SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES: GENERALIZATIONS

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Abstract. We explore generalized versions of four sums involving gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $E = \sqrt{x^2+1}$. Pell and gibonacci polynomials are related by the relationship $b_n(x) = g_n(2x)$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$. [4, 6, 7]

1.1. **Fundamental Gibonacci Identities.**

Using the Binet-like formulas, we can establish the following gibonacci identities [4, 6]:

$$g_{n+k}^2 - g_{n-k}^2 = \begin{cases} f_{2k}f_{2n}, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k}f_{2n}, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise.} \end{cases} \tag{2}$$

1.2. **Sums Involving Gibonacci Polynomial Squares.**

We explored the following gibonacci sums in [7]:

$$\sum_{n=L}^{\infty} \frac{f_{2k}f_{4n}}{[f_{2n}^2 - (-1)^k f_k^2]^2} = \sum_{r=1}^{k} \frac{1}{f_s^2}; \qquad \sum_{n=L}^{\infty} \frac{\Delta^2 f_{2k}f_{4n}}{[l_{2n}^2 + (-1)^k \Delta^2 f_k^2]^2} = \sum_{r=1}^{k} \frac{1}{l_s^2};$$

$$\sum_{n=M}^{\infty} \frac{f_{2k}f_{4n+2}}{[f_{2n+1}^2 + (-1)^k f_k^2]^2} = \sum_{r=1}^{k} \frac{1}{f_t^2}; \qquad \sum_{n=M}^{\infty} \frac{\Delta^2 f_{2k}f_{4n+2}}{[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2]^2} = \sum_{r=1}^{k} \frac{1}{l_t^2},$$

where k is a positive integer; $1 \leq r \leq k$;

$$L = \begin{cases} (k+1)/2,\ k \geq 1, & \text{if } k \text{ is odd;} \\ k/2+1,\ k \geq 2, & \text{otherwise;} \end{cases} \quad \text{and} \quad s = \begin{cases} 2r-1, & \text{if } k \text{ is odd;} \\ 2r, & \text{otherwise; and} \end{cases}$$

$$M = \begin{cases} (k+1)/2,\ k \geq 1, & \text{if } k \text{ is odd;} \\ k/2,\ k \geq 2, & \text{otherwise;} \end{cases} \quad \text{and} \quad t = \begin{cases} 2r, & \text{if } k \text{ is odd;} \\ 2r-1, & \text{otherwise.} \end{cases}$$

1.3. **Telescoping Gibonacci Sums.** The objective of our discourse is to explore generalizations of variations of these sums. To this end and in the interest of brevity, we will first investigate four telescoping sums in the following lemmas.

Lemma 1. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g^{\lambda}_{(2n-1)k}} - \frac{1}{g^{\lambda}_{(2n+1)k}} \right] = \frac{1}{g^{\lambda}_{k}}. \tag{3}$$

Proof. With recursion [4, 6, 7], we will first confirm that

$$\sum_{n=1}^{m} \left[\frac{1}{g^{\lambda}_{(2n-1)k}} - \frac{1}{g^{\lambda}_{(2n+1)k}} \right] = \frac{1}{g^{\lambda}_{k}} - \frac{1}{g^{\lambda}_{(2m+1)k}}. \tag{4}$$

Letting A_m denote the left side (LHS) of this equation and B_m its right side (RHS) , we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g^{\lambda}_{(2m-1)k}} - \frac{1}{g^{\lambda}_{(2m+1)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, establishing the validity of equation (4).

Because $\lim_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$, the given result now follows from equation (3). □

Lemma 2. *Let k and λ be positive integers. Then*

$$\sum_{n=2}^{\infty} \left[\frac{1}{g^{\lambda}_{(2n-2)k}} - \frac{1}{g^{\lambda}_{(2n+2)k}} \right] = \frac{1}{g^{\lambda}_{2k}} + \frac{1}{g^{\lambda}_{4k}}. \tag{5}$$

Proof. By invoking recursion [4, 6, 7], we will first prove that

$$\sum_{n=2}^{m} \left[\frac{1}{g^{\lambda}_{(2n-2)k}} - \frac{1}{g^{\lambda}_{(2n+2)k}} \right] = \frac{1}{g^{\lambda}_{2k}} + \frac{1}{g^{\lambda}_{4k}} - \left[\frac{1}{g^{\lambda}_{2mk}} - \frac{1}{g^{\lambda}_{(2m+2)k}} \right]. \tag{6}$$

By letting A_m = LHS and B_m = RHS of this equation, we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g^{\lambda}_{(2m-2)k}} - \frac{1}{g^{\lambda}_{(2m+2)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

By recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, establishing the truthfulness of equation (6).

