ADDITIONAL SUMS INVOLVING GIBONACCI POLYNOMIALS

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Abstract. We continue the exploration of sums involving gibonacci polynomials and their numeric versions, and their Pell versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4, 5].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2+4}$, and $E = \sqrt{x^2+1}$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \dfrac{1}{g_m} = 0$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [4, 5]:

$$f_{n+k} - f_{n-k} = \begin{cases} f_n l_k, & \text{if } k \text{ is odd;} \\ f_k l_n, & \text{otherwise;} \end{cases} \tag{1}$$

$$l_{n+k} - l_{n-k} = \begin{cases} l_k l_n, & \text{if } k \text{ is odd;} \\ \Delta^2 f_k f_n, & \text{otherwise;} \end{cases} \tag{2}$$

$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n; \tag{3}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise.} \end{cases} \tag{4}$$

These properties can be established using the Binet-like formulas.

2. Gibonacci Polynomial Sums

We begin our explorations with four telescoping sums. Coupled with the above identities, they play a pivotal role in our discourse.

2.1. **Telescoping Sums.**

Lemma 1. *Let k be an odd positive integer. Then,*

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}.$$

Proof. Using recursion [4], we will first establish that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-2r+k}}.$$

To this end, we let A_m denote the left-hand side (LHS) of this equation and B_m its right-hand side (RHS). Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-2(r+1)+k}} - \frac{1}{g_{2m-2r+k}} \right] \\ &= \frac{1}{g_{2m-k}} - \frac{1}{g_{2m+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this implies that

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} \\ &= \left(\frac{1}{g_1} - \frac{1}{g_{2k+1}} \right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-(2r-1)}} \right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$.

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, this yields the desired result. □

Lemma 2. *Let k be an even positive integer. Then,*

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}.$$

Proof. Using recursion [4], we will first confirm that

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{m} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-2r+k}}.$$

Again, we let A_m = LHS of this equation and B_m its RHS. Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-2(r+1)+k}} - \frac{1}{g_{2m-2r+k}} \right] \\ &= \frac{1}{g_{2m-k}} - \frac{1}{g_{2m+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

This implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{k/2+1} - B_{k/2+1} \\ &= \left(\frac{1}{g_2} - \frac{1}{g_{2k+2}}\right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r}} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-2(r-1)}}\right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$, as expected.

The given result follows from this formula. □

Lemma 3. *Let k be an odd positive integer. Then,*

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}.$$

Proof. Using recursion [4], we will first validate the formula

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-(2r-1)+k}}.$$

As before, we let $A_m = \text{LHS}$ and $B_m = \text{RHS}$. Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-(2r+1)+k}} - \frac{1}{g_{2m-(2r-1)+k}}\right] \\ &= \frac{1}{g_{2m+1-k}} - \frac{1}{g_{2m+1+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

This implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} \\ &= \left(\frac{1}{g_2} - \frac{1}{g_{2k+2}}\right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r}} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-2(r-1)}}\right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$.

The given result follows from this formula. □

Lemma 4. *Let k be an even positive integer. Then,*

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}.$$

Proof. To establish this formula, we will first confirm using recursion [4] that

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{m} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-(2r-1)+k}}.$$

Let A_m = LHS and B_m = RHS of this equation. Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1}\left[\frac{1}{g_{2m-(2r+1)+k}} - \frac{1}{g_{2m-(2r-1)+k}}\right] \\ &= \frac{1}{g_{2m+1-k}} - \frac{1}{g_{2m+1+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{k/2} - B_{k/2} \\ &= \left(\frac{1}{g_1} - \frac{1}{g_{2k+1}}\right) - \left[\sum_{r=1}^{k}\frac{1}{g_{2r-1}} - \sum_{r=0}^{k-1}\frac{1}{g_{2k-(2r-1)}}\right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$, establishing the validity of the given formula. □

With these tools at our disposal, we are now ready for the explorations.

