SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES: GRAPH-THEORETIC CONFIRMATIONS

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Abstract. We confirm eight sums involving gibonacci polynomial squares using graph-theoretic techniques.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, and $\Delta = \sqrt{x^2+4}$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ [2, 5, 6].

1.1. Fundamental Gibonacci Identities.

Using Binet-like formulas, we can establish the following gibonacci properties [2, 5]:

$$g_{n+k}^2 - g_{n-k}^2 = \begin{cases} f_{2k}f_{2n}, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k}f_{2n}, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise.} \end{cases} \tag{2}$$

1.2. Telescoping Gibonacci Sums.

In [6], we investigated the following four telescoping sums:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}; \quad \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}} - \frac{1}{g_{2n+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}};$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}}; \quad \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n+1-k}} - \frac{1}{g_{2n+1+k}}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}}.$$

Their proofs depend only on the subscripts of the polynomials g_n, and *not* on the power of g_n. Consequently, we can extend them to any positive integer power λ of g_n. The next four lemmas feature their proofs for the case $\lambda = 2$. Coupled with the above identities, they play a pivotal role in our investigations.

Lemma 1. *Let k be an odd positive integer. Then,*

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n-k}^2} - \frac{1}{g_{2n+k}^2} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}^2}. \tag{3}$$

Proof. With recursion [2], we will first confirm that

$$\sum_{\substack{n=(k+1)/2\\ k\geq 1,\, odd}}^{m} \left(\frac{1}{g_{2n-k}^2} - \frac{1}{g_{2n+k}^2} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-2r+k}^2}.$$

To this end, we let A_m denote the left-hand side (LHS) of this equation and B_m its right-hand side (RHS). Then,

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-2(r+1)+k}^2} - \frac{1}{g_{2m-2r+k}^2} \right] \\ &= \frac{1}{g_{2m-k}^2} - \frac{1}{g_{2m+k}^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

By recursion, it then follows that

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} \\ &= \left(\frac{1}{g_1^2} - \frac{1}{g_{2k+1}^2} \right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r-1}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-(2r-1)}^2} \right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$.

Because $\lim_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$, this yields the desired result. □

Lemma 2. *Let k be an even positive integer. Then,*

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n-k}^2} - \frac{1}{g_{2n+k}^2} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}^2}. \tag{4}$$

Proof. Invoking recursion [2], we will first establish that

$$\sum_{\substack{n=k/2+1\\ k\geq 2,\, even}}^{m} \left(\frac{1}{g_{2n-k}^2} - \frac{1}{g_{2n+k}^2} \right) = \sum_{r=1}^{k} \frac{1}{g_{2r}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-2r+k}^2}.$$

Letting A_m = LHS of this equation and B_m its RHS, we then get

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-2(r+1)+k}^2} - \frac{1}{g_{2m-2r+k}^2} \right] \\ &= \frac{1}{g_{2m-k}^2} - \frac{1}{g_{2m+k}^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, we then have

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{k/2+1} - B_{k/2+1} \\ &= \left(\frac{1}{g_2^2} - \frac{1}{g_{2k+2}^2}\right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-2(r-1)}^2}\right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$.

The given result follows from this formula, as desired. ☐

Lemma 3. *Let k be an odd positive integer. Then,*

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{g_{2n+1-k}^2} - \frac{1}{g_{2n+1+k}^2}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}^2}. \tag{5}$$

Proof. Using recursion [2], we will first establish the formula

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{m} \left(\frac{1}{g_{2n+1-k}^2} - \frac{1}{g_{2n+1+k}^2}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-(2r-1)+k}^2}.$$

Letting A_m = LHS of this equation and B_m its RHS, we get

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g_{2m-(2r+1)+k}^2} - \frac{1}{g_{2m-(2r-1)+k}^2}\right] \\ &= \frac{1}{g_{2m+1-k}^2} - \frac{1}{g_{2m+1+k}^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

This yields,

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{(k+1)/2} - B_{(k+1)/2} \\ &= \left(\frac{1}{g_2^2} - \frac{1}{g_{2k+2}^2}\right) - \left[\sum_{r=1}^{k} \frac{1}{g_{2r}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2k-2(r-1)}^2}\right] \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$.

