SUMS OF k-BONACCI NUMBERS

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Abstract. We give a combinatorial proof of a formula giving the partial sums of the k-bonacci sequence as alternating sums of powers of two multiplied by binomial coefficients. As a corollary, we obtain a formula for the k-bonacci numbers.

1. Introduction

For $k \geq 1$, define k-bonacci numbers $f_n^{(k)}$ by the initial values and recursion[1]

$$f_n^{(k)} = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ \sum_{i=1}^{k} f_{n-i}^{(k)}, & \text{if } n \geq 1. \end{cases} \tag{1.1}$$

The k-bonacci numbers were introduced under the name k-generalized Fibonacci numbers by Miles [8] in 1960. According to [6], the Miles paper may be the earliest paper on the topic to have appeared in a widely available journal.[2] It appears that the terminology "k-bonacci" was in use as early as 1973 by V. E. Hoggatt, Jr. and Marjorie Bicknell [3]. That terminology has continued to be used since then, although not universally.

This paper is about the following formula for the partial sums of the sequence of k-bonacci numbers: *For $k \geq 1$ and $n \geq 0$, the sum of the first $n+1$ k-bonacci numbers is given by*[3]

$$\sum_{i=0}^{n} f_i^{(k)} = \sum_{j=0}^{\lfloor n/(k+1) \rfloor} (-1)^j \binom{n-jk}{j} 2^{n-j(k+1)} . \tag{1.2}$$

We obtained formula (1.2) in the process of developing growth estimates for sums of k-bonacci numbers, which we had hoped to apply to another problem. Because this formula seemed to be new, we put aside the original problem to more fully understand the formula. We found an algebraic proof for (1.2), but we wondered if the formula could be understood at a deeper level—or at least be proved in another way—by using a combinatorial approach.

Thanks to an anonymous referee (who used [10] to track down the reference), we have learned that formula (1.2) should be credited to a 1925 paper by Otto Dunkel. In Dunkel's paper [4], if one compares the equation for $P_2(n)$ in Section 6 to the equation for $P_2(n)$ in Section 10, then one sees that (1.2) holds.

[1]Note that the $k = 2$ case gives the usual Fibonacci numbers with the index shifted by 1. The combinatorial approach to the Fibonacci numbers in [2] employs this shift. Extending that combinatorial approach to all $k \geq 2$, we have shifted onto the negative indices the beginning $k-1$ zeros that usually appear in the k-bonacci sequence. Details are given in Theorem 2.1. Also, note that $f_n^{(1)} = 1$ for all $n \geq 0$.

[2]The 3-bonacci numbers were mentioned by Agronomof in a note that appeared in 1914 in [1].

[3]We use the notation $\lfloor \cdot \rfloor$ for the *floor function*, so $\lfloor x \rfloor$ equals the largest integer less than or equal to x.

Although Dunkel's paper was concerned with various probability problems in coin tossing, his method was to proceed via a difference equation and the roots of that equation, so that at its heart, Dunkel's proof of (1.2) is algebraic.

In this paper, we give a combinatorial proof of (1.2). Also, because one can obtain the k-bonacci numbers from the differences of their partial sums, we obtain a corollary formula (3.2) for the k-bonacci numbers. Our formula for the k-bonacci numbers is similar to equation (12) in Corollary 1 of [6] and the formula in Theorem 2.4 of [5]. It is not related to the formulas for the k-bonacci numbers appearing in [7] and [8]. It also does not occur in Dunkel's paper.

2. Combinatorics

In [2], Benjamin and Quinn begin with the observation that the number of ordered sums of 1s and 2s that add to n is the $(n+1)$th Fibonacci number. They then introduce a visual representation that many, if not most, people will find easier to think about than sums of 1s and 2s. They suggest thinking of the 1s as squares and the 2s as dominoes, so that the number of ordered sums of 1s and 2s that add to n is the number of ways to tile a ruler of length n with squares and dominoes (see Figure 1).

When we generalize to the k-bonacci numbers, we need to consider the number of ordered sums of 1s, 2s, ... , ks that add to n. We will use the metaphor of covering a ruler of length n by tiles of lengths 1 through k (see Figure 2). This metaphorical ruler will run from left to right. The numbers obtained in this way are k-bonacci numbers.

