FIBONACCI OR LUCAS NUMBERS THAT ARE CONCATENATIONS OF TWO g-REPDIGITS

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Abstract. Let $k \geq 1$ and $g \geq 2$ be positive integers. Any positive integer N of the form

$$N = \overline{\underbrace{d_1 \dots d_1}_{m_1 \ times}\underbrace{d_2 \dots d_2}_{m_2 \ times} \dots \underbrace{d_k \dots d_k}_{m_k \ times}}_{(g)},$$

where $d_1, \dots, d_k \in \{0, 1, \dots, g-1\}$ with $d_1 \neq 0$, can be viewed as a concatenation of k repdigits in base g. In this paper, we find all Fibonacci and Lucas numbers that are concatenations of two repdigits in base g for $2 \leq g \leq 9$.

1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, with initial values $F_0 = 0$ and $F_1 = 1$ and let $\{L_n\}_{n\geq 0}$ be the Lucas sequence defined by $L_{n+2} = L_{n+1} + L_n$, where $L_0 = 2$ and $L_1 = 1$. If

$$(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

is the pair of roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci and Lucas numbers, then Binet's formulas for their general terms are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0. \tag{1.1}$$

It can be seen that $1 < \alpha < 2$, $-1 < \beta < 0$, and $\alpha\beta = -1$. The following relations between the nth Fibonacci number F_n, the nth Lucas number L_n, and α are well known

$$\alpha^{n-2} < F_n < \alpha^{n-1} \quad \text{and} \quad \alpha^{n-1} \leq L_n \leq 2\alpha^n \quad \text{for} \quad n \geq 0. \tag{1.2}$$

Notice that there are many papers in the literature that solve Diophantine equations related to Fibonacci numbers and Lucas numbers. For instance in 2011, Luca and Oyono [9] concluded that there is no solution (m, n, s) to the Diophantine equation $F_m^s + F_{m+1}^s = F_n$ for integers $m \geq 2$, $n \geq 1$, and $s \geq 3$ by applying linear forms in logarithms. In 2013, Marques and the fourth author [10] found all solutions (n, a, b, c) to the Diophantine equation $F_n = 2^a + 3^b + 5^c$ and $L_n = 2^a + 3^b + 5^c$ for integers n, a, b, c with $0 \leq \max\{a, b\} \leq c$.

We recall that a positive integer R is called a base g-repdigit if all its digits are the same in base g. That is, R is of the form

$$R = \frac{d(g^m - 1)}{g - 1} = \overline{\underbrace{d \dots d}_{m \ times}}_{(g)}$$

for some positive integers d, m with $1 \leq d \leq g - 1$, and $m \geq 1$. When $g = 10$, we omit the base and we say that R is a repdigit. The problem of searching for repdigits in the Fibonacci and Lucas sequences has been studied by Luca. In [8], he determined the largest repdigits in the Fibonacci and Lucas sequences are $F_{10} = 55$ and $L_5 = 11$. Given $k \geq 1$, we say that N is a concatenation of k repdigits in base g, if N can be written in the form

$$\overline{\underbrace{d_1 \dots d_1}_{m_1 \ times} \underbrace{d_2 \dots d_2}_{m_2 \ times} \cdots \underbrace{d_k \dots d_k}_{m_k \ times}}_{(g)}.$$

In [1], the authors solved the problem of finding the Fibonacci numbers that are concatenations of two repdigits. In [5], Ddamulira studied the problem of finding the Padovan numbers that are concatenations of two repdigits. In [13], Rayaguru and Panda determined all balancing numbers that are concatenations of two repdigits. Motivated by these works, in this study, we address the following two Diophantine equations

$$F_n = \overline{\underbrace{d_1 \dots d_1}_{m_1 \ times} \underbrace{d_2 \dots d_2}_{m_2 \ times}}_{(g)} \quad \text{and} \quad L_n = \overline{\underbrace{d_1 \dots d_1}_{m_1 \ times} \underbrace{d_2 \dots d_2}_{m_2 \ times}}_{(g)}, \tag{1.3}$$

where $d_1, m_1, m_2 \geq 1$ and $d_1, d_2 \in \{0, 1, \dots, g-1\}$ with $d_1 \neq d_2$. That is, we will determine all Fibonacci or Lucas numbers that are concatenations of two repdigits in base g. The main object of this study is to generalize the work of the authors of [1] and [12]. Here is the outline of this paper. In Section 2, we will give some lemmas, and then we prove our main theorems in Section 3.

2. Useful Tools

In this section, we gather the tools we need to prove Theorems 2 and 3. Let η be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$, and let $\eta = \eta^{(1)}, \dots, \eta^{(d)}$ denote its conjugates. The logarithmic height of α is defined by

$$h(\eta) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right).$$

This height has the following basic properties. For η_1, η_2 algebraic numbers, and $m \in \mathbb{Z}$, we have

$$h(\eta_1 \pm \eta_2) \leq h(\eta_1) + h(\eta_2) + \log 2,$$
$$h(\eta_1 \eta_2^{\pm}) \leq h(\eta_1) + h(\eta_2),$$
$$h(\eta_1^m) = |m| h(\eta_1).$$

Now, let $\mathbb{L}$ a real number field of degree $d_{\mathbb{L}}$, $\eta_1, \dots, \eta_s \in \mathbb{L}$ and $b_1, \dots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \dots, |b_s|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_s^{b_s} - 1.$$

Let $A_1, \dots, A_s$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \dots, s.$$

The first tool we need is the following result due to Matveev [11]. Here, we use the version of Bugeaud, Mignotte, and Siksek [3, Theorem 9.4].

