SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES

THOMAS KOSHY

Abstract. We explore four sums involving gibonacci polynomial squares and their numeric versions, and extract their Pell versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $E = \sqrt{x^2+1}$. Gibonacci and Pell polynomials are linked by the relationship $b_n(x) = g_n(2x)$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ [4, 5, 6].

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [4, 5]; they can be established using Binet-like formulas:

$$f_{2n} = f_n l_n; \tag{1}$$

$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n; \tag{2}$$

$$g_{n+k}^2 - g_{n-k}^2 = \begin{cases} f_{2k}f_{2n}, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k} f_{2n}, & \text{otherwise}; \end{cases} \tag{3}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise}. \end{cases} \tag{4}$$

1.2. Telescoping Gibonacci Sums.

In [6], we investigated the following telescoping sums:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}; \quad \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}};$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}; \quad \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}.$$

1.3. **Generalized Telescoping Gibonacci Sums.** The proofs of the above sums depend only on the subscripts of the polynomials g_n, and *not* on the power of g_n. Consequently, they can be extended to any positive integer power λ of g_n, as the next four lemmas feature; in the interest of brevity, we omit their proofs.

Lemma 1.

$$\sum_{\substack{n=(k+1)/2\\k\geq 1,\, odd}}^{\infty}\left(\frac{1}{g_{2n-k}^{\lambda}}-\frac{1}{g_{2n+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r-1}^{\lambda}}.$$

Lemma 2.

$$\sum_{\substack{n=k/2+1\\k\geq 2,\, even}}^{\infty}\left(\frac{1}{g_{2n-k}^{\lambda}}-\frac{1}{g_{2n+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}^{\lambda}}.$$

Lemma 3.

$$\sum_{\substack{n=(k+1)/2\\k\geq 1,\, odd}}^{\infty}\left(\frac{1}{g_{2n+1-k}^{\lambda}}-\frac{1}{g_{2n+1+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r}^{\lambda}}.$$

Lemma 4.

$$\sum_{\substack{n=k/2\\k\geq 2,\, even}}^{\infty}\left(\frac{1}{g_{2n+1-k}^{\lambda}}-\frac{1}{g_{2n+1+k}^{\lambda}}\right)=\sum_{r=1}^{k}\frac{1}{g_{2r-1}^{\lambda}}.$$

These lemmas with $\lambda = 2$, coupled with identities (1) through (4), play a major role in our explorations of sums involving squares of gibonacci polynomials.

2. Sums Involving Gibonacci Polynomial Squares

We begin our discourse with the following result.

Theorem 1. *Let k be a positive integer; $1 \leq r \leq k$;*

$$L = \begin{cases}(k+1)/2, k\geq 1, & if\ k\ is\ odd;\\ k/2+1, k\geq 2, & otherwise;\end{cases} \quad and \quad s = \begin{cases}2r-1, & if\ k\ is\ odd;\\ 2r, & otherwise.\end{cases}$$

Then,

$$\sum_{n=L}^{\infty}\frac{f_{2k}f_{4n}}{\left[f_{2n}^2-(-1)^k f_k^2\right]^2}=\sum_{r=1}^{k}\frac{1}{f_s^2}.$$

Proof. With identities (3) and (4), we have

$$\begin{aligned}\frac{f_{2k}f_{4n}}{\left[f_{2n}^2-(-1)^k f_k^2\right]^2} &= \frac{f_{2n+k}^2-f_{2n-k}^2}{f_{2n+k}^2 f_{2n-k}^2}\\ &= \frac{1}{f_{2n-k}^2}-\frac{1}{f_{2n+k}^2}.\end{aligned} \tag{5}$$

Suppose k is odd. By Lemma 1, we then get

$$\sum_{\substack{n=(k+1)/2\\k\geq 1,\, odd}}^{\infty}\frac{f_{2k}f_{4n}}{\left(f_{2n}^2+f_k^2\right)^2}=\sum_{r=1}^{k}\frac{1}{f_{2r-1}^2}. \tag{6}$$

