SUMS INVOLVING JACOBSTHAL POLYNOMIAL SQUARES

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Abstract. We explore the Jacobsthal consequences of four infinite sums involving gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 5].

Gibonacci and Jacobsthal polynomials are linked by the relationships
$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4, 5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2+4}$, and $D = \sqrt{4x+1}$, where $c_n = c_n(x)$.

1.1. **Sums Involving Gibonacci Polynomial Squares.**

In Theorems 1–4 of [6], we studied the following sums involving gibonacci polynomial squares:

$$\sum_{n=L}^{\infty} \frac{f_{4n}}{\left[f_{2n}^2 - (-1)^k f_k^2\right]^2} = \frac{1}{f_{2k}} \sum_{r=1}^{k} \frac{1}{f_s^2}; \tag{1}$$

$$\sum_{n=L}^{\infty} \frac{f_{4n}}{\left[l_{2n}^2 + (-1)^k \Delta^2 f_k^2\right]^2} = \frac{1}{\Delta^2 f_{2k}} \sum_{r=1}^{k} \frac{1}{l_s^2}; \tag{2}$$

$$\sum_{n=M}^{\infty} \frac{f_{4n+2}}{\left[f_{2n+1}^2 + (-1)^k f_k^2\right]^2} = \frac{1}{f_{2k}} \sum_{r=1}^{k} \frac{1}{f_t^2}; \tag{3}$$

$$\sum_{n=M}^{\infty} \frac{f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^2} = \frac{1}{\Delta^2 f_{2k}} \sum_{r=1}^{k} \frac{1}{l_t^2}, \tag{4}$$

where k is a positive integer; $1 \leq r \leq k$;

$$L = \begin{cases} (k+1)/2, k \geq 1, & \text{if } k \text{ is odd;} \\ k/2+1, k \geq 2, & \text{otherwise;} \end{cases} \qquad s = \begin{cases} 2r-1, & \text{if } k \text{ is odd;} \\ 2r, & \text{otherwise;} \end{cases}$$

$$M = \begin{cases} (k+1)/2, k \geq 1, & \text{if } k \text{ is odd;} \\ k/2, k \geq 2, & \text{otherwise;} \end{cases} \quad \text{and} \quad t = \begin{cases} 2r, & \text{if } k \text{ is odd;} \\ 2r-1, & \text{otherwise.} \end{cases}$$

2. Jacobsthal Consequences

Our objective is to extract the Jacobsthal versions of the gibonacci sums (1)–(4); we will accomplish this using the Jacobsthal-gibonacci relationships cited above. To this end, in the interest of clarity and convenience, we let A denote the left side of each sum and B its right side; and LHS the left-hand side of the corresponding Jacobsthal sum and RHS its right-hand side.

2.1. **Jacobsthal Version of Equation** (1)**.**

Proof. Let $A = \dfrac{f_{4n}}{\left[f_{2n}^2 - (-1)^k f_k^2\right]^2}$. Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with x^{4n-2}. This yields

$$\begin{aligned} A &= \frac{x^{(4n-1)/2}\left[x^{(4n-1)/2} f_{4n}\right]}{\left\{[x^{(2n-1)/2} f_{2n}]^2 - (-1)^k x^{2n-k}[x^{(k-1)/2} f_k]^2\right\}^2} \\ &= \frac{x^{(4n-1)/2} J_{4n}}{\left[J_{2n}^2 - (-1)^k x^{2n-k} J_k^2\right]^2}; \\ \text{LHS} &= \sum_{n=L}^{\infty} \frac{x^{(4n-1)/2} J_{4n}}{\left[J_{2n}^2 - (-1)^k x^{2n-k} J_k^2\right]^2}, \end{aligned} \tag{5}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Case 1. Suppose k is odd. Now, let $B = \dfrac{1}{f_{2k}} \displaystyle\sum_{r=1}^{k} \frac{1}{f_{2r-1}^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{(4r+2k-5)/2}$ yields

$$\begin{aligned} B &= \frac{x^{(2k-1)/2}}{x^{(2k-1)/2} f_{2k}} \sum_{r=1}^{k} \frac{x^{2r-2}}{\left[x^{(2r-2)/2} f_{2r-1}\right]^2}; \\ \text{RHS} &= \frac{x^{(2k-5)/2}}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{J_{2r-1}^2}, \end{aligned} \tag{6}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (5) with k odd, yields

$$\sum_{n=L}^{\infty} \frac{x^{2n} J_{4n}}{\left(J_{2n}^2 + x^{2n-k} J_k^2\right)^2} = \frac{x^{k-1}}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{J_{2r-1}^2}. \tag{7}$$