Theorem 1. *Assume that $\Lambda \neq 0$. Then,*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

Our second tool is a version of the reduction method of Baker and Davenport [2]. We use a slight variant of the version given by Dujella and Pethő [6]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 1. *Let M be a positive integer, p/q be a convergent of the continued fraction expansion of the irrational number τ such that $q > 6M$, and A, B, and μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\| .$$

If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w} \tag{2.1}$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

We see that Lemma 1 cannot be applied when $\mu = 0$ (because then $\varepsilon < 0$). For this case, we use the following well known technical result from Diophantine approximation known as Legendre's criterion, which is our third tool. This comes from the theory of continued fractions (see [7], pages 30 and 37).

Lemma 2. *Let η be an irrational number.*

(i) *If n and m are positive integers such that*

$$\left|\eta - \frac{n}{m}\right| < \frac{1}{2m^2},$$

then $n/m = p_k/q_k$ is a convergent of η.

(ii) *Let M be a positive real number and p_0/q_0, p_1/q_1, ... be all the convergents of the continued fraction of η. Let N be the smallest positive integer such that $q_N > M$. Put $a(M) := \max\{a_k : k = 0, 1, \ldots, N\}$. Then, the inequality*

$$\left|\eta - \frac{n}{m}\right| > \frac{1}{(a(M)+2)m^2}$$

holds for all pairs (n, m) of integers with $0 < m < M$.

3. Main Results

We use the method in [1] to prove our two results of this paper. Note that in the case $g = 2$, we will just take into account $d_1 = 1$ and $d_2 = 0$ because of the assumption $d_1 \neq d_2$ of (1.3). Moreover, the case $g = 10$ is already studied in [1]. Thus, we only need to investigate what happens with $2 \leq g \leq 9$.

3.1. Fibonacci Numbers as Concatenations of Two g-repdigits.

In this subsection, we will prove the following result.

Theorem 2. *The only Fibonacci numbers that are concatenations of two repdigits in base g with $2 \leq g \leq 9$ are*

2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 377, 1597.

Namely, we have

$$\begin{array}{rcccl}
2 &=& F_3 &=& \overline{10}_2,\\
3 &=& F_4 &=& \overline{10}_3,\\
5 &=& F_5 &=& \overline{12}_3 = \overline{10}_5,\\
8 &=& F_6 &=& \overline{100}_2 = \overline{20}_4 = \overline{13}_5 = \overline{12}_6 = \overline{10}_8,\\
13 &=& F_7 &=& \overline{31}_4 = \overline{23}_5 = \overline{21}_6 = \overline{16}_7 = \overline{15}_8 = \overline{14}_9,\\
21 &=& F_8 &=& \overline{41}_5 = \overline{25}_8 = \overline{23}_9 = \overline{30}_7,\\
34 &=& F_9 &=& \overline{114}_5 = \overline{54}_6 = \overline{46}_7 = \overline{42}_8 = \overline{37}_9,\\
55 &=& F_{10} &=& \overline{67}_8 = \overline{61}_9,\\
89 &=& F_{11} &=& \overline{225}_6 = \overline{115}_7,\\
144 &=& F_{12} &=& \overline{400}_6 = \overline{220}_8,\\
377 &=& F_{14} &=& \overline{111222}_3,\\
1597 &=& F_{17} &=& \overline{4441}_7.
\end{array}$$

The immediate consequence of Theorem 2 is the following result.

Corollary 1. *The largest Fibonacci number that can be representable as a concatenation of two repdigits in base g when $g \in \{2, \dots, 9\}$ is $F_{17} = 1597$. More precisely, we have*

$$F_{17} = \overline{4441}_7.$$

We will prove our result under the assumption that $n > 200$; then we will finish using a computer program for what happens for $n \le 200$. From (1.3), the first equation can be rewritten like this

$$\begin{aligned}
F_n &= \overline{\underbrace{d_1 \dots d_1}_{m_1 \text{ times}}\underbrace{d_2 \dots d_2}_{m_2 \text{ times}}}_{(g)}\\
&= \overline{\underbrace{d_1 \dots d_1}_{m_1 \text{ times}}} \times g^{m_2} + \overline{\underbrace{d_2 \dots d_2}_{m_2 \text{ times}}}\\
&= \frac{1}{g-1}\left(d_1 g^{m_1+m_2} - (d_1 - d_2) g^{m_2} - d_2\right). \qquad (3.1)
\end{aligned}$$

We prove the following lemma, which gives a relation on the size of n versus $m_1 + m_2$.

Lemma 3. *All solutions of the Diophantine equation* (1.3) *satisfy*

$$(m_1 + m_2)\log g - 2 < n \log \alpha < (m_1 + m_2)\log g + 1.$$

Proof. The proof is deduced essentially from the first relation of (1.2). So, one can see from (3.1) that

$$\alpha^{n-2} < F_n < g^{m_1+m_2}.$$

Taking the logarithm on both sides, we get

$$(n-2)\log\alpha < (m_1+m_2)\log g,$$

which leads to

$$n\log\alpha < (m_1+m_2)\log g + 2\log\alpha < (m_1+m_2)\log g + 1. \qquad (3.2)$$

For the lower bound, we have from (3.1) that

$$g^{m_1+m_2-1} < F_n < \alpha^{n-1}.$$

Taking the logarithm on both sides, we get that

$$(m_1+m_2-1)\log g < (n-1)\log\alpha,$$

which leads to

$$(m_1 + m_2)\log g + \log\alpha - \log g < n\log\alpha. \tag{3.3}$$

Because $2 \le g \le 9$, we can easily see that $-2 < \log\alpha - \log 9 \le \log\alpha - \log g$. Thus,

$$(m_1 + m_2)\log g - 2 < n\log\alpha. \tag{3.4}$$

Comparing (3.2) and (3.4) gives the result in the lemma. □

Next, we examine the Diophantine equation (3.1) in two different steps to find the upper bound of n and $m_1 + m_2$. Thus, we need to prove the following result.