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, equation (6) yields the desired result. □

An Interesting Byproduct. It follows by Lemma 2 that

$$\sum_{n=1}^{\infty} \left[\frac{1}{g_{2nk}^{\lambda}} - \frac{1}{g_{(2n+4)k}^{\lambda}} \right] = \frac{1}{g_{2k}^{\lambda}} + \frac{1}{g_{4k}^{\lambda}}. \tag{7}$$

Lemma 3. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g_{(2n-1)k}^{\lambda}} - \frac{1}{g_{(2n+3)k}^{\lambda}} \right] = \frac{1}{g_{k}^{\lambda}} + \frac{1}{g_{3k}^{\lambda}}. \tag{8}$$

Proof. Using recursion [4, 6, 7], we will first establish that

$$\sum_{n=1}^{m} \left[\frac{1}{g_{(2n-1)k}^{\lambda}} - \frac{1}{g_{(2n+3)k}^{\lambda}} \right] = \frac{1}{g_{k}^{\lambda}} + \frac{1}{g_{3k}^{\lambda}} - \left[\frac{1}{g_{(2m+1)k}^{\lambda}} + \frac{1}{g_{(2m+3)k}^{\lambda}} \right]. \tag{9}$$

Let A_m = LHS of this equation and B_m its RHS. Then

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g_{(2m-1)k}^{\lambda}} - \frac{1}{g_{(2m+3)k}^{\lambda}} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, establishing the veracity of equation (9).

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, equation (9) yields the given result. □

Lemma 4. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{1}{g_{(2n+1)k}^{\lambda}} - \frac{1}{g_{(2n+3)k}^{\lambda}} \right] = \frac{1}{g_{3k}^{\lambda}}. \tag{10}$$

Proof. With recursion [4, 6, 7], we will first confirm that

$$\sum_{n=1}^{m} \left[\frac{1}{g_{(2n+1)k}^{\lambda}} - \frac{1}{g_{(2n+3)k}^{\lambda}} \right] = \frac{1}{g_{3k}^{\lambda}} - \frac{1}{g_{(2m+)k}^{\lambda}}. \tag{11}$$

Letting A_m = LHS of this equation and B_m its RHS, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{g_{(2m+1)k}^{\lambda}} - \frac{1}{g_{(2m+3)k}^{\lambda}} \\ &= A_m - A_{m-1}. \end{aligned}$$

By recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, verifying equation (11).

This yields the given result. □

2. Generalized Gibonacci Polynomial Sums

With the above identities and lemmas at our disposal, we now begin our explorations with the restriction $\lambda = 2$. In the interest of brevity, we now let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} -1, & \text{if } g_n = f_n; \\ 1, & \text{otherwise.} \end{cases}$$

The first result invokes Lemma 1.

Theorem 1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu f_{2k} f_{4nk}}{\left[g_{2nk}^2 + (-1)^k \mu\nu f_k^2\right]^2} = \frac{1}{g_k^2}. \tag{12}$$

Proof. It follows by identities (1) and (2) that

$$g_{(2n+1)k}^2 - g_{(2n-1)k}^2 = \begin{cases} f_{2k} f_{4nk}, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k} f_{4nk}, & \text{otherwise;} \end{cases}$$

$$g_{(2n+1)k} g_{(2n-1)k} - g_{2nk}^2 = \begin{cases} (-1)^{k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^k \Delta^2 f_k^2, & \text{otherwise.} \end{cases}$$

Suppose $g_n = f_n$. With these two identities, Lemma 1 then yields

$$\frac{f_{2k} f_{4nk}}{\left[f_{2nk}^2 - (-1)^k f_k^2\right]^2} = \frac{f_{(2n+1)k}^2 - f_{(2n-1)k}^2}{f_{(2n+1)k}^2 f_{(2n-1)k}^2};$$

$$\sum_{n=1}^{\infty} \frac{f_{2k} f_{4nk}}{\left[f_{2nk}^2 - (-1)^k f_k^2\right]^2} = \sum_{n=1}^{\infty} \left[\frac{1}{f_{(2n-1)k}^2} - \frac{1}{f_{(2n+1)k}^2}\right]$$

$$= \frac{1}{f_k^2}. \tag{13}$$

On the other hand, let $g_n = l_n$. Using Lemma 1 and the above two identities, we then get

