A NOTE ON THE FIBONACCI SEQUENCE AND SCHREIER-TYPE SETS

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Abstract. A set A of positive integers is said to be Schreier if either $A = \emptyset$ or $\min A \geq |A|$. We give a bijective map to prove the recurrence of the sequence $(|\mathcal{K}_{n,p,q}|)_{n=1}^{\infty}$ (for fixed $p \geq 1$ and $q \geq 2$), where

$$\mathcal{K}_{n,p,q} \;=\; \{A \subset \{1, \ldots, n\} : \text{either } A = \emptyset \text{ or } (\max A - \max_2 A = p \text{ and } \min A \geq |A| \geq q)\}$$

and $\max_2 A$ is the second largest integer in A, given that $|A| \geq 2$. When $p = 1$ and $q = 2$, we have that $(|\mathcal{K}_{n,1,2}|)_{n=1}^{\infty}$ is the Fibonacci sequence. As a corollary, we obtain a new combinatorial interpretation for the sequence $(F_n + n)_{n=1}^{\infty}$.

A. Bird [2] showed that for each $n \geq 1$, if we let

$$\mathcal{A}_n \;:=\; \{A \subset \{1, \ldots, n\} : n \in A \text{ and } \min A \geq |A|\},$$

then $|\mathcal{A}_n| = F_n$. The condition $\min A \geq |A|$ is called the *Schreier condition*, and a set that satisfies the Schreier condition is called a *Schreier set*. (The empty set satisfies the Schreier condition vacuously.) Schreier sets appeared in a paper by Schreier [9] who used them to solve a problem in Banach space theory. The Schreier condition is also the central concept in a celebrated theorem by Odell [8]. Moreover, Schreier sets were independently discovered in combinatorics and appeared in Ramsey-type theorems for subsets of $\mathbb{N}$. Following the discovery by A. Bird, there has been research on various recurrences produced by counting Schreier-type sets (see [1, 3, 4, 5, 6, 7]). In this short note, we retrieve the Fibonacci sequence from a different counting problem than the one by A. Bird. In particular, for $n \geq 1$, define the set

$$\mathcal{K}_n \;:=\; \{A \subset [n] : \text{either } A = \emptyset \text{ or } (\max A - 1 \in A \text{ and } \min A \geq |A|)\},$$

where $[n] = \{1, \ldots, n\}$. Although we fix the maximum element of sets in $\mathcal{A}_n$, we do not fix the maximum of sets in $\mathcal{K}_n$. Instead, we fix the distance between the largest and the second largest elements of sets in $\mathcal{K}_n$.

Theorem A. *For $n \geq 1$, $|\mathcal{K}_n| = F_n$.*

Let us briefly discuss the proof of Theorem A. It is easy to check that $|\mathcal{K}_1| = |\mathcal{K}_2| = 1$. We need only to show that $|\mathcal{K}_{n+1}| - |\mathcal{K}_n| = |\mathcal{K}_{n-1}|$ for all $n \geq 2$. Fix $n \geq 2$. By definition, $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ and

$$\mathcal{K}_{n+1} \backslash \mathcal{K}_n \;=\; \{A \subset [n+1] : n, n+1 \in A \text{ and } \min A \geq |A|\}.$$

We define a bijection $\pi : \mathcal{K}_{n-1} \to \mathcal{K}_{n+1} \backslash \mathcal{K}_n$: for $A \in \mathcal{K}_{n-1}$,

$$\pi(A) \;:=\; \begin{cases} (A \backslash \{\max A\} + 1) \cup \{n, n+1\}, & \text{if } A \neq \emptyset; \\ \{n, n+1\}, & \text{if } A = \emptyset. \end{cases}$$

Interested readers may verify that π is indeed a bijection or may look at the proof of the more general Theorem C below.

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We have the following immediate corollary, which gives the sequence $(F_n + n)_{n=1}^{\infty}$ (see `https://oeis.org/A002062`).

Corollary B. *Let*

$$\mathcal{K}'_n \ := \ \{A \subset [n] \, : \, \textit{either } |A| \leq 1 \textit{ or } (\max A - 1 \in A \textit{ and } \min A \geq |A|)\}.$$

Then, $|K'_n| = F_n + n$ *for all* $n \geq 1$.

Proof. Clearly, $|\mathcal{K}'_n| - |\mathcal{K}_n| = n$ for all $n \geq 1$. Using Theorem A, the result follows. □

We shall prove a more general result. Let $\max_2 A$ be the second largest number in A if $|A| \geq 2$. For $n, p \geq 1$ and $q \geq 2$, define

$$\mathcal{K}_{n,p,q} \ := \ \{A \subset [n] \, : \, \text{either } A = \emptyset \text{ or } (\max A - \max_2 A = p \text{ and } \min A \geq |A| \geq q)\}.$$

Note that $\mathcal{K}_{n,1,2} = \mathcal{K}_n$.

Theorem C. *Fix* $n, p \geq 1$ *and* $q \geq 2$. *We have*

$$|\mathcal{K}_{n,p,q}| \ = \ \begin{cases} 1, & \text{if } 1 \leq n \leq p + 2q - 3; \\ |\mathcal{K}_{n-1,p,q}| + |\mathcal{K}_{n-2,p,q}| + \binom{n-p-q}{q-2} - 1, & \text{if } n > p + 2q - 3. \end{cases}$$

Proof. We prove Theorem C by recalling that $\mathcal{K}_{n-1,p,q} \subset \mathcal{K}_{n,p,q}$, then writing

$$\mathcal{K}_{n,p,q} \backslash \mathcal{K}_{n-1,p,q} \ = \ \mathcal{S} \cup \mathcal{T}$$

for certain disjoint sets $\mathcal{S}$ and $\mathcal{T}$, and finally verifying that $|\mathcal{S}| = |\mathcal{K}_{n-2,p,q}| - 1$, whereas $|\mathcal{T}| = \binom{n-p-q}{q-2}$.