Lemma 4. *All solutions to the Diophantine equation* (3.1) *satisfy*

$$m_1 + m_2 < 3.8 \times 10^{29} \quad \textit{and} \quad n < 5.4 \times 10^{29}.$$

Proof. Step 1. Substituting Binet's formula for F_n in (3.1), we get that

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{1}{g-1}\left(d_1 g^{m_1+m_2} - (d_1 - d_2)g^{m_2} - d_2\right),$$

which is equivalent to

$$(g-1)\alpha^n - d_1\sqrt{5}g^{m_1+m_2} = (g-1)\beta^n - \sqrt{5}((d_1-d_2)g^{m_2} + d_2),$$

from which we deduce that

$$\begin{aligned} \left|(g-1)\alpha^n - d_1\sqrt{5}g^{m_1+m_2}\right| &= \left|(g-1)\beta^n - \sqrt{5}((d_1-d_2)g^{m_2} + d_2)\right| \\ &\le 8 + \sqrt{5}(8g^{m_2} + 8) \\ &< 27.7 \cdot g^{m_2}. \end{aligned}$$

Thus, dividing both sides by $d_1\sqrt{5}g^{m_1+m_2}$, we get that

$$\left|\frac{g-1}{d_1\sqrt{5}} \cdot \alpha^n \cdot g^{-(m_1+m_2)} - 1\right| < \frac{27.7 \cdot g^{m_2}}{d_1\sqrt{5}g^{m_1+m_2}} < \frac{12.4}{g^{m_1}}. \tag{3.5}$$

Put

$$\Gamma_1 := \frac{g-1}{d_1\sqrt{5}} \cdot \alpha^n \cdot g^{-(m_1+m_2)} - 1. \tag{3.6}$$

Next, we apply Theorem 1 on Γ_1. First, we need to check that $\Gamma_1 \ne 0$. If it were not, then we would get that

$$\alpha^{2n} = \frac{5d_1^2 g^{2(m_1+m_2)}}{(g-1)^2},$$

which is impossible because α^{2n} is irrational for all $n \ge 1$. Therefore, $\Gamma_1 \ne 0$. So, we apply Theorem 1 on Γ_1 with $s := 3$ and

$$(\eta_1, b_1) := \left(\frac{g-1}{d_1\sqrt{5}}, 1\right), \quad (\eta_2, b_2) := (\alpha, n), \quad (\eta_3, b_3) := (g, -m_1 - m_2).$$

Using $g^{m_1+m_2-1} < F_n < \alpha^{n-1} < g^{n-1}$, we get $m_1 + m_2 < n$. Therefore, we can take $B := n$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so $d_{\mathbb{L}} := 2$. We have

$$\begin{aligned} h(\eta_1) &= h\left(\frac{g-1}{d_1\sqrt{5}}\right) \\ &\le h\left(\frac{g-1}{d_1}\right) + h(\sqrt{5}) \\ &= \log(\max\{g-1, d_1\}) + \frac{1}{2}\log 5 \\ &\le \log 8 + \frac{1}{2}\log 5. \end{aligned}$$

Furthermore, $h(\eta_2) = h(\alpha) = \dfrac{1}{2}\log\alpha$ and $h(\eta_3) = h(g) = \log g \le \log 9$. Thus, we can take

$$A_1 := 5.8, \quad A_2 := 0.5, \quad \text{and} \quad A_3 := 4.4.$$

Using the previous data, Theorem 1 tells us that

$$\log|\Gamma_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log n)(5.8)(0.5)(4.4).$$

Comparing the above inequality with (3.5) gives

$$m_1 \log g < 1.24 \cdot 10^{13}(1+\log n). \tag{3.7}$$

Step 2. Combining (3.1) with Binet's formula for F_n, we obtain

$$\alpha^n - \sqrt{5}\left(\frac{d_1 g^{m_1} - (d_1-d_2)}{g-1}\right) g^{m_2} = \beta^n - \frac{\sqrt{5}d_2}{g-1},$$

from which we deduce that

$$\left|\alpha^n - \sqrt{5}\left(\frac{d_1 g^{m_1} - (d_1-d_2)}{g-1}\right) g^{m_2}\right| = \left|\beta^n - \frac{d_2\sqrt{5}}{g-1}\right| \le 1+\sqrt{5} < 3.3.$$

Thus, dividing both sides by α^n, we get that

$$\left|\left(\frac{(d_1 g^{m_1} - (d_1-d_2))\sqrt{5}}{g-1}\right) \cdot \alpha^{-n} \cdot g^{m_2} - 1\right| < \frac{3.3}{\alpha^n}. \tag{3.8}$$

Put

$$\Gamma_2 := \left(\frac{(d_1 g^{m_1} - (d_1-d_2))\sqrt{5}}{g-1}\right) \cdot \alpha^{-n} \cdot g^{m_2} - 1. \tag{3.9}$$