$$\frac{\Delta^2 f_{2k} f_{4nk}}{\left[l_{2nk}^2 + (-1)^k \Delta^2 f_k^2\right]^2} = \frac{l_{(2n+1)k}^2 - l_{(2n-1)k}^2}{l_{(2n+1)k}^2 l_{(2n-1)k}^2};$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{4nk}}{\left[l_{2nk}^2 + (-1)^k \Delta^2 f_k^2\right]^2} = \sum_{n=1}^{\infty} \left[\frac{1}{l_{(2n-1)k}^2} - \frac{1}{l_{(2n+1)k}^2}\right]$$

$$= \frac{1}{l_k^2}. \tag{14}$$

Combining equations (13) and (14), we get the desired result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{F_{4n}}{(F_{2n}^2 + 1)^2} = 1 \text{ [7]}; \qquad \sum_{n=1}^{\infty} \frac{F_{4n}}{(L_{2n}^2 - 5)^2} = \frac{1}{5} \text{ [7]};$$

$$\sum_{n=1}^{\infty} \frac{F_{8n}}{(F_{4n}^2 - 1)^2} = \frac{1}{3} \text{ [7]}; \qquad \sum_{n=1}^{\infty} \frac{F_{8n}}{(L_{4n}^2 + 5)^2} = \frac{1}{135};$$

$$\sum_{n=1}^{\infty} \frac{F_{12n}}{(F_{6n}^2 + 4)^2} = \frac{1}{32}; \qquad \sum_{n=1}^{\infty} \frac{F_{12n}}{(L_{6n}^2 - 20)^2} = \frac{1}{640}.$$

The next result is an application of Lemma 2.

Theorem 2. *Let k be a positive integer. Then*

$$\sum_{n=2}^{\infty} \frac{\mu f_{4k} f_{4nk}}{\left(g_{2nk}^2 + \mu\nu f_{2k}^2\right)^2} = \frac{1}{g_{2k}^2} + \frac{1}{g_{4k}^2}. \tag{15}$$

Proof. Using identities (1) and (2), we get

$$g_{(2n+2)k}^2 - g_{(2n-2)k}^2 = \begin{cases} f_{4k} f_{4nk}, & \text{if } g_n = f_n; \\ \Delta^2 f_{4k} f_{4nk}, & \text{otherwise;} \end{cases}$$

$$g_{(2n+2)k} g_{(2n-2)k} - g_{2nk}^2 = \begin{cases} -f_{2k}^2, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k}^2, & \text{otherwise.} \end{cases}$$

Suppose $g_n = f_n$. By invoking these two identities and Lemma 2, we get

$$\begin{aligned} \frac{f_{4k} f_{4nk}}{\left(f_{2nk}^2 - f_{2k}^2\right)^2} &= \frac{f_{(2n+2)k}^2 - f_{(2n-2)k}^2}{f_{(2n+2)k}^2 f_{(2n-2)k}^2}; \\ \sum_{n=2}^{\infty} \frac{f_{4k} f_{4nk}}{\left(f_{2nk}^2 - f_k^2\right)^2} &= \sum_{n=2}^{\infty} \left[\frac{1}{f_{(2n-2)k}^2} - \frac{1}{f_{(2n+2)k}^2}\right] \\ &= \frac{1}{f_{2k}^2} + \frac{1}{f_{4k}^2}. \end{aligned} \tag{16}$$

Now let $g_n = l_n$. Invoking the above identities, Lemma 2 yields

$$\begin{aligned} \frac{\Delta^2 f_{4k} f_{4nk}}{\left(l_{2nk}^2 + \Delta^2 f_{2k}^2\right)^2} &= \frac{l_{(2n+2)k}^2 - l_{(2n-2)k}^2}{l_{(2n+2)k}^2 l_{(2n-2)k}^2}; \\ \sum_{n=2}^{\infty} \frac{\Delta^2 f_{4k} f_{4nk}}{\left(l_{2nk}^2 + \Delta^2 f_{2k}^2\right)^2} &= \sum_{n=2}^{\infty} \left[\frac{1}{l_{(2n-2)k}^2} - \frac{1}{l_{(2n+2)k}^2}\right] \\ &= \frac{1}{l_{2k}^2} + \frac{1}{l_{4k}^2}. \end{aligned} \tag{17}$$