Theorem 1. *Let k be a positive integer; $1 \le r \le k$;*

$$L = \begin{cases} (k+1)/2, k \ge 1, & \text{if } k \text{ is odd;} \\ k/2+1, k \ge 2, & \text{otherwise;} \end{cases} \qquad a_n = \begin{cases} f_n, & \text{if } k \text{ is odd;} \\ l_n, & \text{otherwise;} \end{cases}$$

$$s = \begin{cases} 2r-1, & \text{if } k \text{ is odd;} \\ 2r, & \text{otherwise;} \end{cases} \qquad \text{and} \quad d_k = \begin{cases} l_k, & \text{if } k \text{ is odd;} \\ f_k, & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{n=L}^{\infty} \frac{a_{2n}}{f_{2n}^2 - (-1)^k f_k^2} = \frac{1}{d_k}\sum_{r=1}^{k}\frac{1}{f_s}. \tag{5}$$

Proof. Suppose k is odd. With identities (1) and (4), and Lemma 1, we have

$$\begin{aligned} \frac{f_{2n}l_k}{f_{2n}^2 + f_k^2} &= \frac{f_{2n+k} - f_{2n-k}}{f_{2n+k}f_{2n-k}}, \\ \sum_{\substack{n=(k+1)/2 \\ k\ge 1,\, odd}}^{\infty} \frac{f_{2n}l_k}{f_{2n}^2 + f_k^2} &= \sum_{\substack{n=(k+1)/2 \\ k\ge 1,\, odd}}^{\infty}\left(\frac{1}{f_{2n-k}} - \frac{1}{f_{2n+k}}\right) \\ &= \sum_{r=1}^{k}\frac{1}{f_{2r-1}}. \end{aligned}$$

On the other hand, let k be even. Using identities (1) and (4), and Lemma 2, we get

$$\begin{aligned} \frac{f_k l_{2n}}{f_{2n}^2 - f_k^2} &= \frac{f_{2n+k} - f_{2n-k}}{f_{2n+k}f_{2n-k}}, \\ \sum_{\substack{n=k/2+1 \\ k\ge 2,\, even}}^{\infty} \frac{f_k l_{2n}}{f_{2n}^2 - f_k^2} &= \sum_{\substack{n=k/2+1 \\ k\ge 2,\, even}}^{\infty}\left(\frac{1}{f_{2n-k}} - \frac{1}{f_{2n+k}}\right) \\ &= \sum_{r=1}^{k}\frac{1}{f_{2r}}. \end{aligned}$$

Combining the two cases yields the desired result. □

In particular, with the identity $f_{2n} = f_n l_n$ [4], we get

$$\sum_{n=1}^{\infty} \frac{f_{2n}}{f_{2n}^2 + 1} = \frac{1}{f_2}; \qquad \sum_{n=1}^{\infty} \frac{F_{2n}}{F_{2n}^2 + 1} = 1;$$
$$\sum_{n=2}^{\infty} \frac{l_{2n}}{f_{2n}^2 - x^2} = \frac{l_3}{f_2^2 f_4}; \qquad \sum_{n=2}^{\infty} \frac{L_{2n}}{F_{2n}^2 - 1} = \frac{4}{3}.$$

With identity (3), we can rewrite equation (5) as

$$\sum_{n=L}^{\infty} \frac{a_{2n}}{l_{2n}^2 - (-1)^k \Delta^2 f_k^2 - 4} = \frac{1}{\Delta^2 d_k} \sum_{r=1}^{k} \frac{1}{f_s}. \tag{6}$$

This implies

$$\sum_{n=1}^{\infty} \frac{f_{2n}}{l_{2n}^2 + \Delta^2 - 4} = \frac{1}{\Delta^2 f_2}; \qquad \sum_{n=1}^{\infty} \frac{F_{2n}}{L_{2n}^2 + 1} = \frac{1}{5};$$
$$\sum_{n=1}^{\infty} \frac{l_{2n}}{l_{2n}^2 - (x^2+2)^2} = \frac{l_3}{\Delta^2 f_2^2 f_4}; \qquad \sum_{n=2}^{\infty} \frac{L_{2n}}{L_{2n}^2 - 9} = \frac{4}{15}.$$

The next result employs identities (1) and (4).

Theorem 2. *Let k be a positive integer; $1 \le r \le k$;*

$$M = \begin{cases} (k+1)/2, k \ge 1, & \text{if } k \text{ is odd;} \\ k/2, k \ge 2, & \text{otherwise;} \end{cases} \qquad a_n = \begin{cases} f_n, & \text{if } k \text{ is odd;} \\ l_n, & \text{otherwise;} \end{cases}$$
$$t = \begin{cases} 2r, & \text{if } k \text{ is odd;} \\ 2r-1, & \text{otherwise;} \end{cases} \qquad \text{and} \quad d_k = \begin{cases} l_k, & \text{if } k \text{ is odd;} \\ f_k, & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{n=M}^{\infty} \frac{a_{2n+1}}{f_{2n+1}^2 + (-1)^k f_k^2} = \frac{1}{d_k} \sum_{r=1}^{k} \frac{1}{f_t}. \tag{7}$$

