ADDITIONAL SUMS INVOLVING GIBONACCI POLYNOMIALS REVISITED

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Abstract. We explore three sums involving gibonacci polynomials and extract their Pell versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, $2\alpha(x) = x + \Delta$, $2\beta(x) = x - \Delta$, $E = \sqrt{x^2+1}$, $\gamma(x) = x + E$, $\delta(x) = x - E$, $\gamma = \gamma(1)$, and $\delta = \delta(1)$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_m} = 0$ and $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [5, 7]:

$$f_{n+k} - f_{n-k} = \begin{cases} f_n l_k, & \text{if } k \text{ is odd;} \\ f_k l_n, & \text{otherwise;} \end{cases} \tag{1}$$

$$l_{n+k} - l_{n-k} = \begin{cases} l_k l_n, & \text{if } k \text{ is odd;} \\ \Delta^2 f_k f_n, & \text{otherwise;} \end{cases} \tag{2}$$

$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n; \tag{3}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{4}$$

$$g_{n+k+1}g_{n-k} - g_{n+k}g_{n-k+1} = \begin{cases} (-1)^{n+k+1} f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_{2k}, & \text{otherwise.} \end{cases} \tag{5}$$

These properties can be confirmed using the Binet-like formulas.

1.2. **Telescoping Sums.** We studied the following telescoping sums in [7]:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}; \tag{6}$$

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}; \tag{7}$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}; \tag{8}$$

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}. \tag{9}$$

The next lemma presents an additional telescoping sum.

Lemma 1. *Let k be an odd positive integer. Then,*

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{g_{2n+1-k}}{g_{2n-k}} - \frac{g_{2n+1+k}}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{g_{2r}}{g_{2r-1}} - k\alpha(x). \tag{10}$$

Proof. Using recursion [5], we will first establish that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{g_{2n+1-k}}{g_{2n-k}} - \frac{g_{2n+1+k}}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{g_{2r}}{g_{2r-1}} - \sum_{r=1}^{k} \frac{g_{2m+1+2r-k}}{g_{2m+2r-k}}.$$

To this end, we let A_m denote the left-hand side of this equation and B_m its right-hand side. Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=1}^{k} \frac{g_{2m-1+2r-k}}{g_{2m-2+2r-k}} - \sum_{r=1}^{k} \frac{g_{2m+1+2r-k}}{g_{2m+2r-k}} \\ &= \frac{g_{2m+1-k}}{g_{2m-k}} - \frac{g_{2m+1+k}}{g_{2m+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} \\ &= \left(\frac{g_2}{g_1} - \frac{g_{2k+2}}{g_{2k+1}}\right) - \left(\frac{g_2}{g_1} - \frac{g_{2k+2}}{g_{2k+1}}\right) \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$.

Because $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$, the given result now follows, as desired. □

This lemma has an interesting consequence:

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{g_{2n-k}}{g_{2n+1-k}} - \frac{g_{2n+k}}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{g_{2r-1}}{g_{2r}} + k\beta(x). \tag{11}$$

Coupled with the above identities, the telescoping sums play a major role in our explorations.

2. Additional Gibonacci Polynomial Sums

With the above tools at our disposal, we are now ready for further explorations.

Theorem 1. *Let k be an odd positive integer and $i = \sqrt{-1}$. Then,*

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{f_{2n} + if_k} = \sum_{r=1}^{k} \frac{1}{f_{2r-1}} + i\sum_{r=1}^{k} \frac{f_{2r}}{f_{2r-1}} - ik\alpha(x). \tag{12}$$