The given result now follows by equations (16) and (17). □

This theorem yields

$$\sum_{n=2}^{\infty} \frac{F_{4n}}{(F_{2n}^2 - 1)^2} = \frac{10}{27} \text{ [7]}; \qquad \sum_{n=2}^{\infty} \frac{F_{4n}}{(L_{2n}^2 + 5)^2} = \frac{58}{6,615};$$

$$\sum_{n=2}^{\infty} \frac{F_{8n}}{(F_{4n}^2 - 9)^2} = \frac{50}{9,261}; \qquad \sum_{n=2}^{\infty} \frac{F_{8n}}{(L_{4n}^2 + 45)^2} = \frac{2,258}{11,365,305};$$

$$\sum_{n=2}^{\infty} \frac{F_{12n}}{(F_{6n}^2 - 64)^2} = \frac{325}{2,985,984}; \qquad \sum_{n=2}^{\infty} \frac{F_{12n}}{(L_{6n}^2 + 320)^2} = \frac{13,001}{3,023,425,440}.$$

This theorem can be re-stated in a slightly different way, as the next corollary shows.

Corollary 1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu f_{4k} f_{2(2n+2)k}}{\left[g_{(2n+2)k}^2 + \mu\nu f_{2k}^2\right]^2} = \frac{1}{g_{2k}^2} + \frac{1}{g_{4k}^2}. \tag{18}$$

The next result is an application of Lemma 3.

Theorem 3. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu f_{4k} f_{2(2n+1)k}}{\left[g^2_{(2n+1)k} + (-1)^k \mu\nu f^2_{2k}\right]^2} = \frac{1}{g^2_k} + \frac{1}{g^2_{3k}}. \tag{19}$$

Proof. With identities (1) and (2), we get

$$g^2_{(2n+3)k} - g^2_{(2n-1)k} = \begin{cases} f_{4k} f_{2(2n+1)k}, & \text{if } g_n = f_n; \\ \Delta^2 f_{4k} f_{2(2n+1)k}, & \text{otherwise}; \end{cases}$$

$$g_{(2n+3)k} f_{(2n-1)k} - g^2_{(2n+1)k} = \begin{cases} (-1)^{k+1} f^2_{2k}, & \text{if } g_n = f_n; \\ (-1)^k \Delta^2 f^2_{2k}, & \text{otherwise}. \end{cases}$$

Suppose $g_n = f_n$. Lemma 3, coupled with these two identities, then yields

$$\frac{f_{4k} f_{2(2n+1)k}}{\left[f^2_{(2n+1)k} - (-1)^k f^2_{2k}\right]^2} = \frac{f^2_{(2n+3)k} - f^2_{(2n-1)k}}{f^2_{(2n+3)k} f^2_{(2n-1)k}};$$

$$\sum_{n=1}^{\infty} \frac{f_{4k} f_{2(2n+1)k}}{\left[f^2_{(2n+1)k} - (-1)^k f^2_{2k}\right]^2} = \sum_{n=1}^{\infty} \left[\frac{1}{f^2_{(2n-1)k}} - \frac{1}{f^2_{(2n+3)k}}\right]$$

$$= \frac{1}{f^2_k} + \frac{1}{f^2_{3k}}. \tag{20}$$

Now let $g_n = l_n$. Using Lemma 3 and the two identities above, we then have

$$\frac{\Delta^2 f_{4k} f_{2(2n+1)k}}{\left[l^2_{(2n+1)k} + (-1)^k \Delta^2 f^2_{2k}\right]^2} = \frac{l^2_{(2n+3)k} - l^2_{(2n-1)k}}{l^2_{(2n+3)k} l^2_{(2n-1)k}};$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+1)k}}{\left[l^2_{(2n+1)k} + (-1)^k \Delta^2 f^2_{2k}\right]^2} = \sum_{n=1}^{\infty} \left[\frac{1}{l^2_{(2n-1)k}} - \frac{1}{l^2_{(2n+3)k}}\right]$$

$$= \frac{1}{l^2_k} + \frac{1}{l^2_{3k}}. \tag{21}$$

Combining equations (20) and (21), we get the desired result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(F^2_{2n+1} + 1)^2} = \frac{5}{12} \text{ [6]}; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(L^2_{2n+1} - 5)^2} = \frac{17}{240} \text{ [6]};$$

$$\sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[F^2_{2(2n+1)} - 9]^2} = \frac{65}{1,344}; \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[L^2_{2(2n+1)} + 45]^2} = \frac{37}{34,020};$$

$$\sum_{n=1}^{\infty} \frac{F_{6(2n+1)}}{[F^2_{3(2n+1)} + 64]^2} = \frac{145}{83,232}; \qquad \sum_{n=1}^{\infty} \frac{F_{6(2n+1)}}{[L^2_{3(2n+1)} - 320]^2} = \frac{181}{2,079,360}.$$

The next result is an application of Lemma 4.

