SUMS INVOLVING JACOBSTHAL POLYNOMIALS

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ABSTRACT. We explore the Jacobsthal versions of seven gibonacci sums.

1. INTRODUCTION

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 5].

In the interest of brevity and clarity, we *omit* the argument in the functional notation; so z_n will mean $z_n(x)$. In addition, we let $\Delta = \sqrt{x^2+4}, D = \sqrt{4x+1}$, and $\kappa = \begin{cases} -1, & \text{if } c_n = J_n; \\ D^2, & \text{otherwise.} \end{cases}$

Jacobsthal and Jacobsthal-Lucas polynomials satisfy the following Cassin-like identities [5]:

$$J_{n+k}J_{n-k} - J_n^2 = -(-x)^{n-k}J_k^2; \tag{1}$$
$$j_{n+k}j_{n-k} - j_n^2 = (-x)^{n-k}D^2J_k^2. \tag{2}$$

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2}f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$ [3], and [5] on page 566.

1.1. **Gibonacci Polynomial Sums.** In [6], we studied the following gibonacci sums:

$$\sum_{\substack{n=k+1 \\ k\geq 1,\, \text{odd}}}^{\infty} \frac{l_k}{g_{n+k}g_{n-k}} = \sum_{r=1}^{k} \frac{1}{g_{k+r}g_r}; \tag{3}$$
$$\sum_{n=3}^{\infty} \frac{xl_n}{f_{n+2}f_nf_{n-2}} = \frac{1}{f_3f_1} + \frac{1}{f_4f_2}; \tag{4}$$
$$\sum_{n=3}^{\infty} \frac{xf_n}{l_{n+2}l_nl_{n-2}} = \frac{1}{\Delta^2}\left(\frac{1}{l_3l_1} + \frac{1}{l_4l_2}\right); \tag{5}$$

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{g_{n-3}}{g_{n-2}}-\frac{g_{n+1}}{g_{n+2}}\right) = -\frac{g_0}{g_1}+\frac{g_1}{g_2}-\frac{g_2}{g_3}+\frac{g_3}{g_4}; \tag{6}$$

$$\sum_{n=3}^{\infty}\frac{1}{g_{n+2}g_{n-2}} = \begin{cases}-\frac{1}{f_4}S_{f_3}, & \text{if } g_n=f_n;\\ \frac{1}{\Delta^2 f_4}S_{l_3}, & \text{otherwise};\end{cases} \tag{7}$$

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{f_n^4-(-1)^n(x^2-1)f_n^2-x^2} = -\frac{1}{f_3f_4^2}; \tag{8}$$

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{l_n^4+(-1)^n(x^2-1)\Delta^2l_n^2-\Delta^4x^2} = -\frac{1}{f_4l_4l_3l_2}, \tag{9}$$

where $S_{g_3}=\dfrac{g_0}{g_1}-\dfrac{g_1}{g_2}+\dfrac{g_2}{g_3}-\dfrac{g_3}{g_4}$.

2. Jacobsthal Polynomial Sums

Our objective is to find the Jacobsthal versions of the above seven results using the Jacobsthal-gibonacci links.

To this end, in the interest of clarity and convenience, we let A and B denote the left side and right side of each equation, respectively, and LHS and RHS those of the corresponding Jacobsthal equation, again respectively.

2.1. **Jacobsthal Version of Equation** (3)**.**

Proof.

Case 1. Suppose $g_n=f_n$. Let $A=\dfrac{l_k}{f_{n+k}f_{n-k}}$. Now, replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{n-1}. This yields

$$\begin{aligned} A &= \frac{x^{(2n-k-2)/2}(x^{k/2}l_k)}{\left[x^{(n+k-1)/2}f_{n+k}\right]\left[x^{(n-k-1)/2}f_{n-k}\right]};\\ \text{LHS} &= \sum_{\substack{n=k+1\\k\geq 1,\,\text{odd}}}^{\infty}\frac{x^{(2n-k-2)/2}j_k}{J_{n+k}J_{n-k}}, \end{aligned}$$

where $c_n=c_n(x)$.

