CALKIN-WILF SEQUENCES FOR IRRATIONAL NUMBERS

WILLIAM PAULSEN

Abstract. We generalize the Calkin-Wilf sequence by applying its recursion formula to an irrational initial value. We prove that if we start with quadratic surd, we will eventually reach an integer plus our starting value, although we may have to consider the predecessor sequence. We also find conditions for an integer minus the starting value to appear. In particular, if we start with $\sqrt{n}$, then $m - \sqrt{n}$ appears for some m if and only if there is a solution to the negative Pell equation.

1. Introduction

The Calkin-Wilf sequence gives a way to enumerate the positive rational numbers. The sequence can be defined recursively by

$$x_0 = 0, \qquad x_{n+1} = \frac{1}{2\lfloor x_n \rfloor + 1 - x_n} \quad n \geq 0. \tag{1.1}$$

This produces the sequence of rational numbers

$$\left\{0, 1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, 3, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, 4, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}, \frac{8}{5}, \frac{5}{7}, \frac{7}{2}, \frac{2}{7}, \frac{7}{5}, \frac{5}{8}, \frac{8}{3}, \frac{3}{7}, \frac{7}{4}, \frac{4}{5}, 5, \dots\right\}.$$

The literature often begins the sequence with $x_1 = 1$, in which case the sequence contains every positive rational number once and only once [2]. The recursion formula (1.1) for the sequence was actually discovered later by [4].

But the successor function in the recursion formula in (1.1),

$$f(x) = \frac{1}{2\lfloor x \rfloor + 1 - x}$$

is defined for all real numbers. So what would happen if we begin the sequence with x_0 being irrational? For example, if we begin with $x_0 = \sqrt{2}$, we get the sequence

$$\left\{\sqrt{2}, \frac{3+\sqrt{2}}{7}, \frac{4+\sqrt{2}}{2}, \frac{6+\sqrt{2}}{17}, \frac{11+\sqrt{2}}{7}, \frac{10+\sqrt{2}}{14}, 4+\sqrt{2}, \frac{7+\sqrt{2}}{47}, \frac{40+\sqrt{2}}{34}, \frac{62+\sqrt{2}}{113}, \dots\right\}.$$

A curious thing happens. We find that $x_6 = 4 + \sqrt{2}$, which is an integer plus the starting x_0 value. If we continue the sequence, we find that $x_{102} = 8 + \sqrt{2}$. Is there an explanation for this?

The successor function $f(x)$ is also one-to-one and onto $\mathbb{R} \cup \{\infty\}$. We can compute the predecessor function

$$f^{-1}(x) = -2\left\lfloor -\frac{1}{x} \right\rfloor - 1 - \frac{1}{x}.$$

The easiest way to demonstrate this is to let $g(x) = -1/x$, and observe that the graph of $g(f(x)) = x - 2\lfloor x \rfloor - 1$ is symmetric about the line $y = x$. Hence $g(f(x))$ is its own inverse, and so $f^{-1}(x) = g(f(g(x)))$. The predecessor function allows us to define x_n for negative subscripts. If $x_0 = 0$, we find that $x_{-1} = \infty$, $x_{-2} = -1$, and we get the sequence

$$\left\{\ldots, -5, -\frac{1}{4}, -\frac{4}{3}, -\frac{3}{5}, -\frac{5}{2}, -\frac{2}{5}, -\frac{5}{3}, -\frac{3}{4}, -4, -\frac{1}{3}, -\frac{3}{2}, -\frac{2}{3}, -3, -\frac{1}{2}, -2, -1, \infty, 0\right\}.$$

It is easy to prove by induction that $x_{-n-1} = -1/x_n$ for all $n \geq 1$, using the property that $f^{-1}(x) = g(f(g(x)))$. This means that by extending the Calkin-Wilf sequence for negative subscripts, we have an enumeration for $\mathbb{Q} \cup \{\infty\}$.

On the other hand, if we let $x_0 = \sqrt{2}$, we get the predecessor sequence

$$\left\{\ldots, \frac{23-\sqrt{2}}{17}, \frac{28-\sqrt{2}}{46}, \frac{18-\sqrt{2}}{7}, \frac{17-\sqrt{2}}{41}, \frac{24-\sqrt{2}}{14}, \frac{18-\sqrt{2}}{23}, 5-\sqrt{2}, \frac{2-\sqrt{2}}{2}, \sqrt{2}\right\}.$$

We find another curiosity, $x_{-2} = 5 - \sqrt{2}$, which is an integer *minus* our initial value. Also, $x_{-26} = 9 - \sqrt{2}$, so this is not an isolated incident.