Proof. With k odd, equations (1) and (4), and Lemma 3, we have

$$\frac{f_{2n+1} l_k}{f_{2n+1}^2 - f_k^2} = \frac{f_{2n+1+k} - f_{2n+1-k}}{f_{2n+1+k} f_{2n+1-k}},$$
$$\sum_{\substack{n=(k+1)/2 \\ k \ge 1,\, odd}}^{\infty} \frac{f_{2n+1} l_k}{f_{2n+1}^2 - f_k^2} = \sum_{\substack{n=(k+1)/2 \\ k \ge 1,\, odd}}^{\infty} \left(\frac{1}{f_{2n+1-k}} - \frac{1}{f_{2n+1+k}} \right)$$
$$= \sum_{r=1}^{k} \frac{1}{f_{2r}}.$$

Now, let k be even. Using equations (1) and (4), and Lemma 4, we get

$$\begin{aligned} \frac{f_k l_{2n+1}}{f_{2n+1}^2 + f_k^2} &= \frac{f_{2n+1+k} - f_{2n+1-k}}{f_{2n+1+k} f_{2n+1-k}}, \\ \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{f_k l_{2n+1}}{f_{2n+1}^2 + f_k^2} &= \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{f_{2n+1-k}} - \frac{1}{f_{2n+1+k}} \right) \\ &= \sum_{r=1}^{k} \frac{1}{f_{2r-1}}. \end{aligned}$$

By combining both cases, we get the given result, as desired. □

In particular, with the identity $f_{n+1} + f_{n-1} = l_n$ [4], we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{f_{2n+1}}{f_{2n+1}^2 - 1} &= \frac{1}{f_2^2}; & \sum_{n=1}^{\infty} \frac{F_{2n+1}}{F_{2n+1}^2 - 1} &= 1; \\ \sum_{n=1}^{\infty} \frac{l_{2n+1}}{f_{2n+1}^2 + x^2} &= \frac{l_2}{f_2 f_3}; & \sum_{n=1}^{\infty} \frac{L_{2n+1}}{F_{2n+1}^2 + 1} &= \frac{3}{2}. \end{aligned}$$

With identity (3), equation (7) yields

$$\sum_{n=M}^{\infty} \frac{a_{2n+1}}{l_{2n+1}^2 + (-1)^k \Delta^2 f_k^2 + 4} = \frac{1}{\Delta^2 d_k} \sum_{r=1}^{k} \frac{1}{f_t}. \tag{8}$$

This implies

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{f_{2n+1}}{l_{2n+1}^2 - \Delta^2 + 4} &= \frac{1}{\Delta^2 f_2^2}; & \sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_{2n+1}^2 - 1} &= \frac{1}{5}; \\ \sum_{n=1}^{\infty} \frac{l_{2n+1}}{l_{2n+1}^2 + (x^2+2)^2} &= \frac{l_2}{\Delta^2 f_2 f_3}; & \sum_{n=1}^{\infty} \frac{L_{2n+1}}{L_{2n+1}^2 + 9} &= \frac{3}{10}. \end{aligned}$$

The next theorem features the Lucas version of Theorem 1.

Theorem 3. *Let k be a positive integer; $1 \leq r \leq k$;*

$$\begin{aligned} L &= \begin{cases} (k+1)/2, k \geq 1, & \text{if } k \text{ is odd;} \\ k/2 + 1, k \geq 2, & \text{otherwise;} \end{cases} & h_n &= \begin{cases} l_n, & \text{if } k \text{ is odd;} \\ f_n, & \text{otherwise;} \end{cases} \\ s &= \begin{cases} 2r - 1, & \text{if } k \text{ is odd;} \\ 2r, & \text{otherwise;} \end{cases} & \text{and} \quad e_k &= \begin{cases} l_k, & \text{if } k \text{ is odd;} \\ \Delta^2 f_k, & \text{otherwise.} \end{cases} \end{aligned}$$

Then,

$$\sum_{n=L}^{\infty} \frac{h_{2n}}{l_{2n}^2 + (-1)^k \Delta^2 f_k^2} = \frac{1}{e_k} \sum_{r=1}^{k} \frac{1}{l_s}. \tag{9}$$

Proof. Suppose k is odd. With identities (2) and (4), and Lemma 1, we have

$$\begin{aligned}\frac{l_k l_{2n}}{l_{2n}^2-\Delta^2 f_k^2} &= \frac{l_{2n+k}-l_{2n-k}}{l_{2n+k}l_{2n-k}},\\ \sum_{\substack{n=(k+1)/2\\ k\ge 1,\, odd}}^{\infty}\frac{l_k l_{2n}}{l_{2n}^2-\Delta^2 f_k^2} &= \sum_{\substack{n=(k+1)/2\\ k\ge 1,\, odd}}^{\infty}\left(\frac{1}{l_{2n-k}}-\frac{1}{l_{2n+k}}\right)\\ &= \sum_{r=1}^{k}\frac{1}{l_{2r-1}}.\end{aligned}$$