On the other hand, let k be even. Then, by Lemma 2, equation (5) yields

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{\infty} \frac{f_{2k}f_{4n}}{\left(f_{2n}^2 - f_k^2\right)^2} = \sum_{r=1}^{k} \frac{1}{f_{2r}^2}. \tag{7}$$

Combining equations (6) and (7), we get the desired result. □

In particular, they yield

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{f_2 f_{4n}}{\left(f_{2n}^2+1\right)^2} &= 1; & \sum_{n=1}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2+1\right)^2} &= 1; \\ \sum_{n=2}^{\infty} \frac{f_4 f_{4n}}{\left(f_{2n}^2-x^2\right)^2} &= \frac{f_2^2+f_4^2}{f_2^2 f_4^2}; & \sum_{n=2}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2-1\right)^2} &= \frac{10}{27}. \end{aligned}$$

With identity (2), Theorem 1 yields

$$\sum_{n=L}^{\infty} \frac{\Delta^4 f_{2k}f_{4n}}{\left[l_{2n}^2 - (-1)^k\Delta^2 f_k^2 - 4\right]^2} = \sum_{r=1}^{k} \frac{1}{f_s^2}.$$

Consequently, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{f_{4n}}{\left(l_{2n}^2+x^2\right)^2} &= \frac{1}{\Delta^4 f_2}; & \sum_{n=1}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2+1\right)^2} &= \frac{1}{25}; \\ \sum_{n=2}^{\infty} \frac{f_{4n}}{\left[l_{2n}^2-(x^2+2)^2\right]^2} &= \frac{f_2^2+f_4^2}{\Delta^4 f_2^2 f_4^3}; & \sum_{n=2}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2-9\right)^2} &= \frac{2}{135}. \end{aligned}$$

Next, we investigate the Lucas counterpart of Theorem 1.

Theorem 2. *Let k be a positive integer; $1 \leq r \leq k$;*

$$L = \begin{cases} (k+1)/2, k \geq 1, & if\ k\ is\ odd; \\ k/2+1, k \geq 2, & otherwise; \end{cases} \quad and \quad s = \begin{cases} 2r-1, & if\ k\ is\ odd; \\ 2r, & otherwise. \end{cases}$$

Then,

$$\sum_{n=L}^{\infty} \frac{\Delta^2 f_{2k}f_{4n}}{\left[l_{2n}^2 + (-1)^k\Delta^2 f_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{l_s^2}.$$

Proof. With equations (3) and (4), we get

$$\begin{aligned} \frac{\Delta^2 f_{2k}f_{4n}}{\left[l_{2n}^2 + (-1)^k\Delta^2 f_k^2\right]^2} &= \frac{l_{2n+k}^2 - l_{2n-k}^2}{l_{2n+k}^2 l_{2n-k}^2} \\ &= \frac{1}{l_{2n-k}^2} - \frac{1}{l_{2n+k}^2}. \end{aligned} \tag{8}$$

Let k be odd. Using Lemma 1, we then get

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k}f_{4n}}{\left(l_{2n}^2 - \Delta^2 f_k^2\right)^2} = \sum_{r=1}^{k} \frac{1}{l_{2r-1}^2}. \tag{9}$$

Suppose k is even. With Lemma 2, equation (8) yields

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_{2k}f_{4n}}{\left(l_{2n}^2 + \Delta^2 f_k^2\right)^2} = \sum_{r=1}^{k} \frac{1}{l_{2r}^2}. \tag{10}$$