The given result now follows from this formula. ☐

Lemma 4. *Let k be an even positive integer. Then,*

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{g_{2n+1-k}^2} - \frac{1}{g_{2n+1+k}^2}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}^2}. \tag{6}$$

Proof. To establish this formula, using recursion [2] we will first confirm that

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{m} \left(\frac{1}{g_{2n+1-k}^2} - \frac{1}{g_{2n+1+k}^2}\right) = \sum_{r=1}^{k} \frac{1}{g_{2r-1}^2} - \sum_{r=0}^{k-1} \frac{1}{g_{2m-(2r-1)+k}^2}.$$

With A_m = LHS and B_m = RHS of this equation, we then get

$$\begin{aligned} B_m - B_{m-1} &= \sum_{r=0}^{k-1} \left[\frac{1}{g^2_{2m-(2r+1)+k}} - \frac{1}{g^2_{2m-(2r-1)+k}} \right] \\ &= \frac{1}{g^2_{2m+1-k}} - \frac{1}{g^2_{2m+1+k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_{k/2} - B_{k/2} \\ &= \left(\frac{1}{g^2_1} - \frac{1}{g^2_{2k+1}} \right) - \left[\sum_{r=1}^{k} \frac{1}{g^2_{2r-1}} - \sum_{r=0}^{k-1} \frac{1}{g^2_{2k-(2r-1)}} \right] \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, yielding in the validity of the given formula. □

1.3. **Sums Involving Gibonacci Polynomial Squares.** In [7], we investigated the following sums involving gibonacci polynomial squares:

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{2k} f_{4n}}{\left(f^2_{2n} + f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{f^2_{2r-1}}; \tag{7}$$

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{f_{2k} f_{4n}}{\left(f^2_{2n} - f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{f^2_{2r}}; \tag{8}$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n}}{\left(l^2_{2n} - \Delta^2 f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{l^2_{2r-1}}; \tag{9}$$

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n}}{\left(l^2_{2n} + \Delta^2 f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{l^2_{2r}}; \tag{10}$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{2k} f_{4n+2}}{\left(f^2_{2n+1} - f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{f^2_{2r}}; \tag{11}$$

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{f_{2k} f_{4n+2}}{\left(f^2_{2n+1} + f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{f^2_{2r-1}}; \tag{12}$$

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n+2}}{\left(l^2_{2n+1} + \Delta^2 f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{l^2_{2r}}; \tag{13}$$

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n+2}}{\left(l^2_{2n+1} - \Delta^2 f^2_k\right)^2} = \sum_{r=1}^{k} \frac{1}{l^2_{2r-1}}. \tag{14}$$

Our objective is to confirm these gibonacci sums using graph-theoretic techniques. To achieve this goal, we now present the essential graph-theoretic tools.

2. Graph-theoretic Tools

Consider the *Fibonacci digraph* in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [2, 3, 4].

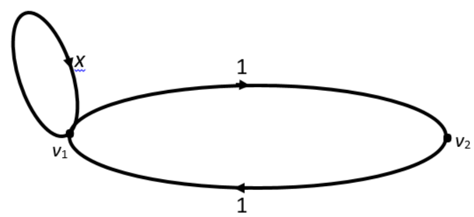

Figure 1. Weighted Fibonacci Digraph

It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [2, 3, 4]. We extend the exponent n to 0, which is consistent with the *Cassini-like formula* $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ [2].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; and *open*, otherwise. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph, where $1 \leq i, j \leq n$ [2, 3, 4]. Consequently, the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$.

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then, the sum of the weights of the elements (a, b) in the product set $A \times B$ is *defined* as the product of the sums of weights from each component [3, 4]. This definition can be extended to any finite number of component sets. In particular, let A, B, C, and D denote the sets of walks of varying lengths originating at a vertex v, respectively. Then, the sum of the weights of the elements (a, b, c, d) in the product set $A \times B \times C \times D$ is the product of the sums of weights from each component [3, 4].

We now make an interesting observation. Let $A = \{u\}$ and $B = \{v\}$, where u denotes the closed walk v_1-v_1 and v denotes the closed walk v_1-v_2-v_1. The weight of the element in $A \times A$ is x^2 and that in $B \times B$ is 1. So, the sum w of the weights of the elements in $C^* = (A \times A) \cup (B \times B) \cup (B \times B) \cup (B \times B) \cup (B \times B)$ is given by $w = x^2 + 4 = \Delta^2$.

These tools play a pivotal role in the graph-theoretic proofs. With them at our finger tips, we are now ready for the proofs.