The goal of this paper is to show that if we start the Calkin-Wilf sequence with a quadratic surd, there will be an infinite number of times that an integer plus the starting value will appear in either the successor sequence or the predecessor sequence. We will also give conditions for which an integer minus the starting value appears in one of the two sequences.

2. The Position Function

There actually is a way of finding the position of a positive rational number q in the Calkin-Wilf sequence, using the continued fraction representation of q. If

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

for integers a_i, we write $q = [a_0; a_1, a_2, \ldots a_n]$. We then define

$$Q(q) = -1 + \sum_{i=0}^{n} (-1)^n 2^{a_0+a_1+\cdots+a_i}. \tag{2.1}$$

However, for rational numbers, there are two possible continued fraction representations: if $a_n > 1$, then $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$. If we use the form with odd length (n even), then $Q(q)$ will be a positive number, otherwise $Q(q)$ will be a negative number. So in (2.1), we use the form in which n is even.

For example, if $q = 3/8$, then the continued fraction of q is $[0; 2, 1, 2]$. But this has $n = 3$, which is odd, so we use the equivalent form $[0; 2, 1, 1, 1]$. Then, $Q(3/8) = -1 + 2^0 - 2^2 + 2^3 - 2^4 + 2^5 = 20$. Looking back at the Calkin-Wilf sequence, we find that $x_{20} = 3/8$.

We can also find $Q(q)$ by considering the continued fraction representation as the run-length encoding of a binary number. Working from right to left, we form the binary number with a_0 1s, followed by a_1 0s, a_2 1s, and so on until we get a_n 1s, with n even. So for $[0; 2, 1, 1, 1]$, we get $10100_2 = 20$.

For positive q, the proof that $Q(q)$ gives the position of q in the Calkin-Wilf sequence is well documented [1, 3]. However, the proofs usually use the Calkin-Wilf *tree*, from which the sequence is usually derived. Instead, we are using the recursion formula in (1.1) to define the sequence, bypassing the tree, so that we can generalize the sequence by choosing different starting values. So, we must eventually reprove this result using only (1.1), but do so in a more generalized setting.

Had we used the continued fraction representation $[0; 2, 1, 2]$ for $3/8$, we would have gotten $Q(3/8) = -1 + 2^0 - 2^2 + 2^3 - 2^5 = -28$. Note that $x_{-28} = -3/8$ in the predecessor sequence. So there is a use for the continued fraction representation with odd n. We can extend the definition of $Q(q)$ for negative rational numbers using (2.1) on the continued fraction representation of $|q|$ which uses odd n. Thus,

$$\text{If} \qquad |q| = [a_0; a_1, a_2, \ldots, a_n] \text{ with } (-1)^n = \operatorname{sgn}(q),$$
$$Q(q) = -1 + \sum_{i=0}^{n} (-1)^n 2^{a_0+a_1+\cdots+a_i}. \tag{2.2}$$

Finally, we define $Q(0) = 0$ and $Q(\infty) = -1$.

We have yet to show that $Q(q)$ gives the position of the integer q for negative rational numbers. To do so, we will need the following lemma.

Lemma 2.1.

$$Q(x) + Q(-1/x) = -1. \tag{2.3}$$

Proof. It is easy to check that (2.3) is true for $x = 0$, ∞, and ± 1. Suppose that $|x| > 1$, and has the continued fraction representation of $[a_0; a_1, a_2, \ldots, a_n]$, with $(-1)^n = \operatorname{sgn}(x)$. Then because $a_0 > 0$, the continued fraction of $|1/x|$ is $[0; a_0, a_1, a_2, \ldots, a_n]$, which increases the length by 1. Thus,

$$Q(x) = -1 + \sum_{i=0}^{n} (-1)^n 2^{a_0+a_1+\cdots+a_i}, \qquad \text{and}$$
$$Q(-1/x) = -1 + 2^0 - \sum_{i=0}^{n} (-1)^n 2^{a_0+a_1+\cdots+a_i}.$$

Adding these together gives us -1. For $0 < |x| < 1$, we let $y = -1/x$, so that $|y| > 1$, and apply the above argument to y. $\square$

We can now show that for negative rational numbers, $Q(q)$ gives the position of q in the predecessor sequence. We already proved that $x_{-n-1} = -1/x_n$. If $q < 0$, then we can let $n = Q(-1/q)$, and because $-1/q > 0$, $x_n = -1/q$. Lemma 2.1 indicates that $Q(q) = -n - 1$, and also $x_{-n-1} = q$, so that $Q(q)$ gives the position in the predecessor sequence.