By combining the two cases, we get the desired result. □

It follows from equations (9) and (10) that

$$\sum_{n=1}^{\infty}\frac{\Delta^2 f_2 f_{4n}}{\left(l_{2n}^2-\Delta^2\right)^2} = \frac{1}{f_2^2}; \qquad \sum_{n=1}^{\infty}\frac{F_{4n}}{\left(L_{2n}^2-5\right)^2} = \frac{1}{5};$$
$$\sum_{n=2}^{\infty}\frac{\Delta^2 f_4 f_{4n}}{\left(l_{2n}^2+\Delta^2 x^2\right)^2} = \frac{l_2^2+l_4^2}{l_2^2 l_4^2}; \qquad \sum_{n=2}^{\infty}\frac{F_{4n}}{\left(L_{2n}^2+5\right)^2} = \frac{58}{6,615}. \text{ [7]}$$

Using identity (3), Theorem 2 yields

$$\sum_{n=L}^{\infty}\frac{\Delta^2 f_{2k} f_{4n}}{\left[\Delta^2 f_{2n}^2+(-1)^k\Delta^2 f_k^2+4\right]^2} = \sum_{r=1}^{k}\frac{1}{l_s^2}.$$

This implies

$$\sum_{n=2}^{\infty}\frac{\Delta^2 f_2 f_{4n}}{\left(\Delta^2 f_{2n}^2-x^2\right)^2} = \frac{1}{l_1^2}; \qquad \sum_{n=2}^{\infty}\frac{F_{4n}}{(5F_{2n}^2-1)^2} = \frac{1}{5};$$
$$\sum_{n=2}^{\infty}\frac{\Delta^2 f_4 f_{4n}}{\left[\Delta^2 f_{2n}^2+(x^2+2)^2\right]^2} = \frac{l_2^2+l_4^2}{l_2^2 l_4^2}; \qquad \sum_{n=2}^{\infty}\frac{F_{4n}}{\left(5F_{2n}^2+9\right)^2} = \frac{58}{6,615}.$$

The next result is also an application of identity (3).

Theorem 3. *Let k be a positive integer; $1 \le r \le k$;*

$$M = \begin{cases}(k+1)/2, k\ge 1, & if\ k\ is\ odd;\\ k/2, k\ge 2, & otherwise;\end{cases} \quad and \quad t = \begin{cases}2r, & if\ k\ is\ odd;\\ 2r-1, & otherwise.\end{cases}$$

Then,

$$\sum_{n=M}^{\infty}\frac{f_{2k} f_{4n+2}}{\left[f_{2n+1}^2+(-1)^k f_k^2\right]^2} = \sum_{r=1}^{k}\frac{1}{f_t^2}.$$

Proof. Using identity (3), we have

$$\begin{aligned}\frac{f_{2k} f_{4n+2}}{\left[f_{2n+1}^2+(-1)^k f_k^2\right]^2} &= \frac{f_{2n+1+k}^2-f_{2n+1-k}^2}{f_{2n+1+k}^2 f_{2n+1-k}^2}\\ &= \frac{1}{f_{2n+1-k}^2}-\frac{1}{f_{2n+1+k}^2}. \end{aligned} \tag{11}$$

Suppose k is odd. With Lemma 3, this yields

$$\sum_{\substack{n=(k+1)/2\\ k\ge 1,\, odd}}^{\infty}\frac{f_{2k} f_{4n+2}}{\left(f_{2n+1}^2-f_k^2\right)^2} = \sum_{r=1}^{k}\frac{1}{f_{2r}^2}. \tag{12}$$

On the other hand, let k be even. By Lemma 4, equation (11) yields

$$\sum_{\substack{n=k/2\\ k\ge 2,\, even}}^{\infty}\frac{f_{2k} f_{4n+2}}{\left(f_{2n+1}^2+f_k^2\right)^2} = \sum_{r=1}^{k}\frac{1}{f_{2r-1}^2}. \tag{13}$$