3. Graph-theoretic Confirmations

Let T_n^* denote the set of closed walks of length n in the digraph originating at v_1, and U_n^* the set of all closed walks of length n in the digraph. Correspondingly, let T_n denote the sum of the weights of elements in T_n^* and U_n that of those in U_n^*. Clearly, $T_n = f_{n+1}$ and $U_n = f_{n+1} + f_{n-1} = l_n$ [2]. With this brief background, we now begin our discourse with the gibonacci sums (7) and (8). Throughout, k denotes a positive integer.

3.1. Confirmations of Equations (7) and (8).

Proof. The sum of the weights of the elements in the product set $T_{2k-1}^* \times T_{4n-1}^*$ is $T_{2k-1}T_{4n-1} = f_{2k}f_{4n}$; the sum of those in $(T_{2n-1}^* \times T_{2n-1}^*) \cup (T_{k-1}^* \times T_{k-1}^*)$ is $T_{2n-1}^2 + T_{k-1}^2 = f_{2n}^2 + f_k^2$; and the sum of those in $(T_{2n-1}^* \times T_{2n-1}^*) - (T_{k-1}^* \times T_{k-1}^*)$ is $T_{2n-1}^2 - T_{k-1}^2 = f_{2n}^2 - f_k^2$.

Combining the two cases, we let

$$S_n = \frac{T_{2k-1}T_{4n-1}}{\left[T_{2n-1}^2 - (-1)^k T_{k-1}^2\right]^2}, \tag{15}$$

where k is odd or even.

Suppose k is odd. Using identities (1) and (2), and Lemma 1, this yields

$$\begin{aligned} S_n &= \frac{f_{2k}f_{4n}}{(f_{2n}^2 + f_k^2)^2} \\ &= \frac{f_{2n+k}^2 - f_{2n+k}^2}{f_{2n+k}^2 f_{2n-k}^2}, \\ \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{f_{2k}f_{4n}}{(f_{2n}^2 + f_k^2)^2} &= \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{f_{2n-k}^2} - \frac{1}{f_{2n+k}^2}\right) \\ &= \sum_{r=1}^{k} \frac{1}{f_{2r-1}^2}, \end{aligned} \tag{16}$$

confirming equation (7), as desired.

On the other hand, let k be even. With identities (1) and (2), and Lemma 2, we get

$$\begin{aligned} S_n &= \frac{f_{2k}f_{4n}}{(f_{2n}^2 - f_k^2)^2} \\ &= \frac{f_{2n+k}^2 - f_{2n+k}^2}{f_{2n+k}^2 f_{2n-k}^2}, \end{aligned}$$

$$\begin{aligned} \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{f_{2k}f_{4n}}{(f_{2n}^2 - f_k^2)^2} &= \sum_{\substack{n=k/2+1 \\ k\geq 1,\, even}}^{\infty} \left(\frac{1}{f_{2n-k}^2} - \frac{1}{f_{2n+k}^2}\right) \\ &= \sum_{r=1}^{k} \frac{1}{f_{2r}^2}. \end{aligned} \tag{17}$$

This gives the desired sum in equation (8). □

It follows from equations (16) and (17) that [7]

$$\sum_{n=1}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2+1\right)^2} = 1; \qquad \sum_{n=2}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2-1\right)^2} = \frac{10}{27};$$
$$\sum_{n=2}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2+4\right)^2} = \frac{129}{800}; \qquad \sum_{n=3}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2-9\right)^2} = \frac{31,865}{592,704}.$$

Next, we focus on the graph-theoretic proofs of sums (9) and (10).

3.2. **Confirmations of Equations** (9) **and** (10)**.**

Proof. Let T_n^*, U_n^*, T_n, and U_n be as before. U_{2n}^* denotes the set of all closed walks of length $2n$ in the digraph and $U_{2n} = l_{2n}$; so U_{2n}^2 gives the sum of the weights of the elements in the product set $U_{2n}^* \times U_{2n}^*$. With the set C^* as in Section 2, the sum of the weight of weight of the elements in the set $C^* \times T_{k-1}^* \times T_{k-1}^*$ is $wT_k^2 = \Delta^2 T_k^2$.