We now let $B=\dfrac{1}{f_{k+r}f_r}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{(k+2r-2)/2}$ yields

$$\begin{aligned} B &= \frac{x^{(k+2r-2)/2}}{\left[x^{(k+r-1)/2}f_{k+r}\right]\left[x^{(r-1)/2}f_r\right]};\\ \text{RHS} &= \sum_{r=1}^{k}\frac{x^{(k+2r-2)/2}}{J_{k+r}J_r}. \end{aligned}$$

Equating the two sides, we get

$$\sum_{\substack{n=k+1\\k\geq 1,\,\text{odd}}}^{\infty}\frac{x^n}{J_{n+k}J_{n-k}}=\frac{x^k}{j_k}\sum_{r=1}^{k}\frac{x^r}{J_{k+r}J_r}.$$

Case 2. Suppose $g_n = l_n$. Let $A = \dfrac{l_k}{l_{n+k}l_{n-k}}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^n yields

$$\begin{aligned} A &= \frac{x^{(2n-k)/2}(x^{k/2}l_k)}{\left[x^{(n+k)/2}l_{n+k}\right]\left[x^{(n-k)/2}l_{n-k}\right]}; \\ \text{LHS} &= \sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{x^{(2n-k)/2}j_k}{j_{n+k}j_{n-k}}, \end{aligned}$$

where $c_n = c_n(x)$.

Let $B = \dfrac{1}{l_{k+r}l_r}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{(k+2r)/2}$ yields

$$\begin{aligned} B &= \frac{x^{(k+2r)/2}}{\left[x^{(k+r)/2}l_{k+r}\right]\left[x^{r/2}l_r\right]}; \\ \text{RHS} &= \sum_{r=1}^{k} \frac{x^{(k+2r)/2}}{j_{k+r}j_r}, \end{aligned}$$

where $c_n = c_n(x)$.

Equating the two sides yields

$$\sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{x^n}{j_{n+k}j_{n-k}} = \frac{x^k}{j_k}\sum_{r=1}^{k} \frac{x^{(k+2r)/2}}{j_{k+r}j_r}.$$

Combining the two cases, we get the desired Jacobsthal version:

$$\sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{x^n}{c_{n+k}c_{n-k}} = \frac{x^k}{j_k}\sum_{r=1}^{k} \frac{x^r}{c_{k+r}c_r}. \tag{10}$$

□

It then follows that [6]

$$\begin{aligned} \sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{L_k}{F_{n+k}F_{n-k}} &= \sum_{r=1}^{k} \frac{1}{F_{k+r}F_r}; & \sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{L_k}{L_{n+k}L_{n-k}} &= \sum_{r=1}^{k} \frac{1}{L_{k+r}L_r}; \\ \sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{2^n}{J_{n+k}J_{n-k}} &= \frac{2^k}{J_k}\sum_{r=1}^{k} \frac{1}{J_{k+r}J_r}; & \sum_{\substack{n=k+1 \\ k\ge1,\,\text{odd}}}^{\infty} \frac{2^n}{j_{n+k}j_{n-k}} &= \frac{2^k}{j_k}\sum_{r=1}^{k} \frac{1}{j_{k+r}j_r}. \end{aligned}$$

In particular, we then have [6]

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{F_n^2 + (-1)^n} &= 1; & \sum_{n=2}^{\infty} \frac{1}{L_n^2 - 5(-1)^n} &= \frac{1}{3}; \\ \sum_{n=2}^{\infty} \frac{2^n}{J_n^2 - (-2)^{n-1}} &= 4; & \sum_{n=2}^{\infty} \frac{2^n}{j_n^2 + 9(-2)^{n-1}} &= \frac{4}{5}, \end{aligned}$$

respectively.