Proof. Using the identity $f_{2n} = f_n l_n$ [5], equations (4), (5), (6), and (10), and k with odd parity, we get

$$\begin{aligned}
\frac{l_k}{f_{2n} + if_k} &= \frac{l_k(f_{2n} - if_k)}{f_{2n}^2 + f_k^2} \\
&= \frac{l_k f_{2n} - if_{2k}}{f_{2n+k} f_{2n-k}} \\
&= \frac{f_{2n+k} - f_{2n-k}}{f_{2n+k} f_{2n-k}} - i\left(\frac{f_{2n+k+1} f_{2n-k} - f_{2n+k} f_{2n-k+1}}{f_{2n+k} f_{2n-k}}\right), \\
\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{f_{2n} + if_k} &= \sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{f_{2n-k}} - \frac{1}{f_{2n+k}}\right) + i\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{f_{2n+1-k}}{f_{2n-k}} - \frac{f_{2n+1+k}}{f_{2n+k}}\right) \\
&= \sum_{r=1}^{k} \frac{1}{f_{2r-1}} + i\sum_{r=1}^{k} \frac{f_{2r}}{f_{2r-1}} - ik\alpha(x),
\end{aligned}$$

as desired. □

In particular, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{x}{f_{2n} + i} &= 1 + i\beta(x); \\
\sum_{n=2}^{\infty} \frac{l_3}{f_{2n} + i(x^2+1)} &= \left(\frac{1}{f_1} + \frac{1}{f_3} + \frac{1}{f_5}\right) + i\left(\frac{f_2}{f_1} + \frac{f_4}{f_3} + \frac{f_6}{f_5}\right) - i3\alpha(x).
\end{aligned}$$

It then follows that [2]

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n} + i} = 1 + i\beta; \qquad \sum_{n=2}^{\infty} \frac{1}{F_{2n} + 2i} = \frac{17}{40} + \frac{26 - 15\sqrt{5}}{40} i.$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}-i} = 1 - i\beta; \qquad \sum_{n=1}^{\infty} \frac{F_{2n}}{F_{2n}^2+1} = 1; \text{ [7]}$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2+1} = -\beta; \text{ [6, 7]} \qquad \sum_{n=2}^{\infty} \frac{1}{F_{2n}-2i} = \frac{17}{40} - \frac{26-15\sqrt{5}}{40} i;$$
$$\sum_{n=2}^{\infty} \frac{F_{2n}}{F_{2n}^2+4} = \frac{17}{20}; \qquad \sum_{n=2}^{\infty} \frac{1}{F_{2n}^2+4} = -\frac{13}{40} + \frac{3\sqrt{5}}{16}.$$

This theorem has an interesting byproduct, as the following corollary shows.

Corollary 1.

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{f_{2n} - i f_k} = \sum_{r=1}^{k} \frac{1}{f_{2r-1}} - i \sum_{r=1}^{k} \frac{f_{2r}}{f_{2r-1}} + ik\alpha(x). \tag{13}$$

Adding equations (12) and (13), we get [7]

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{2n}}{f_{2n}^2 + f_k^2} = \frac{1}{l_k} \sum_{r=1}^{k} \frac{1}{f_{2r-1}}. \tag{14}$$

Its validity can be established independently [7] by using the relationship

$$\frac{f_{2n} l_k}{f_{2n}^2 + f_k^2} = \frac{f_{2n+k} - f_{2n-k}}{f_{2n+k} f_{2n-k}},$$

where k is odd.

It follows from equation (14) that [7]

$$\sum_{n=1}^{\infty} \frac{f_{2n}}{f_{2n}^2+1} = \frac{1}{l_1}; \qquad \sum_{n=2}^{\infty} \frac{f_{2n}}{f_{2n}^2+(x^2+1)^2} = \frac{1}{l_3}\left(\frac{1}{f_1} + \frac{1}{f_3} + \frac{1}{f_5}\right).$$

Consequently, we have [7]

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{F_{2n}^2+1} = 1; \text{ [7]} \qquad \sum_{n=2}^{\infty} \frac{F_{2n}}{F_{2n}^2+4} = \frac{17}{40},$$

respectively.

It also follows by equations (12) and (13) that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{1}{f_{2n}^2 + f_k^2} = \frac{1}{f_{2k}} \left[k\alpha(x) - \sum_{r=1}^{k} \frac{f_{2r}}{f_{2r-1}} \right].$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{f_{2n}^2+1} = -\frac{\beta(x)}{x}; \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2n}^2+1} = -\beta,$$

as found earlier.