The goal is to extend the function Q so that it is defined for all real numbers. To do so, we must introduce the 2-adic integers.

3. The 2-adic Integers

We can extend the ring of integers into a much larger ring by changing what it means for two numbers to be close. Specifically, two 2-adic integers are close if their difference is divisible by a large power of 2. The larger the power of 2, the closer the numbers are. So with this topology, $\lim_{n\to\infty} 2^n = 0$. We then define the 2-adic integers, $\mathbb{Z}_2$, as the set of formal power series of the form

$$\sum_{i=0}^{\infty} s_i 2^i, \qquad \text{where each } s_i = 0 \text{ or } 1.$$

We can represent a 2-adic number with an infinite string of binary digits going to the left:

$$\ldots s_n \ldots s_5 \_ s_4 \_ s_3 \_ s_2 \_ s_1 \_ s_0.$$

Here, the underscore represents the concatenation of digits. Addition is performed as standard binary arithmetic, starting with the rightmost digit and carrying to the next digit as necessary. Likewise, multiplication can be performed as standard binary multiplication. Negatives can be accomplished by taking the "two's complement," that is, switching all 0s to 1s and all 1s to 0s, and finally adding 1. Thus, -7 can be represented by $\ldots 11111111001$. Using these operations, $\mathbb{Z}_2$ forms a ring, and is an integral domain containing a copy of $\mathbb{Z}$ as a subring. For those elements of $\mathbb{Z}_2$ that are in $\mathbb{Z}$, we will use the standard integer notation instead of the binary representation, so $\ldots 11111111001$ will just be -7. The only prime in $\mathbb{Z}_2$ is 2, so every odd integer (one with $s_0 = 1$) has a multiplicative inverse, which can be found using long division.

We now have a way to extend the position function $Q(x)$ to allow for real number values. If x is a positive irrational number, it has a unique continued fraction expansion $[a_0; a_1, a_2, a_3, \ldots]$. We then define

$$Q(x) = -1 + \sum_{i=0}^{\infty} (-1)^n 2^{a_0+a_1+\cdots+a_i}, \tag{3.1}$$

where the sum converges in $\mathbb{Z}_2$. For example, because $\sqrt{2}$ has a continued fraction representation of $[1; 2, 2, 2, \ldots]$, $Q(\sqrt{2}) = \ldots 001100110011001$. Unlike the case for rational numbers, the continued fraction representation for irrational numbers is unique. Thus, if we define for x irrational, $Q(-x) = Q(x)$, then the proof of Lemma 2.1 will hold for irrational x as well (replacing the finite sums with infinite sums).

Whenever the digits of an element $x \in \mathbb{Z}_2$ are periodic, then x can be represented by a rational number. If the digit pattern repeats every m digits, then $(2^m - 1)x$ will be an integer. For example, because

$$\begin{aligned} Q(\sqrt{2}) &= \ldots 001100110011001 \\ 2^4\ Q(\sqrt{2}) &= \ldots \underline{001100110010000} \\ -15\ Q(\sqrt{2}) &= \ldots 000000000001001 \qquad = 9, \end{aligned}$$

we see that $Q(\sqrt{2}) = -3/5$.

A mixed quadratic surd is a number of the form $(a + b\sqrt{n})/c$, where a, b, and c are integers and n is a positive square-free integer. In 1770, Lagrange proved that the continued fraction of a positive mixed quadratic surd will be periodic at some point [5, p. 75]. Thus, $Q(x)$ will send quadratic surds to rational numbers in $\mathbb{Z}_2$. This is similar to Minkowski's question mark function $?(x)$ [9], which is defined on the interval $[0, 1]$:

$$\text{If } x = [0; a_1, a_2, a_3, \ldots], \qquad ?(x) = 2\sum_{n=1}^{\infty} (-1)^{n+1} 2^{-(a_1+a_2+a_3+\cdots+a_n)}.$$

The $Q(x)$ function has a similar definition, but it "goes the other way." The function $?(x)$ is actually continuous on the interval $[0, 1]$, because the two possible continued fraction representations of a rational number produce the same output. We lose continuity by using the ring $\mathbb{Z}_2$, but we still have one sided continuity.