Next, we apply Theorem 1 on Γ_2. First, we need to check that $\Gamma_2 \neq 0$. If not, then we would get that

$$\alpha^{2n} = \frac{5(d_1 g^{m_1} - (d_1-d_2))^2}{(g-1)^2} g^{2m_2},$$

which is impossible because α^{2n} is irrational for $n \ge 1$. Thus, $\Gamma_2 \neq 0$. So, we apply Theorem 1 on Γ_2 with

$$s := 3, \quad \eta_1 := \frac{(d_1 g^{m_1} - (d_1-d_2))\sqrt{5}}{g-1}, \quad \eta_2 := \alpha, \quad \eta_3 := g,$$

and the exponents

$$b_1 := 1, \quad b_2 := -n, \quad b_3 := m_2.$$

As before, we have that $m_2 < n$. Thus, we can take $B := n$. Similarly, $\mathbb{L} = \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so we take $d_{\mathbb{L}} := 2$. Furthermore, we have

$$\begin{aligned} h(\eta_1) &= h\left(\frac{(d_1 g^{m_1} - (d_1 - d_2))\sqrt{5}}{g-1}\right) \\ &\leq h\left(\frac{\sqrt{5}}{g-1}\right) + h(d_1 g^{m_1} - (d_1 - d_2)) \\ &\leq h(\sqrt{5}) + h(g-1) + h(d_1) + h(d_1 - d_2) + \log 2 + m_1 h(g) \\ &\leq \frac{1}{2}\log 5 + 3\log 8 + \log 2 + m_1 \log g \\ &\leq \frac{1}{2}\log 5 + 3\log 8 + \log 2 + 1.24 \cdot 10^{13}(1 + \log n) \quad \text{(by (3.7))} \\ &\leq 1.25 \cdot 10^{13}(1 + \log n). \end{aligned}$$

Thus, we can take

$$A_1 := 2.5 \cdot 10^{13}(1 + \log n), \quad A_2 := 0.5, \quad \text{and} \quad A_3 := 4.4.$$

Theorem 1 tells us that

$$\log|\Gamma_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log n)(2.5 \cdot 10^{13}(1 + \log n))(0.5)(4.4).$$

Comparing the above inequality with (3.8) gives

$$n \log \alpha - \log 3.3 < 5.34 \times 10^{25}(1 + \log n)^2,$$

which is equivalent to

$$n < 1.11 \times 10^{26}(1 + \log n)^2. \tag{3.10}$$

The above inequality gives us

$$n < 5.4 \cdot 10^{29},$$

and Lemma 3.2 implies

$$m_1 + m_2 < 3.8 \cdot 10^{29}.$$

This completes the proof. □

To lower the bounds in Lemma 4, we return to inequality (3.5). Put

$$\begin{aligned} \Lambda_1 &:= -\log(\Gamma_1 + 1) \\ &= (m_1 + m_2)\log g - n \log \alpha - \log\left(\frac{g-1}{d_1\sqrt{5}}\right). \end{aligned}$$

Inequality (3.5) can be written as

$$\left|e^{-\Lambda_1} - 1\right| < \frac{12.4}{g^{m_1}}.$$

Assume that $m_1 \geq 5$. Because $2 \leq g \leq 9$, we get $\left|e^{-\Lambda_1} - 1\right| < \frac{12.4}{g^{m_1}} < \frac{1}{2}$, which implies that $\frac{1}{2} < e^{-\Lambda_1} < \frac{3}{2}$. If $\Lambda_1 > 0$, then

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = e^{\Lambda_1}(1 - e^{-\Lambda_1}) < \frac{24.8}{g^{m_1}}.$$

If $\Lambda_1 < 0$, then

$$0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = e^{-\Lambda_1} - 1 < \frac{12.4}{g^{m_1}}.$$

In any case, it is always true that $0 < |\Lambda_1| < \dfrac{24.8}{g^{m_1}}$, which implies

$$0 < \left| (m_1 + m_2)\frac{\log g}{\log \alpha} - n - \frac{\log\left((g-1)/d_1\sqrt{5}\right)}{\log \alpha} \right| < 51.6 \cdot g^{-m_1}. \tag{3.11}$$

It is easy to see that $\dfrac{\log g}{\log \alpha}$ is irrational. If $\dfrac{\log g}{\log \alpha} = \dfrac{p}{q}$ ($p, q \in \mathbb{Z}$ and $p > 0$, $q > 0$, $\gcd(p, q) = 1$), then $\alpha^p = g^q \in \mathbb{Z}$, which is an absurdity because $2 \le g \le 9$. Now, we apply Lemma 1 with

$$\tau := \frac{\log g}{\log \alpha}, \quad \mu := -\frac{\log\left((g-1)/d_1\sqrt{5}\right)}{\log \alpha}, \quad A := 51.6, \quad B := g.$$

Note that $m_1 + m_2 < 3.8 \cdot 10^{29}$ by Lemma 4, so we take $M := 3.8 \cdot 10^{29}$. The application of Lemma 1 leads to the different results, which are reported in Table 1.

g	2	3	4	5	6	7	8	9
rth convergent	q_{68}	q_{62}	q_{66}	q_{60}	q_{60}	q_{68}	q_{60}	q_{58}
$\varepsilon \ge$	0.451	0.229	0.002	0.104	0.058	0.061	0.026	0.043
$m_1 \le$	110	71	58	47	44	40	37	36

Table 1.

Referring to the above results, it follows that $m_1 \le 110$ is valid in all cases.