Using identity (3), we can rewrite equation (14) as

$$\sum_{n=1}^{\infty} \frac{f_{2n}}{l_{2n}^2 + \Delta^2 f_k^2 - 4} = \frac{1}{\Delta^2\, l_k} \sum_{r=1}^{k} \frac{1}{f_{2r-1}},$$

where k is odd. This implies

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{L_{2n}^2+1} = \frac{1}{5}; \qquad \sum_{n=1}^{\infty}\frac{F_{2n}}{L_{2n}^2+16} = \frac{17}{200}.$$

The next result is an application of identities (2), (4), and (5).

Theorem 2. *Let k be an odd positive integer. Then,*

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{l_{2n}-\Delta f_k} = \sum_{r=1}^{k}\frac{1}{l_{2r-1}} + \frac{1}{\Delta}\sum_{r=1}^{k}\frac{l_{2r}}{l_{2r-1}} - \frac{k\alpha(x)}{\Delta}. \tag{15}$$

Proof. Using the identity $f_{2n}=f_n l_n$ [5], and equations (2), (4), (5), (6), and (10), we get

$$\begin{aligned}
\frac{l_k}{l_{2n}-\Delta f_k} &= \frac{l_k(l_{2n}+\Delta f_k)}{l_{2n}^2-\Delta^2 f_k^2}\\
&= \frac{l_k l_{2n}}{l_{2n+k}l_{2n-k}} + \frac{\Delta f_{2k}}{l_{2n+k}l_{2n-k}}\\
&= \frac{l_{2n+k}-l_{2n-k}}{l_{2n+k}l_{2n-k}} - \frac{1}{\Delta}\left(\frac{l_{2n+1+k}l_{2n-k}-l_{2n+k}l_{2n+1-k}}{l_{2n+k}l_{2n-k}}\right),\\
\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\frac{l_k}{l_{2n}-\Delta f_k} &= \sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\left(\frac{1}{l_{2n-k}}-\frac{1}{l_{2n+k}}\right) + \frac{1}{\Delta}\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty}\left(\frac{l_{2n+1-k}}{l_{2n-k}}-\frac{l_{2n+1+k}}{l_{2n+k}}\right)\\
&= \sum_{r=1}^{k}\frac{1}{l_{2r-1}} + \frac{1}{\Delta}\sum_{r=1}^{k}\frac{l_{2r}}{l_{2r-1}} - \frac{k\alpha(x)}{\Delta},
\end{aligned}$$

as expected. □

This theorem also has an interesting implication, as the next corollary shows.

Corollary 2.

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{l_{2n}+\Delta f_k} = \sum_{r=1}^{k}\frac{1}{l_{2r-1}} - \frac{1}{\Delta}\sum_{r=1}^{k}\frac{l_{2r}}{l_{2r-1}} + \frac{k\alpha(x)}{\Delta}. \tag{16}$$

It follows from equations (15) and (16) that

$$\sum_{n=1}^{\infty}\frac{1}{l_{2n}-\Delta} = \frac{\Delta+l_2-l_1\alpha(x)}{\Delta\, l_1^2}; \qquad \sum_{n=1}^{\infty}\frac{1}{l_{2n}+\Delta} = \frac{\Delta-l_2+l_1\alpha(x)}{\Delta\, l_1^2},$$

respectively. They yield

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}-\sqrt{5}} = \frac{1+\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{2n}+\sqrt{5}} = \frac{3-\sqrt{5}}{2},$$

again respectively.