Lemma 3.1. *The function $Q(x)$ is continuous at every irrational number, and if $a \in \mathbb{Q}$,*

$$\lim_{x \to a^+} Q(x) = Q(a).$$

Proof. Because of the topology of $\mathbb{Z}_2$, we need to show that for every large M, there is a sufficiently small interval around a such that $Q(x)$ agrees with $Q(a)$ for the rightmost M digits. If $a \in \mathbb{Q}^+$ with continued fraction representation $[a_0; a_1, a_2, \ldots, a_n]$ (n even), we let

$N = a_0 + a_1 + a_2 + \cdots a_n$. For $M > N$, we let $b = [a_0; a_1, a_2, \ldots, a_n, M - N]$. Because n is even, $b > a$, and for every x in the half open interval $[a, b)$, $Q(x)$ agrees with $Q(a)$ for the rightmost M digits. Thus, we have continuity from the right. If a is a negative rational, the argument is similar except we use n odd, so $b < |a|$. But then, $-b > a$, so we use the half open interval $[a, -b)$.

For irrational a, the proof is easier because the continued fraction expansion for $|a|$ is infinite. We pick n such that $a_0 + a_1 + a_2 + \cdots a_n > M$, and consider the open interval between $[a_0; a_1, a_2, \ldots, a_n]$ and $[a_0; a_1, a_2, \ldots, a_n + 1]$. For all x in this interval, the rightmost M digits of $Q(x)$ will agree with $Q(a)$. □

4. Relating $Q(x)$ to the Successor Function

The goal of this section is to prove that $Q(f(x)) = Q(x) + 1$ for all real x, where $f(x)$ is the successor function

$$f(x) = \frac{1}{2\lfloor x \rfloor + 1 - x}.$$

This is well known to be true for positive rational x, but we want to generalize this, using the continued fraction representation of x. We begin with the following proposition.

Lemma 4.1. *If $0 \le x < 1$, then $Q(x - 1) + Q(x) = -2$.*

Proof. The proposition is clearly true for $x = 0$. If x is irrational and $1/2 < x < 1$, the continued fraction representation of x is $[0; 1, a_2, a_3, \ldots]$. If $y = [0; a_3, a_4, \ldots]$, then

$$x = \cfrac{1}{1 + \cfrac{1}{a_2 + y}} = \frac{a_2 + y}{1 + a_2 + y}.$$

Thus, $1 - x = 1/(1 + a_2 + y)$ has the continued fraction representation $[0; 1 + a_2, a_3, a_4, \ldots]$. (This identity is used to prove $?(x) + ?(1 - x) = 1$.) Then,

$$\begin{aligned} Q(x) + Q(1-x) &= -1 + 2^0 - 2^1 + \sum_{n=2}^{\infty} (-1)^n 2^{1+a_2+a_2+\cdots a_n} \\ &+ -1 + 2^0 - 2^{1+a_2} - \sum_{n=3}^{\infty} (-1)^n 2^{1+a_2+a_2+\cdots a_n} = -2. \end{aligned}$$

For x irrational and $0 < x < 1/2$, we let $z = 1 - x$, and apply the result to z. Because x is irrational, $Q(x - 1) = Q(1 - x)$, so

$$Q(x) + Q(x - 1) = -2.$$

For $q \in \mathbb{Q}$ and $0 < q < 1$, we have from Lemma 3.1 that

$$\lim_{x \to q^+} Q(x) + Q(x - 1) = Q(q) + Q(q - 1).$$

Because $Q(x) + Q(x - 1) = -2$ for irrational x, we have that $Q(q) + Q(q - 1) = -2$ as well. □

Lemma 4.2. *If n is a nonnegative integer, then*

$$\begin{aligned} &\textit{if } x \ge 0, \quad Q(x + n) = 2^n Q(x) + 2^n - 1, \\ &\textit{if } x < 0, \quad Q(x - n) = 2^n Q(x) + 2^n - 1. \end{aligned}$$