For fixed $1 \le m_1 \le 110$ and $d_1, d_2 \in \{0, 1, \ldots, g-1\}$ with $d_1 \ne d_2$ and $d_1 \ne 0$, we return to (3.8) and put

$$\begin{aligned}\Lambda_2 &:= \log(\Gamma_2 + 1)\\ &= m_2 \log g - n \log \alpha + \log\left(\frac{(d_1 g^{m_1} - (d_1 - d_2))\sqrt{5}}{g-1}\right).\end{aligned}$$

From inequality (3.8) and $n \ge 200$, we conclude that

$$\left|e^{\Lambda_2} - 1\right| < \frac{3.3}{\alpha^n} < \frac{1}{2},$$

which implies that $\dfrac{1}{2} < e^{\Lambda_2} < \dfrac{3}{2}$. If $\Lambda_2 > 0$, then $0 < \Lambda_2 < e^{\Lambda_2} - 1 < \dfrac{3.3}{\alpha^n}$. If $\Lambda_2 < 0$, then

$$0 < |\Lambda_2| < e^{|\Lambda_2|} - 1 = e^{-\Lambda_2} - 1 = e^{-\Lambda_2}(1 - e^{\Lambda_2}) < \frac{6.6}{\alpha^n}.$$

In any case, because $0 < |\Lambda_2| < \dfrac{6.6}{\alpha^n}$, we have

$$0 < \left| m_2 \frac{\log g}{\log \alpha} - n + \frac{\log\left(\sqrt{5}(d_1 g^{m_1} - (d_1 - d_2))/(g-1)\right)}{\log \alpha} \right| < \frac{13.8}{\alpha^n}. \tag{3.12}$$

Again, we apply Lemma 1 with

$$\tau := \frac{\log g}{\log \alpha}, \quad \mu := \frac{\log\left(\sqrt{5}(d_1 g^{m_1} - (d_1 - d_2))/(g-1)\right)}{\log \alpha}, \quad A := 13.8, \quad B := \alpha$$

and $M := 3.8 \cdot 10^{29}$. With the help of Maple, the results obtained are presented in Table 2.

g	2	3	4	5	6	7	8	9
rth convergent	q_{69}	q_{62}	q_{70}	q_{69}	q_{68}	q_{72}	q_{62}	q_{58}
$\varepsilon \geq$	0.004	0.001	0.0001	0.00009	0.000009	0.001	0.0008	0.0001
$n \leq$	118	74	61	55	52	43	40	38

Table 2.

Thus, $n \leq 118$ is valid in all cases, contradicting that $n > 200$. Now, we search for the solutions to the first Diophantine equation of (1.3) with

$$0 \leq n \leq 200, \ 1 \leq m_1 \leq 110, \ 1 \leq m_2 \leq 140,$$

$$2 \leq g \leq 9, \ 1 \leq d_1 \leq g-1, \text{ and } 0 \leq d_2 \leq g-1,$$

by applying a program written in Maple. The only solutions we obtain are listed in Theorem 2. This completes its proof.

3.2. **Lucas Numbers as Concatenations of Two g-repdigits.** In this subsection, we will follow the method in Subsection 3.1. For the sake of completeness, we will give most of the details.

Theorem 3. *The only Lucas numbers that are concatenations of two repdigits in base g with $2 \leq g \leq 9$ are*

$$2, \ 3, \ 4, \ 7, \ 11, \ 18, \ 29, \ 47, \ 76, \ 123, \ 521, \ 843, \ 1364.$$

Namely, we have

$$\begin{array}{rcccl}
2 & = & L_0 & = & \overline{10}_2, \\
3 & = & L_2 & = & \overline{10}_3, \\
4 & = & L_3 & = & \overline{100}_2 = \overline{10}_4, \\
7 & = & L_4 & = & \overline{13}_4 = \overline{12}_5 = \overline{10}_7, \\
11 & = & L_5 & = & \overline{23}_4 = \overline{21}_5 = \overline{15}_6 = \overline{14}_7 = \overline{13}_8 = \overline{12}_9, \\
18 & = & L_6 & = & \overline{200}_3 = \overline{30}_6 = \overline{24}_7 = \overline{20}_9, \\
29 & = & L_7 & = & \overline{45}_6 = \overline{41}_7 = \overline{35}_8 = \overline{32}_9, \\
47 & = & L_8 & = & \overline{233}_4 = \overline{115}_6 = \overline{65}_7 = \overline{57}_8 = \overline{52}_9, \\
76 & = & L_9 & = & \overline{2211}_3 = \overline{114}_8 = \overline{84}_9, \\
123 & = & L_{10} & = & \overline{443}_5, \\
521 & = & L_{13} & = & \overline{2225}_6, \\
843 & = & L_{14} & = & \overline{11333}_5, \\
1364 & = & L_{15} & = & \overline{111110}_4.
\end{array}$$

The next result is a straightforward consequence of the above theorem.

Corollary 2. *The largest Lucas number that can be represented as a concatenation of two repdigits in base g when $g \in \{2, \ldots, 9\}$ is $L_{15} = 1364$. Namely we have*

$$L_{15} = \overline{111110}_4.$$

For the proof, we assume that $n > 510$. We rewrite the second Diophantine equation of (1.3) as

$$L_n = \frac{1}{g-1}\left(d_1 g^{m_1+m_2} - (d_1 - d_2)b^{m_2} - d_2\right). \tag{3.13}$$

The next lemma relates the sizes of n and $m_1 + m_1$.

Lemma 5. *All solutions of the Diophantine equation* (3.13) *satisfy*

$$(m_1 + m_2)\log g - 2.68 < n \log \alpha < (m_1 + m_2)\log g + 0.5.$$

Proof. The proof follows from (1.2). One can see from (3.13) that

$$\alpha^{n-1} \leq L_n < g^{m_1+m_2}.$$

Taking the logarithm of the extreme sides, we get

$$(n-1)\log\alpha < (m_1+m_2)\log g,$$

which leads to

$$n\log\alpha < (m_1+m_2)\log g + \log\alpha < (m_1+m_2)\log g + 0.5. \tag{3.14}$$

For the lower bound, from (3.13) we have

$$g^{m_1+m_2-1} \leq L_n \leq \alpha^{n+1}.$$

Taking the logarithm of extreme sides, we get

$$(m_1+m_2-1)\log g < (n+1)\log\alpha,$$

which leads to

$$(m_1+m_2)\log g - 2.68 < (m_1+m_2-1)\log g - \log\alpha < n\log\alpha. \tag{3.15}$$

Comparing (3.14) and (3.15) gives the result in the lemma. □

Next, we examine (3.13) in two different steps.