Because $c_{n+k}c_{n-k} - c_n^2 = (-x)^{n-k}\kappa\, J_k^2$, we can rewrite equation (10) as

$$\sum_{\substack{n=k+1\\ k\geq 1,\,\text{odd}}}^{\infty} \frac{x^n}{c_n^2 + (-x)^{n-k}\kappa\, J_k^2} = \frac{x^k}{j_k}\sum_{r=1}^{k}\frac{x^r}{c_{k+r}c_r}.$$

2.2. **Jacobsthal Version of Equation** (4).

Proof. With $A = \dfrac{xl_n}{f_{n+2}f_nf_{n-2}}$, replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{(3n-3)/2}$ yields

$$\begin{aligned} A &= \frac{x^{(2n-3)/2}(x^{n/2}l_n)}{\sqrt{x}[x^{(n+1)/2}f_{n+2}][x^{(n-1)/2}f_n][x^{(n-3)/2}f_{n-2}]}\\ &= \frac{x^{n-2}j_n}{J_{n+2}J_nJ_{n-2}};\\ \text{LHS} &= \sum_{n=3}^{\infty}\frac{x^{n-2}j_n}{J_{n+2}J_nJ_{n-2}}, \end{aligned}$$

where $c_n = c_n(x)$.

We now let $B = \dfrac{1}{f_3f_1} + \dfrac{1}{f_4f_2}$. Replace x with $1/\sqrt{x}$, and multiply each numerator and denominator with x^2. This gives

$$\begin{aligned} B &= \frac{x^2}{(x^{2/2}f_3)(x^{0/2}f_1)} + \frac{1}{(x^{3/2}f_4)(x^{1/2}f_2)}\\ \text{RHS} &= \frac{x}{J_3J_1} + \frac{1}{J_4J_2}, \end{aligned}$$

where $c_n = c_n(x)$.

Combining the two sides, we get the desired Jacobsthal version:

$$\sum_{n=3}^{\infty}\frac{x^nj_n}{J_{n+2}J_nJ_{n-2}} = \frac{x^3}{J_3J_1} + \frac{x^2}{J_4J_2}. \tag{11}$$

□

It then follows that [6]

$$\sum_{n=3}^{\infty}\frac{L_n}{F_{n+2}F_nF_{n-2}} = \frac{5}{6}; \qquad \sum_{n=3}^{\infty}\frac{2^nj_n}{J_{n+2}J_nJ_{n-2}} = \frac{52}{15}.$$

2.3. **Jacobsthal Version of Equation** (5).

Proof. With $A = \dfrac{xf_n}{l_{n+2}l_nl_{n-2}}$, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{3n/2}$ yields

$$\begin{aligned} A &= \frac{x^n[x^{(n-1)/2}f_n]}{[x^{(n+2)/2}l_{n+2}][x^{n/2}l_n][x^{(n-2)/2}l_{n-2}]} \\ &= \frac{x^nJ_n}{j_{n+2}j_nj_{n-2}}; \\ \text{LHS} &= \sum_{n=3}^{\infty} \frac{x^nJ_n}{j_{n+2}j_nj_{n-2}}, \end{aligned}$$

where $c_n = c_n(x)$.

With $B = \dfrac{1}{\Delta^2}\left(\dfrac{1}{l_3l_1} + \dfrac{1}{l_4l_2}\right)$, replace x with $1/\sqrt{x}$, and multiply each numerator and denominator with x^3. This yields

$$\begin{aligned} B &= \frac{x}{D^2}\left(\frac{1}{l_3l_1} + \frac{1}{l_4l_2}\right) \\ &= \frac{x}{D^2}\left(\frac{x^2}{(x^{3/2}l_3)(x^{1/2}l_1)} + \frac{x^3}{(x^{4/2}l_4)(x^{2/2}l_2)}\right); \\ \text{RHS} &= \frac{x^3}{D^2}\left(\frac{1}{j_3j_1} + \frac{x}{j_4j_2}\right), \end{aligned}$$

where $c_n = c_n(x)$.