Proof. The result is trivial for $n = 0$. If $x = 0$, then the continued fraction representation of the positive integer n is simply $[n;]$, so $Q(n) = -1 + 2^n$. If $x > 0$, and has the continued fraction representation $[a_0; a_1, a_2, a_3, \dots]$, then $x + n$ has the continued fraction representation $[n + a_0; a_1, a_2, a_3, \dots]$, which has the same parity. So

$$\begin{aligned} Q(x+n) &= -1 + \sum_{n=0}^{\infty} (-1)^n 2^{n+a_0+a_1+\cdots+a_n} = -1 + 2^n \sum_{n=0}^{\infty} (-1)^n 2^{a_0+a_1+\cdots+a_n} \\ &= -1 + 2^n(1 + Q(x)) = -1 + 2^n + 2^n Q(x). \end{aligned}$$

If $x < 0$, then $|x - n| = |x| + n$, so we can apply the same argument to show that $Q(x - n) = -1 + 2^n + 2^n Q(x)$. □

Proposition 4.3. *If*

$$f(x) = \frac{1}{2\lfloor x \rfloor + 1 - x},$$

then $Q(f(x)) = Q(x) + 1$.

Proof. First, we consider the case where $x \geq 0$. Let $n = \lfloor x \rfloor$. Then $0 \leq x - n < 1$, so by Lemma 4.1, $Q(x - n) + Q(x - n - 1) = -2$. Multiplying by 2^n, we get

$$2^n Q(x-n) + 2^n Q(x-n-1) = -2 \cdot 2^n. \tag{4.1}$$

Because $x - n \geq 0$, we can use Lemma 4.2 to show that $Q(x) = 2^n Q(x - n) + 2^n - 1$, so

$$2^n Q(x-n) = Q(x) + 1 - 2^n. \tag{4.2}$$

Because $x - n - 1 < 0$, we can also use Lemma 4.2 to show that $Q(x - 2n - 1) = 2^n Q(x - n - 1) + 2^n - 1$, so

$$2^n Q(x-n-1) = Q(x-2n-1) + 1 - 2^n. \tag{4.3}$$

Replacing the left hand terms in (4.1) with the values in (4.2) and (4.3), we get

$$(Q(x) + 1 - 2^n) + (Q(x-2n-1) + 1 - 2^n) = -2 \cdot 2^n,$$

or

$$Q(x) + Q(x-2n-1) = -2. \tag{4.4}$$

Finally, we can use Lemma 2.1 to show that

$$Q(1/(1+2n-x)) + Q(x-2n-1) = -1,$$

so $Q(x - 2n - 1) = -1 - Q(f(x))$. Thus, $Q(x) - 1 - Q(f(x)) = -2$, so $Q(f(x)) = Q(x) + 1$.

For the case $x = -1$, $f(-1) = \infty$, but we defined $Q(\infty) = -1$, and we can check that the proposition is true for this case. If $x < 0$ and $x \neq -1$, then $-1/f(x) > 0$, so we can apply the result to $-1/f(x)$:

$$Q(f(-1/f(x))) = Q(-1/f(x)) + 1. \tag{4.5}$$

Now, $f(-1/f(x)) = -1/x$, and by Lemma 2.1, $Q(-1/x) = -Q(x) - 1$. Likewise, $Q(-1/f(x)) = -Q(f(x)) - 1$. Thus, (4.5) becomes

$$-Q(x) - 1 = -Q(f(x)) - 1 + 1,$$

which becomes $Q(f(x)) = Q(x) + 1$. □

5. Finding an Integer Plus the Starting Value in the Sequence

It is clear by Proposition 4.3 that if we begin the sequence with an irrational x_0, then $Q(x_n) = Q(x_0) + n$. For the remainder of the paper, we can assume that $x_0 > 0$, because starting the sequence with a negative irrational number merely produces the negative of the sequence starting with the absolute value of x_0. The plan is to use the $Q(x)$ function to determine whether x_n is ever equal to x_0+m with $m \in \mathbb{Z}^+$ for either positive or negative n. The first thing to establish is that this could *only* happen if the continued fraction representation of x_0 is periodic, which means x_0 is a quadratic surd. If $Q(x_0 + m) = Q(x_0) + n$, then by Lemma 4.2,

$$2^m Q(x_0) + 2^m - 1 = Q(x_0) + n.$$

But $2^m - 1$ is odd, and hence invertible in the ring $\mathbb{Z}_2$. Thus, we can solve for $Q(x_0)$:

$$Q(x_0) = \frac{n + 1 - 2^m}{2^m - 1},$$

so the sequence of digits for $Q(x_0)$ eventually repeat, forcing x_0 to be a quadratic surd.