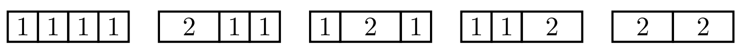


Figure 1. The $f_4^{(2)} = 5$ ways of tiling a ruler of length 4 with squares and dominoes.

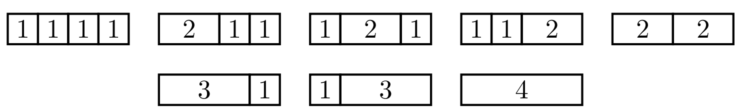


Figure 2. The $f_4^{(4)} = 8$ ways of tiling a ruler of length 4 with tiles of lengths 1, 2, 3, 4.

Theorem 2.1. *For $k \geq 1$ and $n \geq 0$, each $f_n^{(k)}$ equals the number of ways to cover a length n ruler using tiles of lengths 1 through k, where by definition there are 0 ways to cover a ruler of negative length.*

Proof. Fix $k \geq 1$.

Because there is exactly one way to tile a ruler of length 0 (and that is to use no tiles), and because there are 0 ways to tile a ruler of negative length, we see that the number of ways to tile a ruler of length n using tiles of lengths 1 through k satisfies the initial values of (1.1) for $n \leq 0$.

To complete the proof, we will show that for $n \geq 1$, the number of ways to tile a ruler of length n using tiles of lengths 1 through k satisfies the recurrence relation of (1.1). We argue inductively as follows: For each tiling of the ruler of length n, there must be a rightmost tile. That rightmost tile has a length $\ell \in \{1, 2, \ldots, k\}$, and in case $n < k$, we also know that that rightmost tile has length $\ell \leq n$. The part of the ruler to the left of the last tile can be tiled $f_{n-\ell}^{(k)}$ ways and we have $f_{n-\ell}^{(k)} = 0$ in case $\ell > n$. Thus, it holds that

$$f_n^{(k)} = \sum_{\ell=1}^{k} f_{n-\ell}^{(k)} .$$

□

Theorem 2.2. *For $k \geq 1$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} f_i^{(k)} = \sum_{i=0}^{\lfloor n/(k+1) \rfloor} (-1)^i \binom{n-ik}{i} 2^{n-i(k+1)} . \tag{2.1}$$

Proof. Observe that $\sum_{i=0}^{n} f_i^{(k)}$ is the number of ways to place tiles of lengths 1 through k end-to-end with total length not exceeding n. To prove the theorem, we will count the number of ways that this can be done.

Let U be the set of tilings of length not exceeding n, where there is no restriction on the sizes of the tiles that are used. The cardinality of U is easily obtained as follows: On a ruler of length n, place a hash mark at the 0 position and at each of the integer positions 1 to n, either do or do not place a hash mark. Then, between any two hash marks, place a tile that fills the space. The last tile ends at the last hash mark. Because there are 2^n ways to place or not place the hash marks, we have

$$\#(U) = 2^n .$$

In case $k \geq n$, no tile has length exceeding k, so this value, 2^n, equals the left side of (2.1). Also, when $k \geq n$, the summation on the right side of (2.1) reduces to the single term when $i = 0$, that is, to 2^n. Thus, we see that (2.1) holds. From here on, we assume that $n > k$.

To obtain the value on the left side of (2.1), we must subtract from 2^n the number of tilings in U that include at least one tile of length greater than k. To this end, define U_j to be the set of tilings in U for which a tile that has length greater than k has its right end at integer position j. Note that this tile does not need to be the rightmost tile with length greater than k. We need to evaluate

$$\#\left(\bigcup_{1 \leq j \leq n} U_j \right) .$$

The Principle of Inclusion-Exclusion (see for example [9] or [11]) tells us that

$$\#\left(\bigcup_{1 \leq j \leq n} U_j \right) = \sum_{1 \leq j \leq n} \#\Big(U_j\Big) - \sum_{1 \leq j_1 < j_2 \leq n} \#\Big(U_{j_1} \cap U_{j_2}\Big) + \cdots . \tag{2.2}$$

To evaluate the right side of (2.2), we will need to compute

$$\sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} \#\Big(U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_i}\Big) . \tag{2.3}$$

Note that the set $U_{j_1} \cap U_{j_2} \cap \cdots \cap U_{j_i}$ will be empty unless

$$k < j_1 \quad \text{and} \quad j_\ell + k < j_{\ell+1} \quad \text{for} \quad \ell = 1, 2, \ldots, i-1 , \tag{2.4}$$

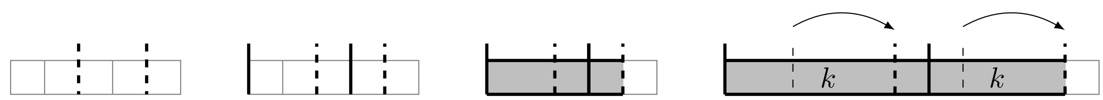
because no two tiles are allowed to overlap. Note that (2.4) tells us that $i(k+1) \leq n$, so only when $i \leq \lfloor n/(k+1) \rfloor$ will any of the summands in (2.3) be nonzero.

Working with a ruler of length $n-ik$, we will show how to evaluate (2.3). (The construction we are about to describe is illustrated in Figure 3.) Choose i numbers from $\{1,2,\dots,n-ik\}$. This can be done in $\binom{n-ik}{i}$ ways. Label the chosen numbers $r_1, r_2, \dots, r_i$ so that

$$0 < r_1 \quad \text{and} \quad r_\ell < r_{\ell+1} \quad \text{for} \quad \ell = 1,2,\dots,i-1\,. \tag{2.5}$$

For $\ell = 1,2,\dots,i$, put a dashed hash mark at integer position r_ℓ.

Next, place a normal hash mark at 0. There remain $n-i(k+1)$ unmarked positive integer positions between 1 and $n-ik$. At each of these remaining unmarked positions, either do or do not place a normal hash mark. This can be done in $2^{n-i(k+1)}$ ways. Between any two hash marks (dashed and dashed, normal and normal, or dashed and normal), place a tile that fills the space. The last tile ends at the last hash mark, and the tiling has total length not exceeding $n-ik$.



Place $i = 2$ dashed hash marks.

Insert a normal hash mark at 0 and up to $n-i(k+1) = 3$ more.

Place tiles between the hash marks.

Replace tiles that have a dashed hash mark on their right end with tiles k units longer.

FIGURE 3. Example of the construction with $n = 11,\ k = 3,\ i = 2$. One of three possible normal hash marks is inserted.

Finally, each tile that has its right end at a dashed hash mark is replaced with a tile that is k units longer, whereas the tiles to its right are moved along k units to accommodate the longer tile. Thus, we have created a tiling the length of which does not exceed n and that for each $\ell = 1,2,\dots,i$ has a tile of length greater than or equal to $k+1$ with its right end at integer position $j_\ell := r_\ell + \ell k$. The j_ℓ satisfy (2.4) if and only if the r_ℓ satisfy condition (2.5). Thus, we see that

$$\sum_{1\le j_1<j_2<\cdots<j_i\le n} \#\Big(U_{j_1}\cap U_{j_2}\cap\cdots\cap U_{j_i}\Big) = \binom{n-ik}{i} 2^{n-i(k+1)}\,.$$

□

3. Corollaries

Corollary 3.1. *For $k \ge 1$, $n \ge 0$, and m with $\lfloor n/k\rfloor \ge m \ge \lfloor n/(k+1)\rfloor$, it holds that*

$$\sum_{i=0}^{n} f_i^{(k)} = \sum_{j=0}^{m} (-1)^j \binom{n-jk}{j} 2^{n-j(k+1)}\,. \tag{3.1}$$

Proof. The summands that are on the right side of (3.1), but not on the right side of (1.2), are those for which

$$\lfloor n/(k+1)\rfloor < j \le \lfloor n/k\rfloor$$

holds. Note that the left side inequality is equivalent to $n-jk < j$ and the right side inequality is equivalent to $n-jk \ge 0$, so that for such a j, the binomial coefficient $\binom{n-jk}{j}$ equals 0. □

As a second corollary, we obtain a formula for the k-bonacci numbers in terms of binomial coefficients and powers of 2.

Corollary 3.2. *For $k \geq 1$ and $n \geq 0$, we have*[4]

$$f_n^{(k)} = \sum_{j=0}^{\lfloor n/(k+1) \rfloor} (-1)^j \, \frac{(n-jk)+j+\delta_{n,0}}{2(n-jk)+\delta_{n,0}} \binom{n-jk}{j} 2^{n-j(k+1)} . \tag{3.2}$$

Proof. For $n = 0$, the summation on the right side of (3.2) consist of only the $j = 0$ term, so the result is confirmed by (1.1).

For $k \geq 1$ and $n \geq 1$, using (1.2), we have:

$$f_n^{(k)} = \sum_{i=0}^{n} f_i^{(k)} - \sum_{i=1}^{n-1} f_i^{(k)}$$

and we will use (3.1) to replace the partial sums of k-bonacci numbers in this equation. We will need to examine the upper limits in the summations that occur in (3.1).

Set $q = \lfloor n/(k+1) \rfloor$. Then $n = q(k+1) + r$, where r is an integer with $0 \leq r < k+1$. We see that

$$\frac{n-1}{k} = q + \frac{q+r-1}{k} .$$

Because $n \geq 1$, one or both of q and r must be positive. We conclude that $\lfloor (n-1)/k \rfloor \geq q$, so when (3.1) is applied with n replaced by $n-1$, we may use q as the upper limit of summation.

We have

$$\begin{aligned} f_n^{(k)} &= \sum_{j=0}^{q} (-1)^j \binom{n-jk}{j} 2^{n-j(k+1)} - \sum_{j=0}^{q} (-1)^j \binom{(n-1)-jk}{j} 2^{(n-1)-j(k+1)} \\ &= \sum_{j=0}^{q} (-1)^j \left[2\binom{n-jk}{j} - \binom{n-jk-1}{j} \right] \frac{1}{2}\, 2^{n-j(k+1)} \\ &= \sum_{j=0}^{q} (-1)^j \frac{1}{2} \left[\frac{2(n-jk)}{n-jk} - \frac{n-jk-j}{n-jk} \right] \binom{n-jk}{j} 2^{n-j(k+1)} \\ &= \sum_{j=0}^{q} (-1)^j \frac{(n-jk)+j}{2(n-jk)} \binom{n-jk}{j} 2^{n-j(k+1)}. \end{aligned}$$

□

References

[1] M. Agronomof, *Sur une suite récurrente*, Mathesis: Recueil Mathématique, **4** (1914), 125–126.

[2] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003.

[3] M. Bicknell and V. E. Hoggatt, Jr., *Generalized Fibonacci polynomials*, The Fibonacci Quarterly, **11.5** (1973), 457–465.

[4] O. Dunkel, *Solutions of a probability difference equation*, The American Mathematical Monthly, **32.7** (1925), 354–370.

[5] F. T. Howard and C. Cooper, *Some identities for r-Fibonacci numbers*, The Fibonacci Quarterly, **49.3** (2011), 231–243.

[6] D. Kessler and J. Schiff, *The combinatoric proof and generalization of Ferguson's formula for k-generalized Fibonacci numbers*, The Fibonacci Quarterly, **42.3** (2004), 266–273.

[4]To avoid 0/0 when $n = j = 0$, we use the Kronecker δ defined by $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

[7] G.-Y. Lee, S.-G. Lee, J.-S. Kim, and H.-K. Shin, *The Binet formula and representations of k-generalized Fibonacci numbers*, The Fibonacci Quarterly, **39.2** (2001), 158–164.
[8] E. P. Miles, Jr., *Generalized Fibonacci numbers and associated matrices*, The American Mathematical Monthly, **67.8** (1960), 745–752.
[9] R. Nelson, *A Brief Journey in Discrete Mathematics*, Springer International Publishing, 2020.
[10] OEIS Foundation Inc. (2022), *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org/A172119`.
[11] R. P. Stanley, *Enumerative Combinatorics*, Volume 1, Cambridge University Press, 2012.

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