Step 1. Substituting the second relation of (1.1) in (3.13), we get

$$(g-1)\alpha^n - d_1 g^{m_1+m_2} = -(g-1)\beta^n - (d_1-d_2)g^{m_2} - d_2,$$

from which we deduce that

$$\begin{aligned}\left|(g-1)\alpha^n - d_1 g^{m_1+m_2}\right| &\leq (g-1)|\beta|^n + |d_1-d_2|g^{m_2} + d_2\\ &\leq 3(g-1)g^{m_2} \leq 24\cdot g^{m_2}.\end{aligned}$$

Thus, dividing both sides by $d_1 b^{m_1+m_2}$, we get

$$\left|\frac{g-1}{d_1}\cdot\alpha^n\cdot g^{-(m_1+m_2)} - 1\right| \leq \frac{24\cdot g^{m_2}}{d_1 g^{m_1+m_2}} \leq \frac{24}{g^{m_1}}. \tag{3.16}$$

Let

$$\Gamma_3 := \frac{g-1}{d_1}\cdot\alpha^n\cdot g^{-(m_1+m_2)} - 1. \tag{3.17}$$

Next, we apply Theorem 1 on Γ_3. First, we need to check that $\Gamma_3 \neq 0$. If it is not, then we would get that

$$\alpha^n = \frac{d_1 g^{m_1+m_2}}{g-1} \in \mathbb{Q},$$

which is impossible. We conclude that $\Gamma_3 \neq 0$. So, we apply Theorem 1 on (3.17) with $s := 3$ and

$$(\eta_1, b_1) := \left(\frac{g-1}{d_1}, 1\right), \quad (\eta_2, b_2) := (\alpha, n), \quad (\eta_3, b_3) := (g, -m_1-m_2).$$

Thus, we have $\mathbb{L} = \mathbb{Q}(\alpha)$, $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. Note that

$$h(\eta_1) := h\left(\frac{g-1}{d_1}\right) = \log\left(\max\{g-1, d_1\}\right) \leq \log 8,$$

and from the previous subsection, we can take

$$A_1 = 4.2, \quad A_2 = 0.5, \quad \text{and} \quad A_3 := 4.4.$$

Because $g^{m_1+m_2-1} < L_n \le 2\alpha^n < g^{n+1}$, we have that $m_1+m_2 < n+2$. As $B \ge \max\{|1|, |n|, |-(m_1+m_2)|\}$, we can take $B := n+2$. Hence, we get

$$|\Gamma_3| > \exp(-8.97 \cdot 10^{12}(1 + \log(n+2))). \tag{3.18}$$

Thus, from (3.16) and (3.18), we obtain

$$m_1 \log g < 8.98 \cdot 10^{12}(1 + \log(n+2)). \tag{3.19}$$

Step 2. We rewrite equation (3.13). Then, we get

$$\left|\alpha^n - \left(\frac{d_1 g^{m_1} - (d_1 - d_2)}{g-1}\right) g^{m_2}\right| = |\beta|^n + \frac{d_2}{g-1} < 2.$$

It follows that

$$\left|\left(\frac{d_1 g^{m_1} - (d_1 - d_2)}{g-1}\right) \cdot \alpha^{-n} \cdot g^{m_2} - 1\right| \le \frac{2}{\alpha^n}. \tag{3.20}$$

Let

$$\Gamma_4 := \left(\frac{d_1 g^{m_1} - (d_1 - d_2)}{g-1}\right) \cdot \alpha^{-n} \cdot g^{m_2} - 1. \tag{3.21}$$

Next, we apply Theorem 1 on (3.21). First, we need to check that $\Gamma_4 \neq 0$. If not, then we would get that

$$\alpha^n = \left(\frac{d_1 g^{m_1} - (d_1 - d_2)}{g-1}\right) \cdot g^{m_2} \in \mathbb{Q},$$

which is false. It follows that $\Gamma_4 \neq 0$. According to Theorem 1, we can consider the following data:

$$s := 3, \quad \eta_1 := \frac{d_1 g^{m_1} - (d_1 - d_2)}{g-1}, \quad \eta_2 := \alpha, \quad \eta_3 := g,$$

and

$$b_1 := 1, \quad b_2 := -n, \quad b_3 := m_2.$$

Thus, we have $\mathbb{L} = \mathbb{Q}(\alpha)$, $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. From (3.19), we can get

$$\begin{aligned} h(\eta_1) &= h\left(\frac{d_1 b^{m_1} - (d_1 - d_2)}{g-1}\right) \\ &\le h\left(d_1 g^{m_1} - (d_1 - d_2)\right) + h(g-1) \\ &\le 3\log(g-1) + m_1 \log g + \log 2 \\ &\le 9 \cdot 10^{12}(1 + \log(n+2)). \end{aligned}$$

Thus, as above, we take

$$A_1 = 1.8 \cdot 10^{13}(1 + \log(n+2)), \quad A_2 = 0.5, \quad \text{and} \quad A_3 = 4.4.$$

As before, we have that $m_2 < n+2$. Thus, we take $B := n+2$. Hence, we get

$$|\Gamma_4| > \exp(-3.85 \cdot 10^{25}(1 + \log(n+2))^2). \tag{3.22}$$

Thus, from (3.20) and (3.22), we get

$$n < 8.01 \cdot 10^{25}(1 + \log(n+2))^2.$$

This implies that $n < 3.9 \cdot 10^{29}$. Hence, we conclude that

$$m_1 + m_2 < \frac{n \log \alpha + 2.68}{\log 2} \le 2.8 \cdot 10^{29}.$$

To sum up, we have the following lemma.

