ϕ-EXPANSIONS OF RATIONALS

KING SHUN LEUNG

Abstract. Let ϕ denote the golden ratio $(\sqrt{5}+1)/2$ and x be a positive rational number. We study how the ϕ-expansion of x can be found using some known results on Fibonacci numbers. We characterize those numbers in $(0,1)$ with finite ϕ-expansions. If $x \in \mathbb{Q} \cap (0,1)$, we give a precise expression for its ϕ-expansion. In this case, the computation involves only simple operations on integers.

1. Introduction

Let ϕ denote the golden ratio $(\sqrt{5}+1)/2 = 1.6180339887\ldots$. We see that ϕ satisfies

$$\phi^2 = \phi + 1. \tag{1.1}$$

It follows that

$$\phi^{n+2} = \phi^{n+1} + \phi^n \tag{1.2}$$

for all integers n. It is known ([7, 8]) that any positive real number x can be expressed in the form

$$x = \sum_{i=-h}^{\infty} a_i \phi^{-i}, \tag{1.3}$$

where $a_i \in \mathcal{D} := \{0, 1\}$. A shorthand for (1.3) is

$$x = a_{-h}a_{-h+1}\ldots a_{-1}a_0.a_1a_2\ldots_{\phi}\,. \tag{1.4}$$

This is not the usual notation (positive (negative) powers of ϕ with positive (negative) indices, see, e.g., [6]), but it is convenient in this paper because we consider mainly numbers in $(0,1)$. The subscript ϕ is omitted in the rest of this note. We may think of this notion of number representation as a number system with the irrational base ϕ and the digit set $\mathcal{D}$. Repeating digits are overbarred as in $10.0\overline{001} = 10.0001001001001001\ldots$. The expression (1.3) (or (1.4)), called a ϕ-expansion of x, is unique if we impose two conditions on the expression. The first one is that

$$a_i a_{i+1} = 0 \tag{1.5}$$

for all i. In other words, two consecutive 1s will not appear in a ϕ-expansion. We can see from (1.2) that $\ldots 100 \ldots = \ldots 011 \ldots$. Because of (1.5) we accept only the former. The second condition is that a tail of $\overline{01}$ is replaced by a 1 followed by a tail of (hidden) 0s. For example, by repeated application of (1.2), we have $0.1 = 0.011 = 0.01011 = 0.0101011 = \cdots = 0.\overline{01}$. We can also prove this result by summing a geometric series: $0.\overline{01} = \phi^{-2} + \phi^{-4} + \phi^{-6} + \phi^{-8} + \cdots = \phi^{-2}/(1-\phi^{-2}) = 1/(\phi^2 - 1) = 1/\phi = \phi^{-1} = 0.1$.

Indeed the representations of numbers in noninteger bases are also studied in the setting of *β-expansions* ([7, 8]), of which the notion of ϕ-expansions is a special case. Although in this note, we confine our study to the ϕ-expansions of numbers, some of the techniques used here can be adapted to find their β-expansions, especially when β satisfies $\beta^2 = n\beta + 1$, where $n(> 1)$ is an integer.

Definition 1. We define the *sequence of Fibonacci numbers* $\{F_n\}_{n\in\mathbb{Z}}$ as follows. For all $n \in \mathbb{Z}$, $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. When n is negative, F_n is called a *negaFibonacci number.*

Table 1 lists the F_ns for $n = -9, -8, \ldots, 9, 10$.

n	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
F_n	34	−21	13	−8	5	−3	2	−1	1	0
n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55

TABLE 1. A list of Fibonacci numbers

It can be shown that $F_{2m-1} = F_{-2m+1}$ and $F_{2m} = -F_{-2m}$ for all integers m.

We list below a number of results on Fibonacci numbers which will be used to find the ϕ-expansions of numbers in the subsequent sections. The first one is:

$$\phi^n = F_n\phi + F_{n-1} \tag{1.6}$$

for all integers n. It can be proved by applying induction twice.

Then, we have two theorems on the representations of integers as a sum of Fibonacci numbers.

Theorem 2. *(Zeckendorf's Theorem [3]) Let N be a positive integer. Then, there exist positive integers $i_j \geq 2$ with $i_{j+1} > i_j + 1$ such that $N = \sum_{j=1}^{m} F_{i_j}$. This representation (called the Zeckendorf representation) of N is unique.*

For example, the Zeckendorf representation of 99 can be found by the Greedy algorithm as follows: $99 = 89 + 10 = 89 + 8 + 2 = F_{11} + F_6 + F_3$.