Equating the two sides yields the Jacobsthal version of equation (5):

$$\sum_{n=3}^{\infty} \frac{x^nJ_n}{j_{n+2}j_nj_{n-2}} = \frac{x^3}{D^2}\left(\frac{1}{j_3j_1} + \frac{x}{j_4j_2}\right). \tag{12}$$

□

This implies [6]

$$\sum_{n=3}^{\infty} \frac{F_n}{L_{n+2}L_nL_{n-2}} = \frac{5}{84}; \qquad \sum_{n=3}^{\infty} \frac{2^nJ_n}{j_{n+2}j_nj_{n-2}} = \frac{88}{595}.$$

2.4. **Jacobsthal Version of Equation** (6).

Proof.

Case 1. Suppose $g_n = f_n$. Let $A = \dfrac{f_{n-3}}{f_{n-2}} - \dfrac{f_{n+1}}{f_{n+2}}$. Replace x with $1/\sqrt{x}$, and then multiply the numerators and denominators with $x^{(n+1)/2}$. We then get

$$\begin{aligned} A &= \frac{x^{5/2}\left[x^{(n-4)/2}f_{n-3}\right]}{x^2\left[x^{(n-3)/2}f_{n-2}\right]} - \frac{x^{1/2}\left(x^{n/2}f_{n+1}\right)}{x^{(n+1)/2}f_{n+2}}; \\ \text{LHS} &= \sqrt{x}\sum_{n=3}^{\infty}(-1)^n\left(\frac{J_{n-3}}{J_{n-2}} - \frac{J_{n+1}}{J_{n+2}}\right), \end{aligned}$$

where $c_n = c_n(x)$.

Now, let $B = -\left(\frac{f_0}{f_1} - \frac{f_1}{f_2} + \frac{f_2}{f_3} - \frac{f_3}{f_4}\right)$. Replacing x with $1/\sqrt{x}$, and multiplying the numerators and denominators with $x^{3/2}$, we get

$$\text{RHS} = -\sqrt{x}\left(\frac{J_0}{J_1} - \frac{J_1}{J_2} + \frac{J_2}{J_3} - \frac{J_3}{J_4}\right),$$

where $c_n = c_n(x)$.

Equating the two sides yields

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{J_{n-3}}{J_{n-2}} - \frac{J_{n+1}}{J_{n+2}}\right) = -\left(\frac{J_0}{J_1} - \frac{J_1}{J_2} + \frac{J_2}{J_3} - \frac{J_3}{J_4}\right).$$

Case 2. Let $g_n = l_n$. With $A = \frac{l_{n-3}}{l_{n-2}} - \frac{l_{n+1}}{l_{n+2}}$, replace x with $1/\sqrt{x}$, and then multiply each numerator and denominator with $x^{(n+2)/2}$. We then get

$$\begin{aligned} A &= \frac{x^{5/2}\left[x^{(n-3)/2}l_{n-3}\right]}{x^2\left[(x^{(n-2)/2}l_{n-2}\right]} - \frac{x^{1/2}\left[x^{(n+1)/2}l_{n+1}\right]}{x^{(n+2)/2}l_{n+2}}; \\ \text{LHS} &= \sqrt{x}\sum_{n=3}^{\infty}(-1)^n\left(\frac{j_{n-3}}{j_{n-2}} - \frac{j_{n+1}}{j_{n+2}}\right), \end{aligned}$$

where $c_n = c_n(x)$.

Now, with $B = -\left(\frac{l_0}{l_1} - \frac{l_1}{l_2} + \frac{l_2}{l_3} - \frac{l_3}{l_4}\right)$, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^2. This yields

$$\text{RHS} = -\sqrt{x}\left(\frac{j_0}{j_1} - \frac{j_1}{j_2} + \frac{j_2}{j_3} - \frac{j_3}{j_4}\right),$$

where $c_n = c_n(x)$.