Lemma 6. *All solutions to the Diophantine equation* (3.13) *satisfy*

$$m_1 + m_2 < 2.8 \cdot 10^{29} \quad \text{and} \quad n < 3.9 \cdot 10^{29}.$$

We note that the bounds in Lemma 6 are too large for computational purposes. However, with the help of Lemmas 1 and 2, they can be considerably sharpened. The rest of the proof is dedicated towards this goal. Put

$$\begin{aligned}\Lambda_3 &:= -\log(\Gamma_3 + 1)\\ &= (m_1 + m_2)\log g - n\log\alpha - \log\left(\frac{g-1}{d_1}\right).\end{aligned}$$

From (3.16), we conclude that

$$\left|e^{-\Lambda_3} - 1\right| < \frac{24}{g^{m_1}}. \tag{3.23}$$

Note that if $m_1 \geq 6$ and $2 \leq g \leq 9$, then $|e^{-\Lambda_3} - 1| < \dfrac{24}{g^{m_1}} < \dfrac{1}{2}$, which implies that $\dfrac{1}{2} < e^{-\Lambda_3} < \dfrac{3}{2}$. If $\Lambda_3 > 0$, then

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = e^{\Lambda_3}(1 - e^{-\Lambda_3}) < \frac{48}{g^{m_1}}.$$

If $\Lambda_3 < 0$, then

$$0 < |\Lambda_3| < e^{|\Lambda_3|} - 1 = e^{-\Lambda_3} - 1 < \frac{24}{g^{m_1}}.$$

In all cases, we have $0 < |\Lambda_3| < \dfrac{48}{g^{m_1}}$, which implies

$$0 < \left|(m_1 + m_2)\frac{\log g}{\log\alpha} - n - \frac{\log((g-1)/d_1)}{\log\alpha}\right| < 100 \cdot g^{-m_1}. \tag{3.24}$$

Note that $m_1 + m_2 < 2.8 \cdot 10^{29}$ by Lemma 6. According to (3.24) and Lemma 1, we take $M := 2.8 \cdot 10^{29}$. To apply Lemma 1 for $3 \leq g \leq 9$ and $1 \leq d_1 \leq g - 2$, we define the following quantities:

$$\tau := \frac{\log g}{\log\alpha}, \quad \mu := -\frac{\log((g-1)/d_1)}{\log\alpha}, \quad A := 100, \quad B := g.$$

The results obtained following the application of the Lemma 1 are presented in Table 3.

g	3	4	5	6	7	8	9
rth convergent	q_{62}	q_{66}	q_{60}	q_{58}	q_{68}	q_{60}	q_{58}
$\varepsilon \geq$	0.07	0.18	0.14	0.01	0.14	0.05	0.03
$m_1 \leq$	72	55	47	44	40	37	36

Table 3.

Therefore, the inequalities

$$m_1 \leq \frac{\log(100q/\varepsilon)}{\log\alpha} \leq 72 \tag{3.25}$$

hold in all cases. Now, in the case $d_1 = g-1$, we have that $\mu = 0$. In this case, we will apply Lemma 2. Inequality (3.24) can be rewritten as

$$0 < \left|(m_1+m_2)\frac{\log g}{\log \alpha} - n\right| < \frac{100}{g^{m_1}}.$$

According to Lemma 2, we take $M := 2.8\times 10^{29}$ because $m_1+m_2 < 2.8\times 10^{29}$. For $2 \le g \le 9$, we use Maple to find the first convergent q_N such that $q_N > M$, and then we get $a(M) := \max\{a_i : i = 0,\dots,N\}$. Thus, Lemma 2 tells us that

$$\begin{aligned}\frac{100}{g^{m_1}} > \left|(m_1+m_2)\frac{\log g}{\log \alpha} - n\right| &> \frac{1}{(a(M)+2)(m_1+m_2)}\\ &> \frac{1}{(a(M)+2)\cdot 2.8\times 10^{29}},\end{aligned}$$

which implies

$$m_1 < \frac{\log\left(100\cdot(a(M)+2)\cdot 2.8\cdot 10^{29}\right)}{\log g}.$$

We thus arrive at the results in Table 4.

g	2	3	4	5	6	7	8	9
$q_N > M$	q_{68}	q_{61}	q_{64}	q_{58}	q_{56}	q_{66}	q_{59}	q_{57}
$a(M)$	134	161	66	59	347	35	44	80
$m_1 \le$	112	71	55	47	44	39	36	35

Table 4.

So we have

$$m_1 \le 112. \tag{3.26}$$

Combining (3.25) and (3.26), we can consider $1 \le m_1 \le 112$.

