu_{n+3}}^2 - F_{\mu_0}^2 F_{\mu_1}^2 F_{\mu_2}^2, \tag{6.4}$$

where $\{\mu_n\}$ is a sequence of integers, with the property that elements, a distance of three apart, have the same parity. For $\mu_n = F_{n-1}$, the parity condition is satisfied and gives

$$\sum_{k=0}^{n} F_{F_k}^3 F_{F_{k+1}}^3 L_{F_k} L_{F_{k+1}} = F_{F_n}^2 F_{F_{n+1}}^2 F_{F_{n+2}}^2, \tag{6.5}$$

where we used $F_{n+3} + F_n = 2F_{n+2}$, $F_{n+3} - F_n = 2F_{n+1}$, and $F_{2n} = F_n L_n$.

7. Epilogue

Lucas' "Question 494" and Mangon's answer made it into Dickson's *History of the Theory of Numbers* [4, p. 402], albeit with the originally published and incorrect answer. One cannot fault Dickson for not picking up the erratum. Just goes to show that one should always verify a formula or result before usage. Lucas went on to shape the way we think about recurrent sequences, was instrumental in attaching the name Fibonacci to the eponymous sequence, and also made his mark in recreational mathematics. Mangon went on to become a lieutenant and published a study on the efficacy of artillery fire a few years later [9]. Catalan's NCM was short lived, the journal saw its last issue in December 1880 and ceased publication, due to lack of subscriptions. In 1881, Mansion and Neuberg filled the void left behind by the disappearance of the NCM and started the journal *Mathesis*, with support from Catalan. With the current article, Fibonacci aficionados that have an interest in solving or creating Fibonacci summation problems have gained additional insights and a new tool in Theorem 5.1 and its generalization, Theorem A.1. Ohtsuka, I hope, will forgive my tongue-in-cheek article and continue to provide the Fibonacci community with interesting and well-crafted "problems for solution."

8. Acknowledgment

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Appendix A. Telescoping Sums

Telescoping sums are normally based on the first-order difference of a function, where subsequent terms cancel each other, leaving only the initial and final terms. One can also construct telescoping sums that are based on the difference of terms that are two apart, resulting in the first two and last two terms being left. The same goes for taking terms three, four or those further apart. An interesting variation, in the context of constructing problems for solution, is obtained by taking $g(n) = f(n+2) - f(n)$, multiplying both sides by $f(n+1)$, resulting in the telescoping sum

$$\sum_{k=0}^{n} f(k+1)g(k) = \sum_{k=0}^{n} \left(f(k+1)f(k+2) - f(k)f(k+1)\right) = f(n+1)f(n+2) - f(0)f(1), \quad \text{(A.1)}$$

that leaves two components that each is the product of two consecutive terms. The natural generalization, taking $g(n) = f(n+r) - f(n)$, where r is a positive integer, is given by

$$\sum_{k=0}^{n} \prod_{j=1}^{r-1} f(k+j)g(k) = \prod_{j=1}^{r} f(n+j) - \prod_{j=0}^{r-1} f(j). \quad \text{(A.2)}$$

This leads to the following generalization of Theorem 5.1 that does not require further proof.

Theorem A.1. *Let $\{\mu_n\}$ be a sequence of integers, where μ_n and μ_{n+r} have the same parity, and r is a positive integer. Then,*

$$\sum_{k=0}^{n} \prod_{j=1}^{r-1} F^2_{\mu_{k+j}} F_{\mu_{k+r}+\mu_k} F_{\mu_{k+r}-\mu_k} = \prod_{j=1}^{r} F^2_{\mu_{n+j}} - \prod_{j=0}^{r-1} F^2_{\mu_j}, \quad \text{(A.3)}$$

with the usual convention that an empty product evaluates to unity.

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