Now, let k be even. Using identities (2) and (4), and Lemma 2, we have

$$\begin{aligned}\frac{\Delta^2 f_k f_{2n}}{l_{2n}^2+\Delta^2 f_k^2} &= \frac{l_{2n+k}-l_{2n-k}}{l_{2n+k}l_{2n-k}},\\ \sum_{\substack{n=k/2+1\\ k\ge 2,\, even}}^{\infty}\frac{\Delta^2 f_k f_{2n}}{l_{2n}^2+\Delta^2 f_k^2} &= \sum_{\substack{n=k/2+1\\ k\ge 2,\, even}}^{\infty}\left(\frac{1}{l_{2n-k}}-\frac{1}{l_{2n+k}}\right)\\ &= \sum_{r=1}^{k}\frac{1}{l_{2r}}.\end{aligned}$$

The given result now follows by combining the two cases. □

With the identities $f_{2n}=f_n l_n$ and $l_{n+1}+l_{n-1}=\Delta^2 f_n$ [4], it follows from this theorem that

$$\sum_{n=1}^{\infty}\frac{l_{2n}}{l_{2n}^2-\Delta^2} = \frac{1}{l_1^2}; \qquad \sum_{n=1}^{\infty}\frac{L_{2n}}{L_{2n}^2-5} = 1;$$
$$\sum_{n=2}^{\infty}\frac{f_{2n}}{l_{2n}^2+\Delta^2 x^2} = \frac{f_3}{f_8}; \qquad \sum_{n=2}^{\infty}\frac{F_{2n}}{L_{2n}^2+5} = \frac{2}{21}.$$

Using the identity $l_n^2-\Delta^2 f_n^2=4(-1)^n$, we can rewrite equation (9) as

$$\sum_{n=L}^{\infty}\frac{h_{2n}}{\Delta^2 f_{2n}^2+(-1)^k\Delta^2 f_k^2+4} = \frac{1}{e_k}\sum_{r=1}^{k}\frac{1}{l_s}. \tag{10}$$

It then follows that

$$\sum_{n=1}^{\infty}\frac{l_{2n}}{\Delta^2 f_{2n}^2-x^2} = \frac{1}{l_1^2}; \qquad \sum_{n=1}^{\infty}\frac{L_{2n}}{5F_{2n}^2-1} = 1;$$
$$\sum_{n=2}^{\infty}\frac{f_{2n}}{\Delta^2 f_{2n}^2+(x^2+2)^2} = \frac{f_3}{f_8}; \qquad \sum_{n=2}^{\infty}\frac{F_{2n}}{5F_{2n}^2+9} = \frac{2}{21}.$$

Finally, we present the Lucas version of Theorem 2.

Theorem 4. *Let k be a positive integer; $1\le r\le k$;*

$$M = \begin{cases}(k+1)/2, k\ge 1, & \text{if } k \text{ is odd;}\\ k/2, k\ge 2, & \text{otherwise;}\end{cases} \qquad t = \begin{cases}2r, & \text{if } k \text{ is odd;}\\ 2r-1, & \text{otherwise;}\end{cases}$$
$$h_n = \begin{cases}l_n, & \text{if } k \text{ is odd;}\\ f_n, & \text{otherwise;}\end{cases} \qquad \text{and} \quad e_k = \begin{cases}l_k, & \text{if } k \text{ is odd;}\\ \Delta^2 f_k, & \text{otherwise.}\end{cases}$$

Then,

$$\sum_{n=M}^{\infty} \frac{h_{2n+1}}{l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2} = \frac{1}{e_k} \sum_{r=1}^{k} \frac{1}{l_t}. \tag{11}$$

Proof. Suppose k is odd. Using identities (2) and (4), and Lemma 3, we have

$$\begin{aligned}
\frac{l_k l_{2n+1}}{l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2} &= \frac{l_{2n+1+k} - l_{2n+1-k}}{l_{2n+1+k} l_{2n+1-k}}, \\
\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{l_k l_{2n+1}}{l_{2n+1}^2 + \Delta^2 f_k^2} &= \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{l_{2n+1-k}} - \frac{1}{l_{2n+1+k}} \right) \\
&= \sum_{r=1}^{k} \frac{1}{l_{2r}}.
\end{aligned}$$