Let

$$\begin{aligned}\Lambda_4 &:= \log(\Gamma_4+1)\\ &= m_2\log g - n\log\alpha + \log\left(\frac{d_1 g^{m_1} - (d_1-d_2)}{g-1}\right).\end{aligned}$$

From (3.20) and $n > 510$, we conclude that

$$\left|e^{\Lambda_4} - 1\right| < \frac{2}{\alpha^n} < \frac{1}{2},$$

which implies that $\frac{1}{2} < e^{\Lambda_4} < \frac{3}{2}$. If $\Lambda_4 > 0$, then

$$0 < \Lambda_4 < e^{\Lambda_4} - 1 < \frac{2}{\alpha^n}.$$

If $\Lambda_4 < 0$, then

$$0 < |\Lambda_4| < e^{|\Lambda_4|} - 1 = e^{-\Lambda_4} - 1 = e^{-\Lambda_4}(1-e^{\Lambda_4}) < \frac{4}{\alpha^n}.$$

In any case, because $0 < |\Lambda_4| < \frac{4}{\alpha^n}$, we have

$$0 < \left|m_2\frac{\log g}{\log\alpha} - n + \frac{\log\left((d_1 g^{m_1} - (d_1-d_2))/(g-1)\right)}{\log\alpha}\right| < 8.4\cdot\alpha^{-n}. \tag{3.27}$$

If $g = 2$, then $d_1 = 1$ and $d_2 \in \{0, 1\}$. Assuming $d_2 = 0$ and $m_1 \neq 1$, (3.27) becomes

$$0 < \left| m_2 \frac{\log 2}{\log \alpha} - n + \frac{\log (2^{m_1} - 1)}{\log \alpha} \right| < 8.4 \cdot \alpha^{-n}. \tag{3.28}$$

So, in this case, we apply Lemma 1 with the data:

$$\tau := \frac{\log 2}{\log \alpha}, \quad \mu := \frac{\log (2^{m_1} - 1)}{\log \alpha}, \quad A := 8.4, \quad B := \alpha.$$

We take $M := 2.8 \cdot 10^{29}$. Using Maple, we find that q_{68} of $\log 2/\log \alpha$ satisfies $q_{68} > 6M$ and $\varepsilon \geq 0.003$. So according to Lemma 1, we get

$$n < \frac{\log(8.4 q_{68}/0.003)}{\log \alpha} < 166. \tag{3.29}$$

Next, assume that $d_2 = 0$ and $m_1 = 1$ or $d_2 = 1$. In this case, (3.27) becomes

$$0 < \left| a \frac{\log 2}{\log \alpha} - n \right| < 8.4 \cdot \alpha^{-n}, \quad \text{where } a \in \{m_2, m_1 + m_2\}. \tag{3.30}$$

Because q_{67} of $\log 2/\log \alpha$ satisfies $q_{68} > M$ and $a(M) = 134$, by Lemma 2, we get

$$\left| a \frac{\log 2}{\log \alpha} - n \right| > \frac{1}{136a} > \frac{1}{136 \cdot 2.8 \cdot 10^{29}}. \tag{3.31}$$

From (3.30) and (3.31), we deduce that

$$n < \frac{\log(8.4 \cdot 136 \cdot 2.8 \cdot 10^{29})}{\log \alpha} < 156. \tag{3.32}$$

From now on, we assume that $3 \leq g \leq 9$. For inequality (3.27), we study the following two cases.

Case $(d_1, m_1, d_2) \neq (1, 1, 0)$.

Here we have $\mu \neq 0$ and we apply Lemma 1 with

$$\tau := \frac{\log g}{\log \alpha}, \ \mu := \frac{\log ((d_1 g^{m_1} - (d_1 - d_2))/(g-1))}{\log \alpha}, \ A := 8.4, \ B := \alpha,$$

and $M := 2.8 \times 10^{29}$. With the help of Maple, we get the following results in Table 5.

g	3	4	5	6	7	8	9
rth convergent	q_{75}	q_{98}	q_{110}	q_{114}	q_{122}	q_{120}	q_{58}
$\varepsilon \geq$	10^{-12}	10^{-18}	10^{-25}	10^{-29}	10^{-32}	10^{-35}	10^{-34}
$n \leq$	260	327	380	415	451	482	509

Table 5.

So, we have in all cases

$$n \leq 509. \tag{3.33}$$

Case $(d_1, m_1, d_2) = (1, 1, 0)$.

In this case, inequalities (3.27) become

$$0 < \left| m_2 \frac{\log g}{\log \alpha} - n \right| < \frac{8.4}{\alpha^n}. \tag{3.34}$$

Once again, we apply Lemma 2 with $M := 2.8 \cdot 10^{29}$, while finding q_N such that $q_N > M$ and $a(M) := \{a_i : i = 0, 1, \ldots, N\}$. It follows that

$$\left| m_2 \frac{\log g}{\log \alpha} - n \right| > \frac{1}{(a(M)+2) \cdot m_2} > \frac{1}{(a(M)+2) \cdot 2.8 \times 10^{29}}. \tag{3.35}$$

Combining (3.34) and (3.35), we get

$$n < \frac{\log\left(8.4 \cdot (a(M)+2) \cdot 2.8 \cdot 10^{29}\right)}{\log \alpha}.$$

Therefore, for $3 \le g \le 9$ and using Maple, we get the following results in Table 6.

g	3	4	5	6	7	8	9
$q_N > M$	q_{61}	q_{66}	q_{59}	q_{56}	q_{67}	q_{60}	q_{57}
$a(M)$	161	66	59	347	35	44	80
$n \le$	157	155	155	158	154	154	155

Table 6.

which lead in all cases to

$$n \le 158. \tag{3.36}$$

In summary, from (3.29), (3.32), (3.33), and (3.36), we have $n \le 509$. This contradicts the assumption $n > 510$. Finally, we search for the solutions to the second Diophantine equation of (1.3) with

$$0 \le n \le 510,\ 1 \le m_1 \le 112,\ 1 \le m_2 \le 360,$$
$$2 \le g \le 9,\ 1 \le d_1 \le g-1, \text{ and } 0 \le d_2 \le g-1,$$

by applying a program written in Maple. The only solutions we obtained are listed in Theorem 3. This completes the proof.

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