Equating the two sides gives

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{j_{n-3}}{j_{n-2}} - \frac{j_{n+1}}{j_{n+2}}\right) = -\left(\frac{j_0}{j_1} - \frac{j_1}{j_2} + \frac{j_2}{j_3} - \frac{j_3}{j_4}\right).$$

Combining the two cases, we get the desired Jacobsthal version:

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{c_{n-3}}{c_{n-2}} - \frac{c_{n+1}}{c_{n+2}}\right) = -\left(\frac{c_0}{c_1} - \frac{c_1}{c_2} + \frac{c_2}{c_3} - \frac{c_3}{c_4}\right). \tag{13}$$

□

This implies [6]

$$\begin{aligned} \sum_{n=3}^{\infty}(-1)^n\left(\frac{G_{n-3}}{G_{n-2}} - \frac{G_{n+1}}{G_{n+2}}\right) &= -\left(\frac{G_0}{G_1} - \frac{G_1}{G_2} + \frac{G_2}{G_3} - \frac{G_3}{G_4}\right); \\ \sum_{n=3}^{\infty}(-1)^n\left(\frac{C_{n-3}}{C_{n-2}} - \frac{C_{n+1}}{C_{n+2}}\right) &= -\left(\frac{C_0}{C_1} - \frac{C_1}{C_2} + \frac{C_2}{C_3} - \frac{C_3}{C_4}\right). \end{aligned}$$

In particular, we then get [6]

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{F_{n-3}}{F_{n-2}}-\frac{F_{n+1}}{F_{n+2}}\right) = \frac{7}{6}; \qquad \sum_{n=3}^{\infty}(-1)^n\left(\frac{L_{n-3}}{L_{n-2}}-\frac{L_{n+1}}{L_{n+2}}\right) = -\frac{155}{84};$$
$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{J_{n-3}}{J_{n-2}}-\frac{J_{n+1}}{J_{n+2}}\right) = \frac{19}{15}; \qquad \sum_{n=3}^{\infty}(-1)^n\left(\frac{j_{n-3}}{j_{n-2}}-\frac{j_{n+1}}{j_{n+2}}\right) = -\frac{1251}{595}.$$

2.5. **Jacobsthal Version of Equation** (7)**.**

Proof.

Case 1. Let $g_n = f_n$. Suppose $A = \dfrac{1}{f_{n+2}f_{n-2}}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^{n-1}. This gives

$$\begin{aligned} A &= \frac{x^{n-1}}{\left[x^{(n+1)/2}f_{n+2}\right]\left[x^{(n-3)/2}f_{n-2}\right]}; \\ \text{LHS} &= \sum_{n=3}^{\infty}\frac{x^{n-1}}{J_{n+2}J_{n-2}}, \end{aligned}$$

where $c_n = c_n(x)$.

Now, with $B = -\dfrac{1}{f_4}\left(\dfrac{f_0}{f_1}-\dfrac{f_1}{f_2}+\dfrac{f_2}{f_3}-\dfrac{f_3}{f_4}\right)$, replace x with $1/\sqrt{x}$, and multiply each numerator and denominator with $x^{3/2}$. This yields

$$\text{RHS} = -\frac{x^2}{J_4}\left(\frac{J_0}{J_1}-\frac{J_1}{J_2}+\frac{J_2}{J_3}-\frac{J_3}{J_4}\right),$$

where $c_n = c_n(x)$.

Equating the two sides, we get

$$\sum_{n=3}^{\infty}\frac{x^n}{J_{n+2}J_{n-2}} = -\frac{x^3}{J_4}\left(\frac{J_0}{J_1}-\frac{J_1}{J_2}+\frac{J_2}{J_3}-\frac{J_3}{J_4}\right).$$

Case 2. Let $g_n = l_n$. With $A = \dfrac{1}{l_{n+2}l_{n-2}}$, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^n. This yields

$$\begin{aligned} A &= \frac{x^{n}}{\left[x^{(n+2)/2}l_{n+2}\right]\left[x^{(n-2)/2}l_{n-2}\right]}; \\ \text{LHS} &= \sum_{n=3}^{\infty}\frac{x^{n}}{j_{n+2}j_{n-2}}, \end{aligned}$$

where $c_n = c_n(x)$.

Next, we let $B = \dfrac{1}{\Delta^2 f_4}\left(\dfrac{l_0}{l_1}-\dfrac{l_1}{l_2}+\dfrac{l_2}{l_3}-\dfrac{l_3}{l_4}\right)$. Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^2. This yields

$$\begin{aligned} B &= \frac{D^2}{xf_4}\left(\frac{l_0}{l_1}-\frac{l_1}{l_2}+\frac{l_2}{l_3}-\frac{l_3}{l_4}\right); \\ \text{RHS} &= \frac{D^2\sqrt{x}}{J_4}\left(\frac{x^2 j_0}{x\sqrt{x}j_1}-\frac{x\sqrt{x}j_1}{xj_2}+\frac{xj_2}{\sqrt{x}j_3}-\frac{\sqrt{x}j_3}{j_4}\right) \\ &= \frac{D^2 x}{J_4}\left(\frac{j_0}{j_1}-\frac{j_1}{j_2}+\frac{j_2}{j_3}-\frac{j_3}{j_4}\right). \end{aligned}$$

Equating the two sides gives

$$\sum_{n=3}^{\infty} \frac{x^n}{j_{n+2}j_{n-2}} = \frac{D^2x}{J_4}\left(\frac{j_0}{j_1} - \frac{j_1}{j_2} + \frac{j_2}{j_3} - \frac{j_3}{j_4}\right).$$

Combining the two cases, we get the Jacobsthal version of equation (7):

$$\sum_{n=3}^{\infty} \frac{x^n}{c_{n+2}c_{n-2}} = \begin{cases} -\frac{x^3}{J_4}S_{c_3}, & \text{if } c_n = J_n; \\ \frac{D^2x}{J_4}S_{c_3}, & \text{otherwise;} \end{cases} \tag{14}$$

where $S_{c_3} = \dfrac{c_0}{c_1} - \dfrac{c_1}{c_2} + \dfrac{c_2}{c_3} - \dfrac{c_3}{c_4}$. □

It follows from equation (14) that [6]

$$\sum_{n=3}^{\infty} \frac{1}{F_{n+2}F_{n-2}} = \frac{7}{18}; \qquad \sum_{n=3}^{\infty} \frac{1}{L_{n+2}L_{n-2}} = \frac{31}{252};$$
$$\sum_{n=3}^{\infty} \frac{2^n}{J_{n+2}J_{n-2}} = \frac{152}{75}; \qquad \sum_{n=3}^{\infty} \frac{2^n}{j_{n+2}j_{n-2}} = \frac{22,518}{2,975}.$$

Using the identities $g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases}$
and $c_{n+k}c_{n-k} - c_n^2 = (-x)^{n-k}\kappa\, J_k^2$, we can rewrite these equations as follows:

$$\sum_{n=3}^{\infty} \frac{1}{F_n^2 - (-1)^n} = \frac{7}{18}; \qquad \sum_{n=3}^{\infty} \frac{1}{L_n^2 + 5(-1)^n} = \frac{31}{252};$$
$$\sum_{n=3}^{\infty} \frac{2^n}{J_n^2 - (-2)^{n-2}} = \frac{152}{75}; \qquad \sum_{n=3}^{\infty} \frac{2^n}{j_n^2 + 9(-2)^{n-2}} = \frac{22,518}{2,975},$$

respectively.

2.6. **Jacobsthal Version of Equation** (8).

Proof. Let $A = \dfrac{(-1)^n}{f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with x^{2n-3} then yields

$$\begin{aligned} A &= \frac{(-1)^n x}{xf_n^4 + (-1)^n(x-1)f_n^2 - 1} \\ &= \frac{(-1)^n x^{2n-2}}{[x^{(n-1)/2}f_n]^4 + (-1)^n(x-1)x^{n-2}[x^{(n-1)/2}f_n]^2 - x^{2n-3}} \\ &= \frac{(-1)^n x^{2n-2}}{J_n^4 + (-1)^n(x-1)x^{n-2}J_n^2 - x^{2n-3}}; \\ \text{LHS} &= \sum_{n=3}^{\infty} \frac{(-1)^n x^{2n-2}}{J_n^4 + (-1)^n(x-1)x^{n-2}J_n^2 - x^{2n-3}}, \end{aligned}$$

where $c_n = c_n(x)$.

With $B = -\dfrac{1}{f_3 f_4^2}$, replacing x with $1/\sqrt{x}$ and multiplying the numerator and denominator with x^4, gives

$$\begin{aligned} B &= -\frac{x^4}{(x^{2/2}f_3)(x^{3/2}f_4)^2}; \\ \text{RHS} &= -\frac{x^4}{J_3 J_4^2}, \end{aligned}$$

where $c_n = c_n(x)$.

Combining the two sides, we get the desired Jacobsthal version:

$$\sum_{n=3}^{\infty} \frac{(-1)^n x^{2n}}{J_n^4 + (-1)^n (x-1)x^{n-2}J_n^2 - x^{2n-3}} = -\frac{x^6}{J_3 J_4^2}. \tag{15}$$

□

This yields [6]

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{F_n^4 - 1} = -\frac{1}{18}; \qquad \sum_{n=3}^{\infty} \frac{(-4)^n}{J_n^4 + (-1)^n 2^{n-2} J_n^2 - 2^{2n-3}} = -\frac{64}{75}.$$

Finally, we explore the Jacobsthal consequence of equation (9).

2.7. Jacobsthal Version of Equation (9).

Proof. Let $A = \dfrac{(-1)^n}{l_n^4 + (-1)^n (x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2}$. Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^{2n-3}. This yields

$$\begin{aligned} A &= \frac{(-1)^n}{x^3 l_n^4 - (-1)^n (x^2 - x) D^2 l_n^2 - D^4} \\ &= \frac{(-1)^n x^{2n}}{(x^{n/2} l_n)^4 - (-1)^n (x-1) x^{n-2} D^2 j_n^2 - D^4 x^{2n-3}}; \\ \text{LHS} &= \sum_{n=3}^{\infty} \frac{(-1)^n x^{2n}}{(x^{n/2} l_n)^4 - (-1)^n (x-1) x^{n-2} D^2 j_n^2 - D^4 x^{2n-3}}, \end{aligned}$$

where $c_n = c_n(x)$.

Now, let $B = -\dfrac{1}{f_4 l_4 l_3 l_2}$. Replacing x with $1/\sqrt{x}$ and multiplying the numerator and denominator with x^6 yields

$$\begin{aligned} B &= -\frac{x^6}{(x^{3/2}f_4)(x^{4/2}l_4)(x^{3/2}l_3)(x^{2/2}l_2)}; \\ \text{RHS} &= -\frac{x^6}{J_4 j_4 j_3 j_2}, \end{aligned}$$

where $c_n = c_n(x)$.

Equating the two sides, we get the Jacobsthal version of equation (9):

$$\sum_{n=3}^{\infty} \frac{(-1)^n x^{2n}}{(x^{n/2} l_n)^4 - (-1)^n (x-1) x^{n-2} D^2 j_n^2 - D^4 x^{2n-3}} = -\frac{x^6}{J_4 j_4 j_3 j_2}. \tag{16}$$

□

This implies [6]

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{L_n^4 - 25} = -\frac{1}{252}; \qquad \sum_{n=3}^{\infty} \frac{(-4)^n}{j_n^4 - 9(-2)^{n-2} j_n^2 - 81 \cdot 2^{2n-3}} = -\frac{64}{2,975}.$$

3. Vieta and Chebyshev Implications

Finally, we can find the Vieta and Chebyshev versions of equations (3) through (9) using the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$ [3, 4, 5], where $i = \sqrt{-1}$. In the interest of brevity, we omit them and encourage gibonacci enthusiasts to explore them.

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