It also follows by equations (15) and (16) that [7]

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_{2n}}{l_{2n}^2 - \Delta^2 f_k^2} = \frac{1}{l_k}\sum_{r=1}^{k} \frac{1}{l_{2r-1}}; \tag{17}$$

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{1}{l_{2n}^2 - \Delta^2 f_k^2} = \frac{1}{\Delta^2 f_{2k}}\left[\sum_{r=1}^{k} \frac{l_{2r}}{l_{2r-1}} - k\alpha(x)\right].$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{L_{2n}}{L_{2n}^2 - 5} = 1; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2 - 5} = \frac{5-\sqrt{5}}{10}.$$

With identity (3), we can rewrite equation (17) as

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_{2n}}{\Delta^2\left(f_{2n}^2 - f_k^2\right) + 4} = \frac{1}{l_k}\sum_{r=1}^{k} \frac{1}{l_{2r-1}}.$$

This implies

$$\sum_{n=1}^{\infty} \frac{l_{2n}}{\Delta^2 f_{2n}^2 - x^2} = \frac{1}{l_1^2}; \qquad \sum_{n=1}^{\infty} \frac{L_{2n}}{5F_{2n}^2 - 1} = 1.$$

The next result invokes the telescoping sums (8) and (11).

Theorem 3. *Let k be an odd positive integer and $i = \sqrt{-1}$. Then,*

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{l_{2n+1} + i\Delta f_k} = \sum_{r=1}^{k} \frac{1}{l_{2r}} - \frac{i}{\Delta}\left[\sum_{r=1}^{k} \frac{l_{2r-1}}{l_{2r}} + k\beta(x)\right]. \tag{18}$$

Proof. Using the identity $f_{2n} = f_n l_n$, and equations (2), (4), (5), (8), and (11), we have

$$\begin{aligned}
\frac{l_k}{l_{2n+1} + i\Delta f_k} &= \frac{l_k(l_{2n+1} - i\Delta f_k)}{l_{2n+1}^2 + \Delta^2 f_k^2}\\
&= \frac{l_{2n+1+k} - l_{2n+1-k}}{l_{2n+1+k}l_{2n+1-k}} - \frac{i}{\Delta}\left(\frac{l_{2n+1+k}l_{2n-k} - l_{2n+k}l_{2n+1-k}}{l_{2n+1+k}l_{2n+1-k}}\right),\\
\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{l_{2n+1} + i\Delta f_k} &= \sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{l_{2n+1-k}} - \frac{1}{l_{2n+1+k}}\right) - \frac{i}{\Delta}\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{l_{2n-k}}{l_{2n+1-k}} - \frac{l_{2n+k}}{l_{2n+1+k}}\right)\\
&= \sum_{r=1}^{k} \frac{1}{l_{2r}} - \frac{i}{\Delta}\left[\sum_{r=1}^{k} \frac{l_{2r-1}}{l_{2r}} + k\beta(x)\right],
\end{aligned}$$

as desired. □

The next result follows from equation (18).

Corollary 3. *Let k be an odd positive integer and $i = \sqrt{-1}$. Then,*

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{l_k}{l_{2n+1} - i\Delta f_k} = \sum_{r=1}^{k} \frac{1}{l_{2r}} + \frac{i}{\Delta}\left[\sum_{r=1}^{k} \frac{l_{2r-1}}{l_{2r}} + k\beta(x)\right].$$

Theorem 3, coupled with Corollary 3, yields [7]

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{l_{2n+1}}{l_{2n+1}^2 + \Delta^2 f_k^2} = \frac{1}{l_k} \sum_{r=1}^{k} \frac{1}{l_{2r}}. \tag{19}$$

This can be confirmed independently [7] using the equation

$$\frac{l_k l_{2n+1}}{l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2} = \frac{l_{2n+1+k} - l_{2n+1-k}}{l_{2n+1+k} l_{2n+1-k}},$$

where k is odd.