Case 2. Suppose k is even. We then have $B = \dfrac{1}{f_{2k}} \displaystyle\sum_{r=1}^{k} \frac{1}{f_{2r}^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator of the resulting expression with $x^{(4r+2k-3)/2}$. This gives

$$\begin{aligned} B &= \frac{x^{(2k-1)/2}}{x^{(2k-1)/2} f_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{\left[x^{(2r-1)/2} f_{2r}\right]^2}; \\ \text{RHS} &= \frac{x^{(2k-3)/2}}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{J_{2r}^2}, \end{aligned} \tag{8}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

It then follows by equations (5) and (8) that

$$\sum_{n=L}^{\infty} \frac{x^{2n} J_{4n}}{\left(J_{2n}^2 - x^{2n-k} J_k^2\right)^2} = \frac{x^{k-1}}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{J_{2r}^2}.$$

Combining the two cases, we get the Jacobsthal version of equation (1):

$$\sum_{n=L}^{\infty} \frac{x^{2n} J_{4n}}{\left[J_{2n}^2 - (-1)^k x^{2n-k} J_k^2\right]^2} = \frac{x^{k-1}}{J_{2k}} \sum_{r=1}^{k} \frac{x^s}{J_s^2}. \tag{9}$$

□

In particular, this yields

$$\sum_{n=L}^{\infty} \frac{F_{4n}}{\left[F_{2n}^2 - (-1)^k F_k^2\right]^2} = \frac{1}{F_{2k}} \sum_{r=1}^{k} \frac{1}{F_s^2}; \qquad \sum_{n=L}^{\infty} \frac{4^n J_{4n}}{\left[J_{2n}^2 - (-1)^k 2^{2n-k} J_k^2\right]^2} = \frac{2^{k-1}}{J_{2k}} \sum_{r=1}^{k} \frac{2^s}{J_s^2}.$$

Consequently, we have [6]

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2 + 1\right)^2} &= 1; & \sum_{n=1}^{\infty} \frac{F_{4n}}{\left(F_{2n}^2 - 1\right)^2} &= \frac{10}{27}; \\ \sum_{n=1}^{\infty} \frac{4^n J_{4n}}{\left(J_{2n}^2 + 2^{2n-1}\right)^2} &= 2; & \sum_{n=2}^{\infty} \frac{4^n J_{4n}}{\left(J_{2n}^2 - 2^{2n-2}\right)^2} &= \frac{232}{125}. \end{aligned}$$

2.2. **Jacobsthal Version of Equation** (2).

Proof. Let $A = \dfrac{f_{4n}}{\left[l_{2n}^2 + (-1)^k \Delta^2 f_k^2\right]^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with x^{4n-2} yields

$$\begin{aligned} A &= \frac{x^2 f_{4n}}{\left[x l_{2n}^2 + (-1)^k D^2 f_k^2\right]^2} \\ &= \frac{x^{(4n-3)/2}\left[x^{(4n-1)/2} f_{4n}\right]}{\left\{(x^{2n/2} l_{2n})^2 + (-1)^k D^2 x^{2n-k}[x^{(k-1)/2} f_k]^2\right\}^2} \\ &= \frac{x^{(4n-3)/2} J_{4n}}{\left[j_{2n}^2 + (-1)^k D^2 x^{2n-k} J_k^2\right]^2}; \\ \text{LHS} &= \sum_{n=L}^{\infty} \frac{x^{(4n-3)/2} J_{4n}}{\left[j_{2n}^2 + (-1)^k D^2 x^{2n-k} J_k^2\right]^2}, \end{aligned} \tag{10}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