By combining equations (12) and (13), we get the desired result. □

It follows from this theorem that

$$\sum_{n=1}^{\infty} \frac{f_2 f_{4n+2}}{\left(f_{2n+1}^2 - 1\right)^2} = \frac{1}{f_2^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{(F_{2n+1}^2 - 1)^2} = 1;$$

$$\sum_{n=1}^{\infty} \frac{f_4 f_{4n+2}}{\left(f_{2n+1}^2 + x^2\right)^2} = \frac{f_1^2 + f_3^2}{f_3^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 + 1\right)^2} = \frac{5}{12}.$$

A Fibonacci Delight. It follows by Theorems 1 and 3 that

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{(F_n^2 - 1)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(F_{2n+1}^2 - 1)^2} + \sum_{n=1}^{\infty} \frac{F_{2(2n)}}{(F_{2n}^2 - 1)^2} \\ &= 1 + \frac{10}{27} \\ &= \frac{37}{27}.\end{aligned}$$

Using identity (3), Theorem 3 yields

$$\sum_{n=M}^{\infty} \frac{\Delta^4 f_{2k} f_{4n+2}}{\left[l_{2n+1}^2 + (-1)^k \Delta^2 f_k^2 + 4\right]^2} = \sum_{r=1}^{k} \frac{1}{f_t^2}.$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{\Delta^4 f_2 f_{4n+2}}{\left(l_{2n+1}^2 - x^2\right)^2} = \frac{1}{f_2^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{(L_{2n+1}^2 - 1)^2} = \frac{1}{25};$$

$$\sum_{n=1}^{\infty} \frac{\Delta^4 f_4 f_{4n+2}}{\left[l_{2n+1}^2 + (x^2+2)^2\right]^2} = \frac{f_1^2 + f_3^2}{f_3^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(L_{2n+1}^2 + 9\right)^2} = \frac{1}{60}.$$

The following result showcases the Lucas version of Theorem 3.

Theorem 4. *Let k be a positive integer; $1 \le r \le k$;*

$$M = \begin{cases} (k+1)/2, k \ge 1, & \textit{if } k \textit{ is odd}; \\ k/2, k \ge 2, & \textit{otherwise}; \end{cases} \quad \textit{and} \quad t = \begin{cases} 2r, & \textit{if } k \textit{ is odd}; \\ 2r-1, & \textit{otherwise}. \end{cases}$$

Then,

$$\sum_{n=M}^{\infty} \frac{\Delta^2 f_{2k} f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{l_t^2}.$$

Proof. With identity (3), we have

$$\begin{aligned}\frac{\Delta^2 f_{2k} f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^2} &= \frac{l_{2n+1+k}^2 - l_{2n+1-k}^2}{l_{2n+1+k}^2 l_{2n+1-k}^2} \\ &= \frac{1}{l_{2n+1-k}^2} - \frac{1}{l_{2n+1+k}^2}.\end{aligned} \tag{14}$$

Suppose k is odd. Using Lemma 3, this yields

$$\sum_{\substack{n=(k+1)/2 \\ k \ge 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n+2}}{\left(l_{2n+1}^2 + \Delta^2 f_k^2\right)^2} = \sum_{r=1}^{k} \frac{1}{l_{2r}^2}. \tag{15}$$

When k is even, by Lemma 4, equation (14) yields

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n+2}}{\left(l_{2n+1}^2 - \Delta^2 f_k^2\right)^2} = \sum_{r=1}^{k} \frac{1}{l_{2r-1}^2}. \tag{16}$$

Equation (15), coupled with equation (16), yields the desired result. □

With identity (1), this theorem yields

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_2 f_{4n+2}}{\left(l_{2n+1}^2 + \Delta^2\right)^2} = \frac{1}{l_2^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{(L_{2n+1}^2 + 5)^2} = \frac{1}{45}; \text{ [7]}$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_4 f_{4n+2}}{\left(l_{2n+1}^2 - \Delta^2 x^2\right)^2} = \frac{l_1^2 + l_3^2}{l_1^2 l_3^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(L_{2n+1}^2 - 5\right)^2} = \frac{17}{240}. \text{ [7]}$$