It also follows by Theorem 3 and Corollary 3 that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{1}{l_{2n+1}^2 + \Delta^2 f_k^2} = \frac{1}{\Delta^2 f_k} \left[\sum_{r=1}^{k} \frac{l_{2r-1}}{l_{2r}} + k\beta(x) \right].$$

Thus, we have

$$\sum_{n=1}^{\infty} \frac{l_{2n+1}}{l_{2n+1}^2 + \Delta^2} = \frac{1}{l_1 l_2}; \qquad \sum_{n=1}^{\infty} \frac{L_{2n+1}}{L_{2n+1}^2 + 5} = \frac{1}{3};$$

$$\sum_{n=1}^{\infty} \frac{1}{l_{2n+1}^2 + \Delta^2} = \frac{1}{\Delta^2} \left[\frac{l_1}{l_2} + \beta(x) \right]; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 + 5} = \frac{1}{6} - \frac{\sqrt{5}}{10}.$$

Using identity (3), we can rewrite equation (19) in a different way:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{l_{2n+1}}{\Delta^2 f_{2n+1}^2 + \Delta^2 f_k^2 - 4} = \frac{1}{l_k} \sum_{r=1}^{k} \frac{1}{l_{2r}}.$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{l_{2n+1}}{\Delta^2 f_{2n+1}^2 + \Delta^2 - 4} = \frac{1}{l_1 l_2}; \qquad \sum_{n=1}^{\infty} \frac{L_{2n+1}}{5F_{2n+1}^2 + 1} = \frac{1}{3}.$$

3. Pell Implications

Using the relationship $b_n(x) = g_n(2x)$, we can find the Pell versions of gibonacci formulas. For example, those of equations (12), (15), and (18) are:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{q_k}{p_{2n} + ip_k} = \sum_{r=1}^{k} \frac{1}{p_{2r-1}} + i \sum_{r=0}^{k-1} \frac{p_{2r}}{p_{2r-1}} - ik\gamma(x);$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{q_k}{q_{2n} - 2Ep_k} = \sum_{r=1}^{k} \frac{1}{q_{2r-1}} + \frac{1}{2E} \sum_{r=1}^{k} \frac{q_{2r}}{q_{2r-1}} - \frac{k\gamma(x)}{2E};$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{q_k}{q_{2n+1} + i2Ep_k} = \sum_{r=1}^{k} \frac{1}{q_{2r}} - \frac{i}{2E} \sum_{r=1}^{k} \frac{q_{2r-1}}{q_{2r}} - \frac{ik\delta(x)}{2E},$$

respectively.

They yield

$$\begin{aligned}\sum_{\substack{n=(k+1)/2\\k\geq 1,\,odd}}^{\infty} \frac{Q_k}{P_{2n}+iP_k} &= \frac{1}{2}\sum_{r=1}^{k}\frac{1}{P_{2r-1}} + \frac{i}{2}\sum_{r=0}^{k-1}\frac{P_{2r}}{P_{2r-1}} - \frac{ik\gamma}{2};\\ \sum_{\substack{n=(k+1)/2\\k\geq 1,\,odd}}^{\infty} \frac{Q_k}{Q_{2n}-\sqrt{2}P_k} &= \frac{1}{2}\sum_{r=1}^{k}\frac{1}{Q_{2r-1}} + \frac{\sqrt{2}}{4}\sum_{r=1}^{k}\frac{Q_{2r}}{Q_{2r-1}} - \frac{\sqrt{2}k\gamma}{4};\\ \sum_{\substack{n=(k+1)/2\\k\geq 1,\,odd}}^{\infty} \frac{Q_k}{Q_{2n+1}+i\sqrt{2}P_k} &= \frac{1}{2}\sum_{r=1}^{k}\frac{1}{Q_{2r}} - \frac{i\sqrt{2}}{4}\sum_{r=1}^{k}\frac{Q_{2r-1}}{Q_{2r}} - \frac{i\sqrt{2}k\delta}{4},\end{aligned}$$

again respectively.

4. Chebyshev and Vieta Implications

Finally, we add that Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $V_n(x) = i^{n-1}f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$ [3, 4, 5], where $i = \sqrt{-1}$. They can be employed to find the Chebyshev and Vieta versions of Theorems 1–3. In the interest of brevity, we omit them and encourage gibonacci enthusiasts to explore them.

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