On the flip side, let k be even. By identities (2) and (4), and Lemma 4, we get

$$\begin{aligned}
\frac{\Delta^2 f_k f_{2n+1}}{l_{2n+1}^2 - \Delta^2 f_k^2} &= \frac{l_{2n+1+k} - l_{2n+1-k}}{l_{2n+1+k} l_{2n+1-k}}, \\
\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_k f_{2n+1}}{l_{2n+1}^2 - \Delta^2 f_k^2} &= \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{l_{2n+1-k}} - \frac{1}{l_{2n+1+k}} \right) \\
&= \sum_{r=1}^{k} \frac{1}{l_{2r-1}}.
\end{aligned}$$

Combining the two cases yields the desired result. □

In particular, using the identity $l_{n+1} + l_{n-1} = \Delta^2 f_n$ [4], we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{l_{2n+1}}{l_{2n+1}^2 + \Delta^2} &= \frac{1}{l_1 l_2}; & \sum_{n=1}^{\infty} \frac{L_{2n+1}}{L_{2n+1}^2 + 5} &= \frac{1}{3}; \\
\sum_{n=1}^{\infty} \frac{f_{2n+1}}{l_{2n+1}^2 - \Delta^2 x^2} &= \frac{1}{l_1 l_3}; & \sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_{2n+1}^2 - 5} &= \frac{1}{4}.
\end{aligned}$$

With identity (3), we can rewrite equation (11) in a slightly different way:

$$\sum_{n=1}^{\infty} \frac{h_{2n+1}}{\Delta^2 f_{2n+1}^2 - (-1)^k \Delta^2 f_k^2 - 4} = \frac{1}{e_k} \sum_{r=1}^{k} \frac{1}{l_t}. \tag{12}$$

This implies

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{l_{2n+1}}{\Delta^2 f_{2n+1}^2 + x^2} &= \frac{1}{l_1 l_2}; & \sum_{n=1}^{\infty} \frac{L_{2n+1}}{5F_{2n+1}^2 + 1} &= \frac{1}{3}; \\
\sum_{n=1}^{\infty} \frac{f_{2n+1}}{\Delta^2 f_{2n+1}^2 - (x^2+2)^2} &= \frac{1}{l_1 l_3}; & \sum_{n=1}^{\infty} \frac{F_{2n+1}}{5F_{2n+1}^2 - 9} &= \frac{1}{4}.
\end{aligned}$$

3. Pell Consequences

With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, Theorems 1–4 yield the following Pell versions:

$$\sum_{n=L}^{\infty} \frac{a_{2n}^*}{p_{2n}^2 - (-1)^k p_k^2} = \frac{1}{d_k^*} \sum_{r=1}^{k} \frac{1}{p_s}; \qquad \sum_{n=M}^{\infty} \frac{a_{2n+1}^*}{p_{2n+1}^2 + (-1)^k p_k^2} = \frac{1}{d_k^*} \sum_{r=1}^{k} \frac{1}{p_t};$$
$$\sum_{n=L}^{\infty} \frac{h_{2n}^*}{q_{2n}^2 + 4(-1)^k E^2 p_k^2} = \frac{1}{e_k^*} \sum_{r=1}^{k} \frac{1}{q_s}; \qquad \sum_{n=M}^{\infty} \frac{h_{2n+1}^*}{q_{2n+1}^2 - 4(-1)^k E^2 p_k^2} = \frac{1}{e_k^*} \sum_{r=1}^{k} \frac{1}{q_t},$$

respectively, where $a_n^* = a_n(2)$, $d_n^* = d_n(2)$, $e_n^* = e_n(2)$, and $h_n^* = h_n(2)$. In the interest of brevity, we omit their numeric versions and encourage gibonacci enthusiasts to explore them.

4. Chebyshev and Vieta Implications

Finally, we add that Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$ [2, 3, 4], where $i = \sqrt{-1}$. They can be employed to find the Chebyshev and Vieta versions of Theorems 1–4. Again, we omit them and encourage gibonacci enthusiasts to pursue them.

5. Acknowledgment

The author thanks the reviewer for the careful reading of the article and encouraging words.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] A. F. Horadam, *Vieta polynomials*, The Fibonacci Quarterly, **40.3** (2002), 223–232.
[3] T. Koshy, *Vieta polynomials and their close relatives*, The Fibonacci Quarterly, **54.2** (2016), 141–148.
[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[5] T. Koshy, *Sums involving gibonacci polynomials*, The Fibonacci Quarterly, **60.4** (2022), 344–351.

MSC2020: Primary 11B37, 11B39, 11C08

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