Using identity (3), it follows from Theorem 4 that

$$\sum_{n=M}^{\infty} \frac{\Delta^2 f_{2k} f_{4n+2}}{\left[\Delta^2 f_{2n+1}^2 - (-1)^k \Delta^2 f_k^2 - 4\right]^2} = \sum_{r=1}^{k} \frac{1}{l_t^2}.$$

It then follows that

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_2 f_{4n+2}}{\left(\Delta^2 f_{2n+1}^2 + x^2\right)^2} = \frac{1}{l_2^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{(5F_{2n+1}^2 + 1)^2} = \frac{1}{45};$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_4 f_{4n+2}}{\left[\Delta^2 f_{2n+1}^2 - (x^2+2)^2\right]^2} = \frac{l_1^2 + l_3^2}{l_1^2 l_3^2}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(5F_{2n+1}^2 - 9\right)^2} = \frac{17}{240}.$$

Next we explore the Pell versions of the theorems.

3. Pell Implications

With the relationship $b_n(x) = g_n(2x)$, Theorems 1–4 yield the following Pell versions:

$$\sum_{n=L}^{\infty} \frac{p_{2k} p_{4n}}{\left[p_{2n}^2 - (-1)^k p_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{p_s^2}; \qquad \sum_{n=L}^{\infty} \frac{4E^2 p_{2k} p_{4n}}{\left[q_{2n}^2 + 4(-1)^k E^2 p_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{q_s^2};$$

$$\sum_{n=M}^{\infty} \frac{p_{2k} p_{4n+2}}{\left[p_{2n+1}^2 + (-1)^k p_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{p_t^2}; \qquad \sum_{n=M}^{\infty} \frac{4E^2 p_{2k} p_{4n+2}}{\left[q_{2n+1}^2 - 4(-1)^k E^2 p_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{q_t^2},$$

respectively. Consequently, we have

$$\sum_{n=L}^{\infty} \frac{P_{2k} P_{4n}}{\left[P_{2n}^2 - (-1)^k P_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{P_s^2}; \qquad \sum_{n=L}^{\infty} \frac{2P_{2k} P_{4n}}{\left[Q_{2n}^2 + 2(-1)^k P_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{Q_s^2};$$

$$\sum_{n=M}^{\infty} \frac{P_{2k} P_{4n+2}}{\left[P_{2n+1}^2 + (-1)^k P_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{P_t^2}; \qquad \sum_{n=M}^{\infty} \frac{2P_{2k} P_{4n+2}}{\left[Q_{2n+1}^2 - 2(-1)^k P_k^2\right]^2} = \sum_{r=1}^{k} \frac{1}{Q_t^2},$$

again respectively.

4. Chebyshev and Vieta Implications

Finally, Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$, where $i = \sqrt{-1}$ [2, 3, 4]. They can be employed to find the Chebyshev and Vieta versions of Theorems 1–4. In the interest of brevity, we omit them; but we encourage gibonacci enthusiasts to pursue them.

5. Acknowledgment

The author thanks the reviewer for a careful reading of the article, and for encouraging and supporting words.

References

[1] M. Bicknell, *A Primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] A. F. Horadam, *Vieta polynomials*, The Fibonacci Quarterly, **40.3** (2002), 223–232.
[3] T. Koshy, *Vieta polynomials and their close relatives*, The Fibonacci Quarterly, **54.2** (2016), 141–148.
[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[5] T. Koshy, *Sums involving gibonacci polynomials*, The Fibonacci Quarterly, **60.4** (2022), 344–351.
[6] T. Koshy, *Additional sums involving gibonacci polynomials*, The Fibonacci Quarterly, **61.1** (2023) 12–20.
[7] T. Koshy, *Infinite sums involving gibonacci polynomials revisited*, The Fibonacci Quarterly, **60.2** (2022), 120–125.

MSC2020: Primary 11B37, 11B39, 11C08.

Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu