# ADDITIONAL SUMS INVOLVING JACOBSTHAL POLYNOMIAL SQUARES

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Abstract. We explore the Jacobsthal versions of four infinite sums involving gibonacci polynomial squares.

## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x)+ b(x)z_n(x)$, where $x$ is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the $n$th *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [1, 5].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 5].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4, 5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so $z_n$ will mean $z_n(x)$. In addition, we let $g_n = f_n$ or $l_n$, $c_n = J_n$ or $j_n$, $\Delta = \sqrt{x^2 + 4}$, and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

### 1.1. Sums Involving Gibonacci Squares.

We studied the following sums involving gibonacci polynomial squares in Theorems 1–4 of [6]:

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} - 4(-1)^k f_{2k} f_{4n}}{\left[f_{2n}^2 - (-1)^k f_k^2\right]^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_s^4}; \tag{1}$$

$$\sum_{n=L}^{\infty} \frac{f_{4k} f_{8n} + 4(-1)^k f_{2k} f_{4n}}{\left[l_{2n}^2 + (-1)^k \Delta^2 f_k^2\right]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_s^4}; \tag{2}$$

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} + 4(-1)^k f_{2k} f_{4n+2}}{\left[f_{2n+1}^2 + (-1)^k f_k^2\right]^4} = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_t^4}; \tag{3}$$

$$\sum_{n=M}^{\infty} \frac{f_{4k} f_{8n+4} - 4(-1)^k f_{2k} f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^4} = \frac{1}{\Delta^2} \sum_{r=1}^{k} \frac{1}{l_t^4}, \tag{4}$$

where $k$ is a positive integer; $1 \leq r \leq k$;

$$L = \begin{cases} (k+1)/2, k \geq 1, & \text{if } k \text{ is odd;} \\ k/2+1, k \geq 2, & \text{otherwise;} \end{cases} \qquad s = \begin{cases} 2r-1, & \text{if } k \text{ is odd;} \\ 2r, & \text{otherwise;} \end{cases}$$

$$M = \begin{cases} (k+1)/2, k \geq 1, & \text{if } k \text{ is odd;} \\ k/2, k \geq 2, & \text{otherwise;} \end{cases} \quad \text{and} \quad t = \begin{cases} 2r, & \text{if } k \text{ is odd;} \\ 2r-1, & \text{otherwise.} \end{cases}$$

## 2. Jacobsthal Consequences

Our objective is to explore the Jacobsthal versions of the gibonacci sums (1)–(4); we will extract them from the above sums using the Jacobsthal-gibonacci relationships in Section 1.

To this end, in the interest of brevity and clarity, we let $A$ denote the left-hand side (LHS) of each equation and $B$ its right-hand side (RHS), and LHS and RHS those of the corresponding Jacobsthal equation, respectively.

### 2.1. **Jacobsthal Version of Equation** (1).

*Proof.* Let $A = \dfrac{f_{4k} f_{8n} - 4(-1)^k f_{2k} f_{4n}}{\left[f_{2n}^2 - (-1)^k f_k^2\right]^4}$. Replacing $x$ with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with $x^{8n-4}$, we get

$$\begin{aligned} A &= \frac{x^{4n-2k-3}\left[x^{(4k-1)/2} f_{4k}\right]\left[x^{(8n-1)/2} f_{8n}\right] - 4(-1)^k x^{6n-k-3}\left[x^{(2k-1)/2} f_{2k}\right]\left[x^{(4n-1)/2} f_{4n}\right]}{\left\{[x^{(2n-1)/2} f_{2n}]^2 - (-1)^k x^{2n-k}[x^{(k-1)/2} f_k]^2\right\}^4} \\ &= \frac{x^{4n-2k-3} J_{4k} J_{8n} - 4(-1)^k x^{6n-k-3} J_{2k} J_{4n}}{\left[J_{2n}^2 - (-1)^k x^{2n-k} J_k^2\right]^4}; \\ \text{LHS} &= \sum_{n=L}^{\infty} \frac{x^{4n-2k-3} J_{4k} J_{8n} - 4(-1)^k x^{6n-k-3} J_{2k} J_{4n}}{\left[J_{2n}^2 - (-1)^k x^{2n-k} J_k^2\right]^4}, \end{aligned} \tag{5}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, let $B = \Delta^2 \displaystyle\sum_{r=1}^{k} \frac{1}{f_s^4}$.

*Case 1.* Suppose $k$ is odd. Replace $x$ with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{4r-4}$; this yields

$$\begin{aligned} B &= \frac{D^2}{x} \sum_{r=1}^{k} \frac{1}{f_{2r-1}^4} \\ &= \frac{D^2}{x} \sum_{r=1}^{k} \frac{x^{4r-4}}{(x^{r-1} f_{2r-1})^4}; \\ \text{RHS} &= D^2 \sum_{r=1}^{k} \frac{x^{4r-5}}{J_{2r-1}^4}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

This, coupled with equation (5) and $k$ odd, yields

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, \text{odd}}}^{\infty} \frac{x^{4n-2k-3} J_{4k} J_{8n} + 4x^{6n-k-3} J_{2k} J_{4n}}{\left(J_{2n}^2 + x^{2n-k} J_k^2\right)^4} = D^2 \sum_{r=1}^{k} \frac{x^{4r-5}}{J_{2r-1}^4}. \tag{6}$$

*Case 2.* Suppose $k$ is even. Replacing $x$ with $1/\sqrt{x}$ in $B$, and then multiplying the numerator and denominator with $x^{4r-2}$, we get

$$\begin{aligned} B &= \frac{D^2}{x}\sum_{r=1}^{k}\frac{1}{f_{2r}^4} \\ &= \frac{D^2}{x}\sum_{r=1}^{k}\frac{x^{4r-2}}{\left[x^{(2r-1)/2}f_{2r}\right]^4}; \\ \text{RHS} &= D^2\sum_{r=1}^{k}\frac{x^{4r-3}}{J_{2r}^4}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Together with equation (5) and $k$ even, this yields

$$\sum_{\substack{n=k/2\\ k\geq 2,\text{ even}}}^{\infty}\frac{x^{4n-2k-3}J_{4k}J_{8n} - 4x^{6n-k-3}J_{2k}J_{4n}}{\left(J_{2n}^2 - x^{2n-k}J_k^2\right)^4} = D^2\sum_{r=1}^{k}\frac{x^{4r-3}}{J_{2r}^4}. \tag{7}$$

Merging equations (6) and (7), we get the desired Jacobsthal version:

$$\sum_{n=L}^{\infty}\frac{x^{4n}J_{4k}J_{8n} - 4(-1)^k x^{6n}J_{2k}J_{4n}}{\left[J_{2n}^2 - (-1)^k x^{2n-k}J_k^2\right]^4} = D^2x^{2k}\sum_{r=1}^{k}\frac{x^{2s}}{J_s^4}. \tag{8}$$

□

This yields

$$\begin{aligned} \sum_{n=L}^{\infty}\frac{F_{4k}F_{8n} - 4(-1)^kF_{2k}F_{4n}}{\left[F_{2n}^2 - (-1)^kF_k^2\right]^4} &= 5\sum_{r=1}^{k}\frac{1}{F_s^4}; \\ \sum_{n=L}^{\infty}\frac{16^nJ_{4k}J_{8n} - 4^{3n+1}(-1)^kJ_{2k}J_{4n}}{\left[J_{2n}^2 - (-1)^k2^{2n-k}J_k^2\right]^4} &= 9\cdot 4^k\sum_{r=1}^{k}\frac{4^s}{J_s^4}. \end{aligned}$$

Consequently, we have [6]

$$\sum_{n=1}^{\infty}\frac{3F_{8n}+4F_{4n}}{\left(F_{2n}^2+1\right)^4} = 5; \qquad \sum_{n=2}^{\infty}\frac{7F_{8n}-4F_{4n}}{\left(F_{2n}^2-1\right)^4} = \frac{410}{243};$$

$$\sum_{n=1}^{\infty}\frac{5\cdot 16^nJ_{8n}+4^{3n+1}J_{4n}}{\left(J_{2n}^2+2^{2n-1}\right)^4} = 144; \qquad \sum_{n=2}^{\infty}\frac{17\cdot 16^nJ_{8n}-4^{3n+1}J_{4n}}{\left(J_{2n}^2-4^{n-1}\right)^4} = \frac{1,476,864}{3,125}.$$

### 2.2. **Jacobsthal Version of Equation** (2)**.**

*Proof.* Let $A = \dfrac{f_{4k}f_{8n} + 4(-1)^kf_{2k}f_{4n}}{\left[l_{2n}^2 + (-1)^k\Delta^2f_k^2\right]^4}$. Replacing $x$ with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with $x^{8n-4}$, we get

$$\begin{aligned}
A &= \frac{x^4\left[f_{4k}f_{8n}+4(-1)^k f_{2k}f_{4n}\right]}{\left[xl_{2n}^2+(-1)^k D^2 f_k^2\right]^4}\\
&= \frac{x^{4n-2k+1}\left[x^{(4k-1)/2}f_{4k}\right]\left[x^{(8n-1)/2}f_{8n}\right]+4(-1)^k x^{6n-k+1}\left[x^{(2k-1)/2}f_{2k}\right]\left[x^{(4n-1)/2}f_{4n}\right]}{\left\{(x^{2n/2}l_{2n})^2+(-1)^k D^2 x^{2n-k}[x^{(k-1)/2}f_k]^2\right\}^4}\\
&= \frac{x^{4n-2k+1}\left[x^{2n}J_{4n}J_{8n}+4(-1)^k x^{2n+k}J_{2k}J_{4n}\right]}{\left[j_{2n}^2+(-1)^k D^2 x^{2n-k}J_k^2\right]^4};\\
\text{LHS} &= \sum_{n=L}^{\infty}\frac{x^{4n-2k+1}\left[x^{2n}J_{4n}J_{8n}+4(-1)^k x^{2n+k}J_{2k}J_{4n}\right]}{\left[j_{2n}^2+(-1)^k D^2 x^{2n-k}J_k^2\right]^4}, \qquad (9)
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Let $B = \dfrac{1}{\Delta^2}\displaystyle\sum_{r=1}^{k}\frac{1}{l_s^4}$. With $k$ odd, replace $x$ with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{4r-2}$. Then

$$\begin{aligned}
B &= \frac{x}{D^2}\sum_{r=1}^{k}\frac{1}{l_{2r-1}^4}\\
&= \frac{1}{D^2}\sum_{r=1}^{k}\frac{x^{4r-1}}{\left[x^{(2r-1)/2}l_{2r-1}\right]^4};\\
\text{RHS} &= \frac{1}{D^2}\sum_{r=1}^{k}\frac{x^{4r-1}}{j_{2r-1}^4},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Using equation (9) with $k$ odd, this yields

$$\sum_{\substack{n=(k+1)/2\\ k\ge 1,\,\text{odd}}}^{\infty}\frac{x^{4n}J_{4k}J_{8n}-4x^{2n+k}J_{2k}J_{4n}}{\left(j_{2n}^2-D^2x^{2n-k}J_k^2\right)^4}=\frac{x^{2k}}{D^2}\sum_{r=1}^{k}\frac{x^{4r-2}}{j_{2r-1}^4}. \qquad (10)$$

On the other hand, let $k$ be even. Replacing $x$ with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{4r}$ yield

$$\begin{aligned}
B &= \frac{x}{D^2}\sum_{r=1}^{k}\frac{1}{l_{2r}^4}\\
&= \frac{1}{D^2}\sum_{r=1}^{k}\frac{x^{4r+1}}{\left(x^{2r/2}l_{2r}\right)^4};\\
\text{RHS} &= \frac{1}{D^2}\sum_{r=1}^{k}\frac{x^{4r+1}}{j_{2r}^4},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Using equation (9) with $k$ even, this yields

$$\sum_{\substack{n=k/2+1\\k\geq 2,\text{ even}}}^{\infty} \frac{x^{4n}\left(J_{4k}J_{8n}+4x^{2n+k}J_{2k}J_{4n}\right)}{\left(j_{2n}^2+D^2x^{2n-k}J_k^2\right)^4} = \frac{x^{2k}}{D^2}\sum_{r=1}^{k}\frac{x^{4r}}{j_{2r}^4}. \tag{11}$$

By combining the equations (10) and (11), we get the Jacobsthal version of equation (2):

$$\sum_{n=L}^{\infty} \frac{x^{4n}\left[J_{4k}J_{8n}+4(-1)^k x^{2n+k}J_{2k}J_{4n}\right]}{\left[j_{2n}^2+(-1)^k D^2x^{2n-k}J_k^2\right]^4} = \frac{x^{2k}}{D^2}\sum_{r=1}^{k}\frac{x^{2s}}{j_s^4}. \tag{12}$$

□

It then follows that

$$\begin{aligned}\sum_{n=L}^{\infty} \frac{F_{4k}F_{8n}+4(-1)^kF_{2k}F_{4n}}{\left[L_{2n}^2+5(-1)^kF_k^2\right]^4} &= \frac{1}{5}\sum_{r=1}^{k}\frac{1}{L_s^4};\\ \sum_{n=L}^{\infty} \frac{16^n[J_{4k}J_{8n}+(-1)^k2^{2n+k+2}J_{2k}J_{4n}]}{\left[j_{2n}^2+9(-1)^k2^{2n-k}J_k^2\right]^4} &= \frac{4^k}{9}\sum_{r=1}^{k}\frac{4^s}{j_s^4}.\end{aligned}$$

This yields [6]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3F_{8n}-4F_{4n}}{\left(L_{2n}^2-5\right)^4} &= \frac{1}{5}; & \sum_{n=2}^{\infty}\frac{7F_{8n}+4F_{4n}}{\left(L_{2n}^2+5\right)^4} &= \frac{2,482}{2,917,215};\\ \sum_{n=1}^{\infty}\frac{16^n(5J_{8n}-2^{2n+3}J_{4n})}{\left(j_{2n}^2-9\cdot 2^{2n-1}\right)^4} &= \frac{16}{9}; & \sum_{n=2}^{\infty}\frac{16^n(17J_{8n}+4^{n+3}J_{4n})}{\left(j_{2n}^2+9\cdot 4^{n-1}\right)^4} &= \frac{23,941,376}{2,349,028,125}.\end{aligned}$$

Next, we investigate the Jacobsthal implication of equation (3).

### 2.3. **Jacobsthal Version of Equation** (3)**.**

*Proof.* Let $A = \dfrac{f_{4k}f_{8n+4}+4(-1)^kf_{2k}f_{4n+2}}{\left[f_{2n+1}^2+(-1)^kf_k^2\right]^4}$. Replace $x$ with $1/\sqrt{x}$, and then multiply the numerator and denominator of the resulting expression with $x^{8n}$. We then get

$$\begin{aligned} A &= \frac{x^{4n-2k-1}\left[x^{(4k-1)/2}f_{4k}\right]\left[x^{(8n+3)/2}f_{8n+4}\right]+4(-1)^kx^{6n-k}\left[x^{(2k-1)/2}f_{2k}\right]\left[x^{(4n+1)/2}f_{4n+2}\right]}{\left\{\left(x^{2n/2}f_{2n+1}\right)^2+(-1)^kx^{2n-k+1}\left[x^{(k-1)/2}f_k\right]^2\right\}^4}\\ &= \frac{x^{4n-2k-1}\left[J_{4k}J_{8n+4}+4(-1)^kx^{2n+k+1}J_{2k}J_{4n+2}\right]}{\left[J_{2n+1}^2+(-1)^kx^{2n-k+1}J_k^2\right]^4};\\ \text{LHS} &= \sum_{n=M}^{\infty}\frac{x^{4n-2k-1}\left[J_{4k}J_{8n+4}+4(-1)^kx^{2n+k+1}J_{2k}J_{4n+2}\right]}{\left[J_{2n+1}^2+(-1)^kx^{2n-k+1}J_k^2\right]^4}, \end{aligned} \tag{13}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, let $B = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_t^4}$. Suppose, $k$ is odd. Replace $x$ with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{4r-2}$. This yields

$$\begin{aligned} B &= \frac{D^2}{x} \sum_{r=1}^{k} \frac{1}{f_{2r}^4} \\ &= \frac{D^2}{x} \sum_{r=1}^{k} \frac{x^{4r-2}}{\left[x^{(2r-1)/2} f_{2r}\right]^4}; \\ \text{RHS} &= D^2 \sum_{r=1}^{k} \frac{x^{4r-3}}{J_{2r}^4}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Together with equation (13) and $k$ odd, this yields

$$\sum_{\substack{n=(k+1)/2 \\ k\geq 1,\, \text{odd}}}^{\infty} \frac{x^{4n}\left(J_{4k}J_{8n+4} - 4x^{2n+k+1}J_{2k}J_{4n+2}\right)}{\left(J_{2n+1}^2 - x^{2n-k+1}J_k^2\right)^4} = D^2 x^{2k+1} \sum_{r=1}^{k} \frac{x^{4r-3}}{J_{2r}^4}. \tag{14}$$

With $k$ even, we have $B = \Delta^2 \sum_{r=1}^{k} \frac{1}{f_{2r-1}^4}$. Replacing $x$ with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{4r-4}$, we get

$$\begin{aligned} B &= \frac{D^2}{x} \sum_{r=1}^{k} \frac{1}{f_{2r-1}^4} \\ &= D^2 \sum_{r=1}^{k} \frac{x^{4r-5}}{\left(x^{r-1} f_{2r-1}\right)^4}; \\ \text{RHS} &= D^2 \sum_{r=1}^{k} \frac{x^{4r-5}}{J_{2r-1}^4}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Coupled with equation (13) and $k$ even, this yields

$$\sum_{\substack{n=k/2 \\ k\geq 2,\, \text{even}}}^{\infty} \frac{x^{4n}\left(J_{4k}J_{8n+4} + 4x^{2n+k+1}J_{2k}J_{4n+2}\right)}{\left(J_{2n+1}^2 + x^{2n-k+1}J_k^2\right)^4} = D^2 x^{2k+1} \sum_{r=1}^{k} \frac{x^{4r-5}}{J_{2r-1}^4}.$$

Combining this with equation (14), we get the desired Jacobsthal version:

$$\sum_{n=M}^{\infty} \frac{x^{4n}\left[J_{4k}J_{8n+4} + 4(-1)^k x^{2n+k+1}J_{2k}J_{4n+2}\right]}{\left[J_{2n+1}^2 + (-1)^k x^{2n-k+1}J_k^2\right]^4} = D^2 x^{2k-2} \sum_{r=1}^{k} \frac{x^{2t}}{J_t^4}. \tag{15}$$

□

This implies

$$\begin{aligned}\sum_{n=M}^{\infty} \frac{F_{4k}F_{8n+4} + 4(-1)^k F_{2k}F_{4n+2}}{\left[F_{2n+1}^2 + (-1)^k F_k^2\right]^4} &= 5\sum_{r=1}^{k} \frac{1}{F_t^4};\\ \sum_{n=M}^{\infty} \frac{16^n[J_{4k}J_{8n+4} + (-1)^k 2^{2n+k+3} J_{2k}J_{4n+2}]}{\left[J_{2n+1}^2 + (-1)^k 2^{2n-k+1} J_k^2\right]^4} &= 9\cdot 4^{k-1}\sum_{r=1}^{k} \frac{4^t}{J_t^4}.\end{aligned}$$