Theorem 4. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu f_{2k} f_{2(2n+2)k}}{\left[g^2_{(2n+2)k} + (-1)^k \mu\nu f^2_{2k}\right]^2} = \frac{1}{g^2_{3k}}. \tag{22}$$

Proof. With identities (1) and (2), we get

$$\begin{aligned}
g^2_{(2n+3)k} - g^2_{(2n+1)k} &= \begin{cases} f_{2k} f_{(2n+2)k}, & \text{if } g_n = f_n; \\ \Delta^2 f_{2k} f_{(2n+2)k}, & \text{otherwise;} \end{cases} \\
g_{(2n+3)k} g_{(2n+1)k} - g^2_{(2n+2)k} &= \begin{cases} (-1)^{k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^k \Delta^2 f_k^2, & \text{otherwise.} \end{cases}
\end{aligned}$$

Let $g_n = f_n$. With these two identities, Lemma 4 then yields

$$\begin{aligned}
\frac{f_{2k} f_{2(2n+2)k}}{\left[f^2_{(2n+2)k} - (-1)^k f^2_{2k}\right]^2} &= \frac{f^2_{(2n+3)k} - f^2_{(2n+1)k}}{f^2_{(2n+3)k} f^2_{(2n+1)k}}; \\
\sum_{n=1}^{\infty} \frac{f_{2k} f_{2(2n+2)k}}{\left[f^2_{(2n+2)k} - (-1)^k f^2_{k}\right]^2} &= \sum_{n=1}^{\infty} \left[\frac{1}{f^2_{(2n+1)k}} - \frac{1}{f^2_{(2n+3)k}}\right] \\
&= \frac{1}{f^2_{3k}}.
\end{aligned} \tag{23}$$

Next, we let $g_n = l_n$. Using Lemma 4 and the two identities above, we then have

$$\begin{aligned}
\frac{\Delta^2 f_{2k} f_{2(2n+2)k}}{\left[l^2_{(2n+2)k} + (-1)^k \Delta^2 f^2_{k}\right]^2} &= \frac{l^2_{(2n+3)k} - l^2_{(2n+1)k}}{l^2_{(2n+3)k} l^2_{(2n+1)k}}; \\
\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+2)k}}{\left[l^2_{(2n+2)k} + (-1)^k \Delta^2 f^2_{k}\right]^2} &= \sum_{n=1}^{\infty} \left[\frac{1}{l^2_{(2n+1)k}} - \frac{1}{l^2_{(2n+3)k}}\right] \\
&= \frac{1}{l^2_{3k}}.
\end{aligned} \tag{24}$$

Combining equations (23) and (24), we get the desired result. □

In particular, we then have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(F^2_{2n+2} + 1)^2} &= \frac{1}{4}; & \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(L^2_{2n+2} - 5)^2} &= \frac{1}{80}; \\
\sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{[F^2_{2(2n+2)} - 1]^2} &= \frac{1}{192}; & \sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{[L^2_{2(2n+2)} + 5]^2} &= \frac{1}{4,860}; \\
\sum_{n=1}^{\infty} \frac{F_{6(2n+2)}}{[F^2_{3(2n+2)} + 4]^2} &= \frac{1}{9,248}; & \sum_{n=1}^{\infty} \frac{F_{6(2n+2)}}{[L^2_{3(2n+2)} - 20]^2} &= \frac{1}{231,040}.
\end{aligned}$$

This theorem also can be reworded in a slightly different way, as the next corollary shows.

Corollary 2. *Let k be a positive integer. Then*

$$\sum_{n=2}^{\infty} \frac{\mu f_{2k} f_{4nk}}{\left[g^2_{2nk} + (-1)^k \mu\nu f^2_{2k}\right]^2} = \frac{1}{g^2_{3k}}. \tag{25}$$

Finally, with identities (1) and (2) and the sum

$$\sum_{n=1}^{m}\left[\frac{1}{g_{2nk}^{\lambda}}-\frac{1}{g_{(2n+2)k}^{\lambda}}\right]=\frac{1}{g_{2k}^{\lambda}}, \tag{26}$$

we can establish the following result. Its proof follows similar steps in previous theorems. So we omit it for brevity.