The ring $\mathbb{Z}_2$ is not an ordered ring. (There is an element

$$\sqrt{-7} \approx \ldots 1101000100100111000110001101100111001111101001011$$

whose square is -7.) However, when x_0 is a quadratic surd, then $Q(x_0) \in \mathbb{Z}_2 \cap \mathbb{Q}$, which *is* an ordered ring. Thus, we can consider whether $Q(x_0) > -1$.

Proposition 5.1. *If x_0 is a quadratic surd, and $Q(x_0) > -1$, then there is an $m \in \mathbb{Z}^+$ such that x_0+m is in the successor sequence $x_1, x_2, x_3, \ldots$. On the other hand, if $Q(x_0) < -1$, then there is an $m \in \mathbb{Z}^+$ such that $x_0 + m$ is in the predecessor sequence $x_{-1}, x_{-2}, x_{-3}, \ldots$.*

Proof. Suppose that x_0 has a continued fraction representation

$$x_0 = [a_0; a_1, a_2, \ldots a_j, \overline{b_1, b_2, \ldots, b_k}],$$

where the overline indicates the repeating portion of the continued fraction. (If the repeating portion starts immediately, as in the case for $\sqrt{2}$, we let $j = 0$.) If the period of the continued fraction k is even, we let $m = b_1 + b_2 + \cdots + b_k$. However, if the period k is odd, we let $m = 2(b_1 + b_2 + \cdots + b_k)$. Then the period of the repeating digits in $Q(x_0)$ will be m, so that $(2^m - 1)Q(x_0)$ will be an integer.

We claim that $x_n = x_0 + m$ for some n, and we can compute the number n. By Lemma 4.2, $Q(x_0 + m) = 2^m Q(x_0) + 2^m - 1$. Thus,

$$Q(x_0) + n = 2^m Q(x_0) + 2^m - 1.$$

Solving for n, we get

$$n = (2^m - 1)(Q(x_0) + 1).$$

Because $(2^m - 1)Q(x_0)$ is an integer, n will be an integer. Furthermore, if $Q(x_0) > -1$, then $n > 0$ so $x_0 + m$ appears in the successor sequence. On the other hand, if $Q(x_0) < -1$, then n will be negative, and $x_0 + m$ appears in the predecessor sequence. □

The next question is whether an integer minus the starting value will appear as x_n for some n. We will find conditions for which this is possible.

Proposition 5.2. *If x_0 has a continued fraction representation*

$$x_0 = [a_0; a_1, a_2, \ldots a_j, \overline{b_1, b_2, \ldots, b_k}],$$

with period k being odd, then if $Q(x_0) > -1$, there is an $m \in \mathbb{Z}^+$ such that $m - x_0$ is in the predecessor sequence $x_{-1}, x_{-2}, x_{-3}, \ldots$. On the other hand, if $Q(x_0) < -1$, there is an $m \in \mathbb{Z}^+$

such that $m - x_0$ is in the successor sequence $x_1, x_2, x_3, \ldots$. If the period k is even, then x_n is never equal to $m - x_0$ for any integer m.

Proof. Let $h = b_1 + b_2 + \cdots + b_k$ be the half period of the repeating digits in $Q(x_0)$. Because k is odd, the second half period will be the binary complement of the first half (every 0 becomes a 1, and every 1 becomes a 0). Thus, $2^h Q(x_0)$ will eventually have the complementary digits as $Q(x_0)$, so $(2^h + 1)Q(x_0)$ will be an integer.

We now let $m = 2\lfloor x_0 \rfloor + 1 + h$. From (4.4), $Q(x_0) + Q(x_0 - 2\lfloor x_0 \rfloor - 1) = -2$. But because x_0 is irrational, $Q(x_0 - 2\lfloor x_0 \rfloor - 1) = Q(2\lfloor x_0 \rfloor + 1 - x_0)$. Thus,

$$Q(2\lfloor x_0 \rfloor + 1 - x_0) = -2 - Q(x_0).$$

Using Lemma 4.2, we have

$$\begin{aligned} Q(2\lfloor x_0 \rfloor + 1 + h - x_0) &= 2^h Q(2\lfloor x_0 \rfloor + 1 - x_0) + 2^h - 1 \\ &= 2^h(-2 - Q(x_0)) + 2^h - 1 \\ &= -2^h Q(x_0) - 2^h - 1. \end{aligned}$$

If $Q(2\lfloor x_0 \rfloor + 1 + h - x_0) = Q(x_n) = Q(x_0) + n$, we can solve for n to obtain

$$n = -(2^h + 1)(Q(x_0) + 1).$$

Because $(2^h + 1)Q(x_0)$ is an integer, n will be an integer. Furthermore, if $Q(x_0) > -1$, n will be negative, so $m - x_0$ appears in the predecessor sequence. If $Q(x_0) < -1$, then n will be positive, so $m - x_0$ will be in the successor sequence.