The sum of the elements in the set $U_{2n}^* \times U_{2n}^* - C^* \times T_k^* \times T_k^*$ equals $U_{2n}^2 - wT_{k-1}^2 = l_{2n}^2 - \Delta^2 f_k^2$; and that of those in the set $U_{2n}^* \times U_{2n}^* \cup C^* \times T_k^* \times T_k^*$ equals $U_{2n}^2 + wT_{k-1}^2 = l_{2n}^2 + \Delta^2 f_k^2$.

Combining the two cases, we now let

$$\begin{aligned} S_n &= \frac{T_{2k-1}T_{4n-1}}{\left[U_{2n}^2 + (-1)^k\, wT_{k-1}^2\right]^2} \\ &= \frac{f_{2k}f_{4n}}{\left[l_{2n}^2 + (-1)^k \Delta^2 f_k^2\right]^2}, \end{aligned} \tag{18}$$

where k is odd or even.

Suppose k is odd. With identities (1) and (2), and Lemma 1, we have

$$\begin{aligned} \frac{\Delta^2 f_{2k} f_{4n}}{(l_{2n}^2 - \Delta^2 f_k^2)^2} &= \frac{l_{2n+k}^2 - l_{2n+k}^2}{l_{2n+k}^2 l_{2n-k}^2}, \\ \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n}}{(l_{2n}^2 - \Delta^2 f_k^2)^2} &= \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{l_{2n-k}^2} - \frac{1}{l_{2n+k}^2} \right) \\ &= \sum_{r=1}^{k} \frac{1}{l_{2r-1}^2}, \end{aligned} \tag{19}$$

as in equation (9).

On the flip side, let k be even. Using identities (1) and (2), and Lemma 2, it follows from equation (18) that

$$\begin{aligned} S_n &= \frac{f_{2k} f_{4n}}{\left(l_{2n}^2 + \Delta^2 f_k^2\right)^2}, \\ \Delta^2 S_n &= \frac{l_{2n+k}^2 - l_{2n+k}^2}{l_{2n+k}^2 l_{2n-k}^2}, \\ \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_{2k} f_{4n}}{\left(l_{2n}^2 + \Delta^2 f_k^2\right)^2} &= \sum_{\substack{n=k/2+1 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{l_{2n-k}^2} - \frac{1}{l_{2n+k}^2} \right) \\ &= \sum_{r=1}^{k} \frac{1}{l_{2r}^2}, \end{aligned} \tag{20}$$

as in equation (10). □

Equations (19) and (20) yield [7]

$$\sum_{n=1}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2 - 5\right)^2} = \frac{1}{5}; \qquad \sum_{n=2}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2 + 5\right)^2} = \frac{58}{6,615};$$

$$\sum_{n=2}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2 - 20\right)^2} = \frac{2,073}{77,440}; \qquad \sum_{n=3}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2 + 45\right)^2} = \frac{4,736,509}{3,682,358,820}.$$

3.3. **Confirmations of Equations** (11) **and** (12)**.**

Proof. With T_n^*, U_n^*, T_n, and U_n as before, we have

$$\frac{T_{2k-1}T_{4n+1}}{\left[T_{2n}^2 + (-1)^k T_{k-1}^2\right]^2} = \frac{f_{2k}f_{4n+2}}{\left[f_{2n+1}^2 + (-1)^k f_k^2\right]^2}, \tag{21}$$

where k is odd or even.

Suppose k is odd. Using identities (1) and (2), and Lemma 3, we have

$$\begin{aligned}
\frac{\Delta^2 f_{2k}f_{4n+2}}{(f_{2n+1}^2 - f_k^2)^2} &= \frac{f_{2n+1+k}^2 - f_{2n+1-k}^2}{f_{2n+1+k}^2 f_{2n+1-k}^2}, \\
\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k}f_{4n+2}}{(f_{2n+1}^2 - f_k^2)^2} &= \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{f_{2n+1-k}^2} - \frac{1}{f_{2n+1+k}^2}\right) \\
&= \sum_{r=1}^{k} \frac{1}{f_{2r}^2},
\end{aligned} \tag{22}$$

confirming equation (11) as desired.