Theorem 5.

$$\sum_{n=1}^{\infty}\frac{\mu f_{2k}f_{2(2n+1)k}}{[g_{(2n+1)k}^2+\mu\nu f_k^2]^2}=\frac{1}{g_{2k}^2}. \tag{27}$$

In particular, we then have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{F_{2(2n+1)}}{(F_{2n+1}^2-1)^2} &= 1\ [5]; & \sum_{n=1}^{\infty}\frac{F_{2(2n+1)}}{(L_{2n+1}^2+1)^2} &= \frac{1}{45};\\
\sum_{n=1}^{\infty}\frac{F_{4(2n+1)}}{[F_{2(2n+1)}^2-1]^2} &= \frac{1}{27}; & \sum_{n=1}^{\infty}\frac{F_{4(2n+1)}}{[L_{2(2n+1)}^2+5]^2} &= \frac{1}{735};\\
\sum_{n=1}^{\infty}\frac{F_{6(2n+1)}}{[F_{3(2n+1)}^2-4]^2} &= \frac{1}{512}; & \sum_{n=1}^{\infty}\frac{F_{6(2n+1)}}{[L_{3(2n+1)}^2+20]^2} &= \frac{1}{12,960}.
\end{aligned}$$

2.1. **Gibonacci Treasures.** The theorems yield interesting gibonacci dividends. Using Theorems 1 and 3, we get

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{F_{2n}}{(F_n^2+1)^2} &= \sum_{n=1}^{\infty}\frac{F_{2(2n+1)}}{\left(F_{2n+1}^2+1\right)^2}+\sum_{n=1}^{\infty}\frac{F_{2(2n)}}{\left(F_{2n}^2+1\right)^2}\\
&= \frac{17}{12};\\
\sum_{n=1}^{\infty}\frac{F_{2n}}{(F_n^2+1)^2} &= \frac{5}{3}\ [6];\\
\sum_{n=2}^{\infty}\frac{F_{2n}}{(L_n^2-5)^2} &= \sum_{n=1}^{\infty}\frac{F_{2(2n+1)}}{(L_{2n+1}^2-5)^2}+\sum_{n=1}^{\infty}\frac{F_{2(2n)}}{(L_{2n}^2-5)^2}\\
&= \frac{13}{48};\\
\sum_{n=1}^{\infty}\frac{F_{2n}}{(L_n^2-5)^2} &= \frac{1}{3}\ [8].
\end{aligned}$$

Likewise, Theorems 2 and 3 yield

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{F_{4n}}{(L_{2n}^2+45)^2} &= \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[L_{2(2n+1)}^2+45]^2} + \sum_{n=1}^{\infty} \frac{F_{4(2n)}}{[L_{2(2n)}^2+45]^2} \\ &= \frac{4,736,509}{3,682,358,820}; \\ \sum_{n=1}^{\infty} \frac{F_{4n}}{(L_{2n}^2+45)^2} &= \frac{236,804}{102,287,745}; \\ \sum_{n=3}^{\infty} \frac{F_{2n}}{(F_n^2+1)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(F_{2n+1}^2+1)^2} + \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(F_{2n+2}^2+1)^2} \\ &= \frac{2}{3}; \\ \sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2+1)^2} &= \frac{5}{3} \text{ [6]},\end{aligned}$$

as obtained above.

Using the result [5, 6]

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(F_{2n+1}^2-1)^2} = 1,$$

it follows by Theorem 2 that

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{(F_n^2-1)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(F_{2n+1}^2-1)^2} + \sum_{n=2}^{\infty} \frac{F_{2(2n)}}{(F_{2n}^2-1)^2} \\ &= \frac{37}{27} \text{ [7]}.\end{aligned}$$

Finally, it follows by Theorems 4 and 5 that

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{4n}}{(L_{2n}^2+5)^2} &= \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{(L_{2(2n+1)}^2+5)^2} + \sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{(L_{2(2n+2)}^2+5)^2} \\ &= \frac{373}{238,140}; \\ \sum_{n=1}^{\infty} \frac{F_{4n}}{(L_{2n}^2+5)^2} &= \frac{13}{540}.\end{aligned}$$

3. Pell, Chebyshev, and Vieta Implicatins

Finally, with the Pell-gibonacci relationship $b_n(x) = g_n(2x)$, we can extract the Pell versions of the theorems. Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$, where $i = \sqrt{-1}$ [2, 3, 4]. They can be employed to find the Chebyshev and Vieta versions of the theorems. In the interest of brevity, we omit them all; but we encourage gibonacci enthusiasts to pursue them.

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