This argument is reversible, so if $x_n = m - x_0$ for some integer m, then letting $h = m - 2\lfloor x_0 \rfloor - 1$, we find that $(2^h + 1)Q(x_0)$ will be an integer. This shows, among other things, that $h > 0$. Because $(2^{2h} - 1)Q(x_0) = (2^h - 1)(2^h + 1)Q(x_0)$ is also an integer, the digits of $Q(x_0)$ will be periodic with period $2h$. But because $(2^h + 1)Q(x_0)$ is an integer, half of the period is the binary complement of the other half. This can only happen if the period of the continued fraction representation of x_0 is odd. Thus, if k is even, then x_n is never equal to $m - x_0$ for any integer m. □

In the case where $x_0 = \sqrt{N}$ for a non-square integer N, we can say even more. In [8], it was proved that the continued fraction of $\sqrt{N}$ had an odd period if and only if the negative Pell equation

$$x^2 - Ny^2 = -1$$

has a solution. Furthermore, the periodic portion of the continued fraction begins immediately,

$$\sqrt{N} = [a_0; \overline{b_1, b_2, \ldots, b_k}].$$

This will cause $Q(\sqrt{N}) - 2^{2k}Q(\sqrt{N})$ to be a positive number, less than $2^{2k} - 1$. This means that $-1 < Q(\sqrt{N}) < 0$. Thus, $m - \sqrt{N}$ will appear in the predecessor sequence if and only if there is a solution to the negative Pell equation.

6. Conclusion

We discovered that if we start with a quadratic surd of the form $x_0 = a + b\sqrt{N}$, where a and b are rational, for the initial value for the Calkin-Wilf recursion formula, then $m + x_0$ will appear in the generalized Calkin-Wilf sequence for an infinite number of m (although one must occasionally consider the predecessor sequence). Also, we proved that $m - x_0$ appears in either the successor sequence or predecessor sequence if and only if the continued fraction of x_0 has an odd period. In particular, if $x_0 = \sqrt{N}$, then $m - \sqrt{N}$ appears in the predecessor sequence if and only if there is a solution to the negative Pell equation. The integer sequence of those N for which $m - \sqrt{N}$ appears in the predecessor sequence is given in [7]. If we are only interested in the square-free N, we use sequence [6].

This paper also introduced the $Q(x)$ function $Q : \mathbb{R} \cup \infty \to \mathbb{Z}_2$, which has many of the properties of Minkowski's question mark function $?(x)$. In particular, every rational number is mapped to an integer, and every quadratic surd is mapped to a rational number. Perhaps this function can be explored further in a future paper.

References

[1] B. Bates, M. Bunder and K. Tognetti, *Linking the Calkin-Wilf and Stern-Brocot trees*, European Journal of Combinatorics, **31.7** (2010), 1637–1661.

[2] N. Calkin and H. S. Wilf, *Recounting the rationals*, Amer. Math. Monthly, **107.4** (2000), 360–363.

[3] L. Gibbons, D. Lester, and R. Bird, *Functional pearl: Enumerating the rationals*, J. Functional Programming, **16** (2006), 281–291.

[4] D. E. Knuth, *Problem 10906*, Solution by Moshe Newman, *Amer. Math. Monthly*, **110.7** (2003), 642–643.

[5] C. G. Moore, *An Introduction to Continued Fractions*, The National Council of Teachers of Mathematics, Washington, D.C., 1964.

[6] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A003654`.

[7] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A003814`.

[8] P. J. Rippon and H. Taylor, *Even and odd periods in continued fractions of square roots*, The Fibonacci Quarterly, **42.2** (2004), 170–180.

[9] P. Viader and J. Paradis, *A new light on Minkowski's* $?(x)$ *Function*, Journal of Number Theory, **73** (1998), 212–227.

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Department of Mathematics and Statistics, Arkansas State University, Jonesboro, AR
Email address: `wpaulsen@astate.edu`