With $B = \dfrac{1}{\Delta^2 f_{2k}} \displaystyle\sum_{r=1}^{k} \dfrac{1}{l_s^2}$, let k be odd. Now, replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{(4r+2k-3)/2}$. Then,

$$\begin{aligned} B &= \frac{x}{D^2 f_{2k}} \sum_{r=1}^{k} \frac{1}{l_{2r-1}^2} \\ &= \frac{x^{(2k+1)/2}}{D^2 \left[x^{(2k-1)/2} f_{2k}\right]} \sum_{r=1}^{k} \frac{x^{2r-1}}{\left[x^{(2r-1)/2} l_{2r-1}\right]^2}; \\ \text{RHS} &= \frac{x^{(2k+1)/2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{j_{2r-1}^2}, \end{aligned} \tag{11}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Using equations (10) and (11), we get

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{x^{2n} J_{4n}}{\left(j_{2n}^2 - D^2 x^{2n-k} J_k^2\right)^2} = \frac{x^{k+2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{j_{2r-1}^2}.$$

Suppose k is even. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{(4r+2k-1)/2}$ yields

$$\begin{aligned} B &= \frac{x}{D^2 f_{2k}} \sum_{r=1}^{k} \frac{1}{l_{2r}^2} \\ &= \frac{x^{(2k+1)/2}}{D^2 \left[x^{(2k-1)/2} f_{2k}\right]} \sum_{r=1}^{k} \frac{x^{2r}}{\left(x^{2r/2} l_{2r}\right)^2}; \\ \text{RHS} &= \frac{x^{(2k+1)/2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{j_{2r}^2}, \end{aligned} \tag{12}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

With equations (10) and (12), we get

$$\sum_{\substack{n=k/2+1 \\ k\geq 2,\text{even}}}^{\infty} \frac{x^{2n} J_{4n}}{\left(j_{2n}^2 + D^2 x^{2n-k} J_k^2\right)^2} = \frac{x^{k+2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{j_{2r}^2}.$$

By combining the two cases, we get the Jacobsthal version of equation (2):

$$\sum_{n=L}^{\infty} \frac{x^{2n} J_{4n}}{\left[j_{2n}^2 + (-1)^k D^2 x^{2n-k} J_k^2\right]^2} = \frac{x^{k+2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^s}{j_s^2}. \tag{13}$$

□

This implies,

$$\sum_{n=L}^{\infty} \frac{F_{4n}}{\left[L_{2n}^2 + 5(-1)^k F_k^2\right]^2} = \frac{1}{5F_{2k}} \sum_{r=1}^{k} \frac{1}{L_s^2}; \qquad \sum_{n=L}^{\infty} \frac{2^{2n} J_{4n}}{\left[j_{2n}^2 + 9(-1)^k 2^{2n-k} J_k^2\right]^2} = \frac{2^{k+2}}{9J_{2k}} \sum_{r=1}^{k} \frac{2^s}{j_s^2}.$$

It then follows that [6]

$$\sum_{n=1}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2 - 5\right)^2} = \frac{1}{5}; \qquad \sum_{n=2}^{\infty} \frac{F_{4n}}{\left(L_{2n}^2 + 5\right)^2} = \frac{58}{6,615};$$
$$\sum_{n=1}^{\infty} \frac{4^n J_{4n}}{\left(j_{2n}^2 - 9 \cdot 2^{2n-1}\right)^2} = \frac{16}{9}; \qquad \sum_{n=2}^{\infty} \frac{4^n J_{4n}}{\left(j_{2n}^2 + 9 \cdot 2^{2n-2}\right)^2} = \frac{24,896}{325,125}.$$

2.3. **Jacobsthal Version of Equation** (3)**.**

Proof. Let $A = \dfrac{f_{4n+2}}{\left[f_{2n+1}^2 + (-1)^k f_k^2\right]^2}$. Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with $x^{(4n+1)/2}$. We then get

$$\begin{aligned} A &= \frac{x^{(4n-1)/2}\left[x^{(4n+1)/2} f_{4n+2}\right]}{\left\{\left(x^{2n/2} f_{2n+1}\right)^2 + (-1)^k x^{2n-k}[x^{k/2} f_k]^2\right\}^2} \\ &= \frac{x^{(4n-1)/2} J_{4n+2}}{\left[J_{2n+1}^2 + (-1)^k x^{2n-k} J_k^2\right]^2}; \\ \text{LHS} &= \sum_{n=M}^{\infty} \frac{x^{(4n-1)/2} J_{4n+2}}{\left[J_{2n+1}^2 + (-1)^k x^{2n-k} J_k^2\right]^2}, \end{aligned} \tag{14}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, let $B = \dfrac{1}{f_{2k}} \displaystyle\sum_{r=1}^{k} \frac{1}{f_t^2}$. With k odd, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{(4r+2k-3)/2}$. This yields

$$\begin{aligned} B &= \frac{x^{(2k-1)/2}}{x^{(2k-1)/2} f_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{\left[x^{(2r-1)/2} f_{2r}\right]^2} \\ &= \frac{x^{(2k+1)/2}}{D^2\left[x^{(2k-1)/2} f_{2k}\right]} \sum_{r=1}^{k} \frac{x^{2r-1}}{\left[x^{(2r-1)/2} f_{2r}\right]^2}; \\ \text{RHS} &= \frac{x^{(2k-1)/2}}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{J_{2r}^2}, \end{aligned} \tag{15}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (14), yields

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\text{odd}}}^{\infty} \frac{x^{2n+1} J_{4n+2}}{\left(J_{2n+1}^2 - x^{2n-k} J_k^2\right)^2} = \frac{x^k}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{J_{2r}^2}.$$

When k is even, $B = \dfrac{1}{f_{2k}} \displaystyle\sum_{r=1}^{k} \frac{1}{f_{2r-1}^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{(4r+2k-3)/2}$, we get

$$B = \frac{x^{(2k-1)/2}}{x^{(2k-1)/2} f_{2k}} \sum_{r=1}^{k} \frac{x^{2r-2}}{\left[x^{(2r-2)/2} f_{2r-1}\right]^2};$$

$$\text{RHS} = \frac{x^{(2k-1)/2}}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-2}}{J_{2r-1}^2}, \tag{16}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

It follows by equations (14) and (16) that

$$\sum_{\substack{n=k/2 \\ k\geq 2,\,\text{even}}}^{\infty} \frac{x^{2n+1} J_{4n+2}}{\left(J_{2n+1}^2 + x^{2n-k} J_k^2\right)^2} = \frac{x^k}{J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{J_{2r-1}^2}.$$

By combining the two cases, we get the Jacobsthal version of equation (3):

$$\sum_{n=M}^{\infty} \frac{x^{2n+1} J_{4n+2}}{\left[J_{2n+1}^2 + (-1)^k x^{2n-k} J_k^2\right]^2} = \frac{x^k}{J_{2k}} \sum_{r=1}^{k} \frac{x^t}{J_t^2}. \tag{17}$$

□

It then follows that

$$\sum_{n=M}^{\infty} \frac{F_{4n+2}}{\left[F_{2n+1}^2 + (-1)^k F_k^2\right]^2} = \frac{1}{F_{2k}} \sum_{r=1}^{k} \frac{1}{F_t^2}; \quad \sum_{n=M}^{\infty} \frac{2^{2n+1} J_{4n+2}}{\left[J_{2n+1}^2 + (-1)^k 2^{2n-k} J_k^2\right]^2} = \frac{2^k}{J_{2k}} \sum_{r=1}^{k} \frac{2^t}{J_t^2}.$$

In particular, we then get [6]

$$\sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 - 1\right)^2} = 1; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(F_{2n+1}^2 + 1\right)^2} = \frac{5}{12};$$

$$\sum_{n=1}^{\infty} \frac{4^n J_{4n+2}}{\left(J_{2n+1}^2 - 2^{2n-1}\right)^2} = 4; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{4n+2}}{\left(J_{2n+1}^2 + 2^{2n-2}\right)^2} = \frac{52}{45}.$$

Finally, we explore the Jacobsthal counterpart of equation (4).

2.4. **Jacobsthal Version of Equation** (4).

Proof. Let $A = \dfrac{f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^2}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with x^{4n}. We then get

$$\begin{aligned}
A &= \frac{x^2 f_{4n+2}}{\left[x l_{2n+1}^2 - (-1)^k D^2 f_k^2\right]^2} \\
&= \frac{x^{(4n+3)/2}\left[x^{(4n+1)/2} f_{4n+2}\right]}{\left\{\left[x^{(2n+1)/2} l_{2n+1}\right]^2 - (-1)^k D^2 x^{2n-k+1}\left[x^{(k-1)/2} f_k\right]^2\right\}^2} \\
&= \frac{x^{(4n+3)/2} J_{4n+2}}{\left[j_{2n+1}^2 - (-1)^k D^2 x^{2n-k+1} J_k^2\right]^2};
\end{aligned}$$

$$\text{LHS} = \sum_{n=M}^{\infty} \frac{x^{(4n+3)/2} J_{4n+2}}{\left[j_{2n+1}^2 - (-1)^k D^2 x^{2n-k+1} J_k^2\right]^2}, \tag{18}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, let $B = \dfrac{1}{\Delta^2 f_{2k}} \sum_{r=1}^{k} \dfrac{1}{l_t^2}$. Suppose k is odd. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{(4r+2k-1)/2}$ yields

$$\begin{aligned} B &= \frac{x}{D^2 f_{2k}} \sum_{r=1}^{k} \frac{1}{l_{2r}^2} \\ &= \frac{x^{(2k+1)/2}}{D^2 \left[x^{(2k-1)/2} f_{2k}\right]} \sum_{r=1}^{k} \frac{x^{2r}}{\left(x^{2r/2} l_{2r}\right)^2}; \\ \text{RHS} &= \frac{x^{(2k+1)/2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{j_{2r}^2}, \end{aligned} \tag{19}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

It then follows by equations (18) and (19) that

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, \text{odd}}}^{\infty} \frac{x^{2n+1} J_{4n+2}}{\left(j_{2n+1}^2 + D^2 x^{2n-k+1} J_k^2\right)^2} = \frac{x^k}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r}}{j_{2r}^2}. \tag{20}$$

With k even, we have $B = \dfrac{1}{\Delta^2 f_{2k}} \sum_{r=1}^{k} \dfrac{1}{l_{2r-1}^2}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{(4r+2k-1)/2}$. This gives

$$\begin{aligned} B &= \frac{x}{D^2 f_{2k}} \sum_{r=1}^{k} \frac{1}{l_{2r-1}^2} \\ &= \frac{x^{(2k+1)/2}}{D^2 \left[x^{(2k-1)/2} f_{2k}\right]} \sum_{r=1}^{k} \frac{x^{2r-1}}{\left[x^{(2r-1)/2} l_{2r-1}\right]^2}; \\ \text{RHS} &= \frac{x^{(2k+1)/2}}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{j_{2r-1}^2}, \end{aligned} \tag{21}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Coupled with equation (18), this gives

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, \text{even}}}^{\infty} \frac{x^{2n+1} J_{4n+2}}{\left(j_{2n+1}^2 - D^2 x^{2n-k+1} J_k^2\right)^2} = \frac{x^k}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^{2r-1}}{j_{2r-1}^2}.$$

Merging equations (20) and (21), we get the desired Jacobsthal version:

$$\sum_{n=M}^{\infty} \frac{x^{2n+1} J_{4n+2}}{\left[j_{2n+1}^2 - (-1)^k D^2 x^{2n-k+1} J_k^2\right]^2} = \frac{x^k}{D^2 J_{2k}} \sum_{r=1}^{k} \frac{x^t}{j_t^2}. \tag{22}$$

□

In particular, we have

$$\sum_{n=M}^{\infty} \frac{F_{4n+2}}{\left[L_{2n+1}^2 - 5(-1)^k F_k^2\right]^2} = \frac{1}{5F_{2k}} \sum_{r=1}^{k} \frac{1}{L_t^2};$$

$$\sum_{n=M}^{\infty} \frac{2^{2n+1} J_{4n+2}}{\left[j_{2n+1}^2 - 9(-1)^k 2^{2n-k+1} J_k^2\right]^2} = \frac{2^k}{9J_{2k}} \sum_{r=1}^{k} \frac{2^t}{j_t^2}.$$

Consequently, we have [6]

$$\sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(L_{2n+1}^2 + 5\right)^2} = \frac{1}{45}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n+2}}{\left(L_{2n+1}^2 - 5\right)^2} = \frac{17}{240};$$

$$\sum_{n=1}^{\infty} \frac{4^n J_{4n+2}}{\left(j_{2n+1}^2 + 9 \cdot 2^{2n}\right)^2} = \frac{4}{225}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{4n+2}}{\left(j_{2n+1}^2 - 9 \cdot 2^{2n-1}\right)^2} = \frac{212}{2,205}.$$

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