Theorem 3. *(Representations of integers by negaFibonacci numbers [4]) Let N be a nonzero integer. Then, there exist positive integers $i_j \geq 1$ with $i_{j+1} > i_j + 1$ such that $N = \sum_{j=1}^{m} F_{-i_j}$. This representation of N is unique.*

For example, the negaFibonacci representation of 99 can be obtained by Bunder's algorithm [4] as follows: $99 = 89 + 10 = 89 + 13 - 3 = F_{-11} + F_{-7} + F_{-4}$.

Similarly, the negaFibonacci representation of −99 is given by

$-99 = -144 + 45 = -144 + 34 + 11 = -144 + 34 + 13 - 2 = -144 + 34 + 13 - 3 + 1 = F_{-12} + F_{-9} + F_{-7} + F_{-4} + F_{-1}$.

Let q be a positive integer greater than one. It can be shown that the sequence $\{F_n \bmod q\}_{n=0}^{\infty}$ is periodic ([11, 12]). The (shortest) period is called the *qth Pisano period* [13] and written $\pi(q)$. Table 2 shows that $\pi(7) = 16$.

n	0	1	2	3	4	5	6	7	8	9	10
$F_n \bmod 7$	0	1	1	2	3	5	1	6	0	6	6
n	11	12	13	14	15	16	17	18	19	20	21
$F_n \bmod 7$	5	4	2	6	1	0	1	1	2	3	5

TABLE 2. A table of F_n mod 7

It can be deduced from the periodicity of $\{F_n \bmod q\}_{n=0}^{\infty}$ that

$$F_{k-1} \equiv F_{k+1} \equiv F_{k+2} \equiv 1 \bmod q, \tag{1.7}$$

where $k := \pi(q)$.

The following result is a particular case of Theorem 3.4 in [10].

Theorem 4. *Let $x = p/q \in \mathbb{Q} \cap (0,1)$ where p and q are positive integers and $(p,q) = 1$. Then the ϕ-expansion of x is strictly periodic with period $k = \pi(q)$, i.e., it takes the form $x = 0.\overline{a_1 a_2 \ldots a_k}$.*

Remark 5. The converse of Theorem 4 is not true in the sense that some numbers with strictly periodic ϕ-expansions are not rationals. For example, $0.\overline{100} = \phi^{-1}/(1-\phi^{-3}) = \phi^2/(\phi^3-1) = \phi^2/(F_3\phi + F_2 - 1) = \phi^2/(2\phi+1-1) = \phi/2 \notin \mathbb{Q}$.

The rest of this note is organized as follows. Section 2 following this introductory section gives a characterization of numbers in $(0,1)$ with finite ϕ-expansions. The main result of this paper, a precise description of the ϕ-expansion of a rational in $(0,1)$ (Theorem 10) is given in Section 3. In this section, we will also illustrate with examples how we can apply the previous results to find the ϕ-expasions of positive rationals.

2. Numbers in $(0,1)$ with Finite ϕ-expansions

It is known [6] that all positive real numbers that have a finite expansion are given by the positive numbers in $\mathbb{Z}[\phi^{-1}]$. We characterize the numbers in $(0,1)$ with finite expansions below.

Theorem 6. *The ϕ-expansion of a number $x \in (0,1)$ is finite if and only if $x = A\phi + B$ for some integers A and B with negaFibonacci representations $A = \sum_{j=1}^{m} F_{-i_j}$ and $B = \sum_{j=1}^{m} F_{-i_j-1}$ for some increasing sequence $\{i_j\}_{j=1}^{m}$ of positive integers satisfying $i_{j+1} > i_j + 1$.*

Proof. If. $x = A\phi + B = (\sum_{j=1}^{m} F_{-i_j})\phi + \sum_{j=1}^{m} F_{-i_j-1} = \sum_{j=1}^{m}(F_{-i_j}\phi + F_{-i_j-1}) = \sum_{j=1}^{m} \phi^{-i_j} < \sum_{j=1}^{m} \phi^{-(2j-1)} < 0.\overline{10} = 1$. It follows from Theorem 3 that (1.5) is satisfied in the expression $x = \sum_{j=1}^{m} \phi^{-i_j}$.

Only if. Let m be the number of 1s in the ϕ-expansion of x. Then, there exists an increasing sequence of positive integers $\{i_j\}_{j=1}^{m}$ such that $i_{j+1} > i_j + 1$ and $x = \sum_{j=1}^{m} \phi^{-i_j} = \sum_{j=1}^{m}(F_{-i_j}\phi + F_{-i_j-1}) = (\sum_{j=1}^{m} F_{-i_j})\phi + \sum_{j=1}^{m} F_{-i_j-1}$. Here, $\sum_{j=1}^{m} F_{-i_j}$ and $\sum_{j=1}^{m} F_{-i_j-1}$ are integers. $\square$

Remark 7. Anderson [1] studies the representations of integers A and B in $A\phi + B > 0$ as sums of Fibonacci numbers with no restriction on the signs of the subscripts.