On the other hand, let k be even. With identities (1) and (2), and Lemma 4, we get

$$\begin{aligned}
\frac{f_{2k}f_{4n+2}}{(f_{2n+1}^2 + f_k^2)^2} &= \frac{f_{2n+1+k}^2 - f_{2n+1-k}^2}{f_{2n+1+k}^2 f_{2n+1-k}^2}, \\
\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{f_{2k}f_{4n+2}}{(f_{2n+1}^2 + f_k^2)^2} &= \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{f_{2n+1-k}^2} - \frac{1}{f_{2n+1+k}^2}\right) \\
&= \sum_{r=1}^{k} \frac{1}{f_{2r-1}^2}.
\end{aligned} \tag{23}$$

This confirms equation (12), as expected. □

It follows from equations (22) and (23) that [7]

$$\sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 - 1\right)^2} = \frac{1}{5}; \qquad \sum_{n=2}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 - 4\right)^2} = \frac{649}{8,640};$$

$$\sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 + 1\right)^2} = \frac{5}{12}; \qquad \sum_{n=2}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 + 9\right)^2} = \frac{21,901}{354,900}.$$

Finally, we turn to the confirmations of sums (13) and (14).

3.4. **Confirmations of Equations** (13) **and** (14)**.**

Proof. Using the product sets $T^*_{2k-1} \times T^*_{4n+1}$, $U^*_{2n+1} \times U^*_{2n+1}$, $C^* \times C^*$, and $T^*_{k-1} \times T^*_{k-1}$, we get

$$\frac{T_{2k-1}T_{4n+1}}{\left[U^2_{2n+1} - (-1)^k w T^2_{k-1}\right]^2} = \frac{f_{2k}f_{4n+2}}{\left[l^2_{2n+1} - (-1)^k \Delta^2 f^2_k\right]^2}, \tag{24}$$

where k is odd or even.

Suppose k is odd. Then,

$$\frac{T_{2k-1}T_{4n+1}}{\left(U^2_{2n+1} + wT^2_{k-1}\right)^2} = \frac{f_{2k}f_{4n+2}}{\left(l^2_{2n+1} + \Delta^2 f^2_k\right)^2}. \tag{25}$$

With identities (1) and (2), and Lemma 4, we have

$$\begin{aligned}
\frac{\Delta^2 f_{2k}f_{4n+2}}{\left(l^2_{2n+1} + \Delta^2 f^2_k\right)^2} &= \frac{l^2_{2n+1+k} - l^2_{2n+1-k}}{l^2_{2n+1+k}l^2_{2n+1-k}}, \\
\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \frac{\Delta^2 f_{2k}f_{4n+2}}{(l^2_{2n+1} + \Delta^2 f^2_k)^2} &= \sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, odd}}^{\infty} \left(\frac{1}{f^2_{2n+1-k}} - \frac{1}{f^2_{2n+1+k}}\right) \\
&= \sum_{r=1}^{k} \frac{1}{l^2_{2r}}, \qquad (26)
\end{aligned}$$

confirming equation (13).

When k is even, we get similarly from equation (24) that

$$\begin{aligned}
\frac{\Delta^2 f_{2k}f_{4n+2}}{\left(l^2_{2n+1} - \Delta^2 f^2_k\right)^2} &= \frac{l^2_{2n+1+k} - l^2_{2n+1-k}}{l^2_{2n+1+k}l^2_{2n+1-k}}, \\
\sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \frac{\Delta^2 f_{2k}f_{4n+2}}{(l^2_{2n+1} - \Delta^2 f^2_k)^2} &= \sum_{\substack{n=k/2 \\ k\geq 2,\, even}}^{\infty} \left(\frac{1}{l^2_{2n+1-k}} - \frac{1}{l^2_{2n+1+k}}\right) \\
&= \sum_{r=1}^{k} \frac{1}{l^2_{2r-1}}. \qquad (27)
\end{aligned}$$

This confirms equation (14), as desired. □

It follows by equations (26) and (27) that [7]

$$\sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(L^2_{2n+1} + 5\right)^2} = \frac{1}{45}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(L^2_{2n+1} - 5\right)^2} = \frac{17}{240};$$

$$\sum_{n=2}^{\infty} \frac{F_{4n+2}}{\left(L^2_{2n+1} + 20\right)^2} = \frac{2,137}{238,140}; \qquad \sum_{n=2}^{\infty} \frac{F_{4n+2}}{\left(L^2_{2n+1} - 45\right)^2} = \frac{1,745,329}{170,958,480}.$$

4. Pell Consequences

With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, we can construct the graph-theoretic proofs of the Pell versions of equations (7) – (14) independently by changing the weight of the loop at v_1 from x to $2x$. We encourage the gibonacci enthusiasts to pursue them.

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