Fix $p \geq 1$ and $q \geq 2$. First, we check that for $1 \leq n \leq p + 2q - 3$, $|\mathcal{K}_{n,p,q}| = 1$. Recall that

$$\mathcal{K}_{n,p,q} \ = \ \{A \subset [n] \, : \, \text{either } A = \emptyset \text{ or } (\max A - \max_2 A = p \text{ and } \min A \geq |A| \geq q)\}.$$

Suppose A is nonempty and $A \in \mathcal{K}_{n,p,q}$. Write $A = \{a_1, \ldots, a_k\}$. Then, $a_1 \geq q$, $a_k \leq p + 2q - 3$, and $a_{k-1} \leq 2q - 3$. Hence,

$$|\{a_1, \ldots, a_{k-1}\}| \ \leq \ q - 2$$

and so, $|A| \leq q - 1$, which contradicts the requirement that $|A| \geq q$. Therefore, for $1 \leq n \leq p + 2q - 3$, $\mathcal{K}_{n,p,q} = \{\emptyset\}$.

For $n \geq p + 2q - 2$, we show that $|\mathcal{K}_{n,p,q}| = |\mathcal{K}_{n-1,p,q}| + |\mathcal{K}_{n-2,p,q}| + \binom{n-p-q}{q-2} - 1$. Let $\mathcal{S} = \{A \in \mathcal{K}_{n,p,q} \backslash \mathcal{K}_{n-1,p,q} : |A| \geq q + 1\}$ and $\mathcal{T} = \{A \in \mathcal{K}_{n,p,q} \backslash \mathcal{K}_{n-1,p,q} : |A| = q\}$. We define a bijection $\pi : \mathcal{K}_{n-2,p,q} \backslash \{\emptyset\} \rightarrow \mathcal{S}$ for a nonempty set $A \in \mathcal{K}_{n-2,p,q}$ by

$$\pi(A) \ := \ (A \backslash \{\max A\} + 1) \cup \{n - p, n\}.$$

First, π is well-defined. Because $n \in \pi(A)$, $\pi(A) \notin \mathcal{K}_{n-1,p,q}$. That $\max A \leq n - 2$ implies that $\max_2 A \leq n - 2 - p$, so $\pi(A)$ does not contain any number strictly between $n - p$ and n. Hence,

$$\max \pi(A) - \max_2 \pi(A) \ = \ n - (n - p) \ = \ p.$$

Also, $|\pi(A)| = |A| + 1 \geq q + 1$ and

$$\min \pi(A) \ = \ \min A + 1 \ \geq \ |A| + 1 = |\pi(A)|.$$

Therefore, $\pi(A) \in \mathcal{S}$.

Next, π is one-to-one. Let $A_1, A_2 \in \mathcal{K}_{n-2,p,q} \backslash \{\emptyset\}$ such that $\pi(A_1) = \pi(A_2)$. Note that

$$\max(A_i \backslash \{\max A_i\} + 1) \ \leq \ (n - 2 - p) + 1 \ = \ n - 1 - p \text{ for } i = 1, 2.$$

Hence, $\pi(A_1) = \pi(A_2)$ implies that $A_1\backslash\{\max A_1\} = A_2\backslash\{\max A_2\}$. So, $\max_2 A_1 = \max_2 A_2$, which, combined with $\max A_i - \max_2 A_i = p$ for $i = 1, 2$ gives $A_1 = A_2$. We conclude that π is one-to-one.

Next, π is onto. Take $A \in \mathcal{S}$. Then $n, n-p \in A$ and $|A| \ge q+1$. Let $B = A\backslash\{n-p, n\} - 1$ and $\ell = \max B$. Let $C = B \cup \{\ell + p\}$. We claim that $C \in \mathcal{K}_{n-2,p,q}$. Indeed,

$$\begin{aligned} \max C &= \max B + p \le n - p - 1 - 1 + p = n - 2, \\ \min C &= \min A - 1 \ge |A| - 1 = |B| + 1 = |C|, \text{ and} \\ |C| &= |B| + 1 = |A| - 1 \ge (q+1) - 1 = q. \end{aligned}$$

It is clear from how we define C that $\max C - \max_2 C = p$. Finally, $\pi(C) = A$ by construction.

We have shown that $|\mathcal{S}| = |\mathcal{K}_{n-2,p,q}\backslash\{\emptyset\}| = |\mathcal{K}_{n-2,p,q}| - 1$. It remains to show that

$$|\mathcal{T}| = \binom{n-p-q}{q-2}.$$

A set A is in $\mathcal{T}$ if and only if $\min A \ge |A| = q$, $\max A = n$, and $\max_2 A = n - p$. Hence, we can write a set A in $\mathcal{T}$ as $A = D \cup \{n-p, n\}$, where $D \subset \{q, \ldots, n-p-1\}$ and $|D| = q - 2$. Therefore, $|\mathcal{T}| = \binom{n-p-q}{q-2}$. This completes our proof as

$$\begin{aligned} |\mathcal{K}_{n,p,q}| &= |\mathcal{K}_{n-1,p,q}| + |\mathcal{K}_{n,p,q}\backslash\mathcal{K}_{n-1,p,q}| \\ &= |\mathcal{K}_{n-1,p,q}| + |\mathcal{S}| + |\mathcal{T}| \\ &= |\mathcal{K}_{n-1,p,q}| + |\mathcal{K}_{n-2,p,q}| + \binom{n-p-q}{q-2} - 1. \end{aligned}$$

□

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