The following simpler characterization of the numbers in $(0,1)$ was suggested by the referee. Let $w(A) := \lfloor \phi A \rfloor$ for $A \in \mathbb{Z}^+$, the famous Wythoff sequence.

Theorem 8. (Referee) *The ϕ-expansion of a number $x \in (0,1)$ is finite if and only if $x = A\phi - w(A)$ for some positive integer A, or $x = A\phi + w(-A) + 1$ for some negative integer A.*

Proof. Only if. Because $x \in (0,1)$, the ϕ-expansion of x can only have nonzero digits with negative indices (because the geometric series starting at $1/\phi$ and multiplier ϕ^{-2} sums to 1). Now, because $\phi^{-1} = \phi - 1$, x can be written as $x = A\phi + B$ for some integers A and B. The condition $x \in (0,1)$ then forces $B = -w(A)$ or $B = w(-A) + 1$, according to the sign of A.

If. According to Theorem 2 in [6], the set of numbers that possess a finite ϕ-expansion are the positive elements of $\mathbb{Z}[\phi^{-1}]$. Because $\phi^{-1} = \phi - 1$, these are the positive elements of the ring $Z(\phi)$, so the numbers $A\phi - w(A)$, and $A\phi + w(-A) + 1$ have finite expansions. $\square$

To find the ϕ-expansion of $x = 10\phi - 16$, we see that the negaFibonacci representation of 10 is $10 = -3 + 13 = F_{-4} + F_{-7}$, whereas that of -16 is $-16 = 5 - 21 = F_{-5} + F_{-8}$. By Theorem 6, we get $x = (F_{-4} + F_{-7})\phi + (F_{-5} + F_{-8}) = (F_{-4}\phi + F_{-5}) + (F_{-7}\phi + F_{-8}) = \phi^{-4} + \phi^{-7} = 0.0001001$. $\square$

Remark 9. We can deduce from Theorem 6 that the ϕ-expansion of any positive integer is finite (see [9] for a different proof). Let N be a positive integer. The assertion is obviously true if $N = 1$. If $N \geq 2$, then there exists a positive integer $n(\geq 2)$ for which $M := N\phi^{-n} < 1$. Using (1.6), we can express M in the form $M = A\phi + B$, where A and B are integers. As $M \in (0, 1)$, it follows from Theorem 6 that the ϕ-expansion of M is finite. As a result, the ϕ-expansion of N is also finite as it can be obtained by shifting the decimal point in the ϕ-expansion of M n places to the right.

3. ϕ-expansions of Rationals

3.1. **ϕ-expansions of Rationals in** $(0, 1)$**.** [2] gives the ϕ-expansions of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$. The main result (Theorem 10) of this note is a precise description of the ϕ-expansion of a rational number $x \in (0, 1)$.

Theorem 10. *Let $x = p/q \in (0, 1)$ and $k = \pi(q)$ be defined as in Theorem 4. Then the ϕ-expansion of x is given by $x = 0.\overline{a_1 a_2 \ldots a_k}$ if and only if the a_is satisfy the Zeckendorf representation of $(F_{k+2} - 1)x$, i.e., $(F_{k+2} - 1)x = a_1 F_{k+1} + a_2 F_k + \cdots + a_k F_2$.*

Proof. Only if. Theorem 4 states that the ϕ-expansion of x takes the form $x = 0.\overline{a_1 a_2 \ldots a_k}$. Then, we have $\phi^k x = a_1 a_2 \ldots a_k.\overline{a_1 a_2 \ldots a_k}$. This implies that
$x = \phi^k x - a_1\phi^{k-1} - a_2\phi^{k-2} - \cdots - a_k$.
Using (1.6), we get
$x = (F_{k-1}x - a_1F_{k-2} - a_2F_{k-3} - \cdots - a_kF_{-1}) + (F_kx - a_1F_{k-1} - a_2F_{k-2} - \cdots - a_kF_0)\phi$.
It follows that

$$(F_{k-1} - 1)x - a_1F_{k-2} - a_2F_{k-3} - \cdots - a_kF_{-1} = 0, \tag{3.1}$$

$$F_kx - a_1F_{k-1} - a_2F_{k-2} - \cdots - a_kF_0 = 0. \tag{3.2}$$

To determine the a_is, we can proceed as follows.
Adding (3.1) and (3.2) gives

$$(F_{k+1} - 1)x - a_1F_k - a_2F_{k-1} - \cdots - a_kF_1 = 0. \tag{3.3}$$

Adding (3.2) and (3.3) gives

$$(F_{k+2} - 1)x - a_1F_{k+1} - a_2F_k - \cdots - a_kF_2 = 0. \tag{3.4}$$

Because $(F_{k+2} - 1)x$ is a positive integer, Theorem 2 gives that $a_1F_{k+1} + \cdots + a_kF_2$ is the Zeckendorf representation of $(F_{k+2} - 1)x$ (where we use (1.5) because $0.\overline{a_1 a_2 \ldots a_k}$ is a ϕ-expansion).

If. Suppose $(F_{k+2} - 1)x = a_1F_{k+1} + a_2F_k + \cdots + a_kF_2$ and the ϕ-expansion of x is $x = 0.\overline{b_1 b_2 \ldots b_k}$. Then by the "only if" part of this theorem, the b_is satisfy $(F_{k+2} - 1)x = b_1F_{k+1} + b_2F_k + \cdots + b_kF_2$. As the Zeckendorf representation of $(F_{k+2} - 1)x$ is unique, $b_i = a_i$ for $i = 1, 2, \ldots, k$. □

To find the ϕ-expansion of 2/7, we have seen from Table 2 that $\pi(7) = 16$. By Theorem 10, we have

$(F_{18} - 1)(2/7) = a_1F_{17} + a_2F_{16} + \cdots + a_{16}F_2$. Applying Theorem 2, we obtain $(F_{18} - 1)(2/7) = 2583(2/7) = 738 = 610 + 89 + 34 + 5 = F_{15} + F_{11} + F_9 + F_5$. That means $a_3 = a_7 = a_9 = a_{13} = 1$ and all other a_is are equal to zero. Hence, $2/7 = 0.\overline{0010001010001000}$.

We can verify our result as follows.

$0.\overline{0010001010001000} = (\phi^{-3} + \phi^{-7} + \phi^{-9} + \phi^{-13})/(1 - \phi^{-16}) = (\phi^{13} + \phi^9 + \phi^7 + \phi^3)/(\phi^{16} - 1) = [(F_{13}\phi + F_{12}) + (F_9\phi + F_8) + (F_7\phi + F_6) + (F_3\phi + F_2)]/(F_{16}\phi + F_{15} - 1) = [(233\phi + 144) + (34\phi + 21) + (13\phi + 8) + (2\phi + 1)]/(987\phi + 610 - 1) = (282\phi + 174)/(987\phi + 609) = [2(141\phi + 87)]/[7(141\phi + 87)] = 2/7$. □

3.2. **ϕ-expansions of Rationals Greater Than One.** We can use the idea of Remark 9 to find the ϕ-expansion of an integer greater than one. We illustrate the idea explicitly with the examples below.

To find the ϕ-expansion of 23, we can proceed as follows. Because $23 = 15 + 8 < 13\phi + 8 = F_7\phi + F_6 = \phi^7$ (by (1.6)), we let $M := 23\phi^{-7}$. Using (1.6) and Theorem 3, we have $M = 23(F_{-7}\phi + F_{-8}) = 23 \cdot 13\phi + 23(-21) = 299\phi - 483 = (F_{-1} + F_{-4} + F_{-8} + F_{-11} + F_{-13})\phi + (F_{-2} + F_{-5} + F_{-9} + F_{-12} + F_{-14}) = (F_{-1}\phi + F_{-2}) + (F_{-4}\phi + F_{-5}) + (F_{-8}\phi + F_{-9}) + (F_{-11}\phi + F_{-12}) + (F_{-13}\phi + F_{-14}) = \phi^{-1} + \phi^{-4} + \phi^{-8} + \phi^{-11} + \phi^{-13}$. Then, $23 = M\phi^7 = \phi^6 + \phi^3 + \phi^{-1} + \phi^{-4} + \phi^{-6} = 1001000.100101$.

We end this note with an example on how to find the ϕ-expansion of a nonintegral rational greater than one.

To find the ϕ-expansion of $\frac{17}{3}$, the first step is to write $\frac{17}{3} = 5\frac{2}{3}$. It is easy to see that 5 has ϕ-expansion 1000.1001, and an application of Theorem 10 gives that the ϕ-expansion of 2/3 is $0.\overline{10000010} = 0.10000\overline{0101000}$. Then, we have $5\frac{2}{3} = 1000.1001 + 0.1000 + 0.00000\overline{0101000}$. The sum of the first two terms can be obtained by repeated use of (1.2) as follows: $1000.1001 + 0.1000 = 1000.0111 + 0.1000 = 1000.1111 = 1001.0011 = 1001.0100$. Finally, we have $5\frac{2}{3} = 1001.0100 + 0.00000\overline{0101000} = 1001.01000\overline{0101000} = 1001.01\overline{00001010}$.

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Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong,

Email address: `ksleung@eduhk.hk`