Consequently, we have [6]

$$\sum_{n=1}^{\infty} \frac{3F_{8n+4} - 4F_{4n+2}}{\left(F_{2n+1}^2 - 1\right)^4} = 5; \qquad \sum_{n=1}^{\infty} \frac{7F_{8n+4} + 4F_{4n+2}}{\left(F_{2n+1}^2 + 1\right)^4} = \frac{85}{48};$$

$$\sum_{n=1}^{\infty} \frac{16^n(5J_{8n+4} - 4^{n+2}J_{4n+2})}{\left(J_{2n+1}^2 - 4^n\right)^4} = 144; \qquad \sum_{n=2}^{\infty} \frac{16^n(17J_{8n+4} + 2^{2n+5}J_{4n+2})}{\left(J_{2n+1}^2 + 2^{2n-1}\right)^4} = \frac{1,552}{45}.$$

Finally, we explore the Jacobsthal consequence of equation (4).

### 2.4. **Jacobsthal Version of Equation** (4).

*Proof.* Let $A = \dfrac{f_{4k}f_{8n+4} - 4(-1)^k f_{2k}f_{4n+2}}{\left[l_{2n+1}^2 - (-1)^k \Delta^2 f_k^2\right]^4}$. Replacing $x$ with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with $x^{8n}$, we get

$$\begin{aligned} A &= \frac{x^4\left[f_{4k}f_{8n+4} - 4(-1)^k f_{2k}f_{4n+2}\right]}{\left[xl_{2n+1}^2 - (-1)^k D^2 f_k^2\right]^4}\\ &= \frac{x^{4n-2k-1}\left[x^{(4k-1)/2}f_{4k}\right]\left[x^{(8n+3)/2}f_{8n+4}\right] - 4(-1)^k x^{6n-k}\left[x^{(2k-1)/2}f_{2k}\right]\left[x^{(4n+1)/2}f_{4n+2}\right]}{\left\{\left[x^{(2n+1)/2}l_{2n+1}\right]^2 - (-1)^k D^2 x^{2n-k+1}\left[x^{(k-1)/2}f_k\right]^2\right\}^4}\\ &= \frac{x^{4n-2k-1}J_{4k}J_{8n+4} - 4(-1)^k x^{6n-k}J_{2k}J_{4n+2}}{\left[j_{2n+1}^2 - (-1)^k D^2 x^{2n-k+1}J_k^2\right]^4};\\ \text{LHS} &= \sum_{n=M}^{\infty} \frac{x^{4n-2k-1}J_{4k}J_{8n+4} - 4(-1)^k x^{6n-k}J_{2k}J_{4n+2}}{\left[j_{2n+1}^2 - (-1)^k D^2 x^{2n-k+1}J_k^2\right]^4}, \end{aligned} \tag{16}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \dfrac{1}{\Delta^2}\displaystyle\sum_{r=1}^{k} \frac{1}{l_t^4}$ and $k$ be odd. Now, replace $x$ with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{4r}$. Then

$$\begin{aligned} B &= \frac{1}{\Delta^2}\sum_{r=1}^{k} \frac{1}{l_{2r}^4}\\ &= \frac{x}{D^2}\sum_{r=1}^{k} \frac{x^{4r}}{\left(x^{2r/2}l_{2r}\right)^4};\\ \text{RHS} &= \frac{x}{D^2}\sum_{r=1}^{k} \frac{x^{4r}}{j_{2r}^4}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Coupled with equation (16) and $k$ odd, this yields

$$\sum_{\substack{n=(k+1)/2\\k\geq 1,\,\text{odd}}}^{\infty} \frac{x^{4n-2k-1}\left(J_{4k}J_{8n+4}+4x^{2n+k+1}J_{2k}J_{4n+2}\right)}{\left(j_{2n+1}^2+D^2x^{2n-k+1}J_k^2\right)^4} = \frac{x}{D^2}\sum_{r=1}^{k}\frac{x^{4r}}{j_{2r}^4}. \tag{17}$$

When $k$ is even, $B = \dfrac{1}{\Delta^2}\displaystyle\sum_{r=1}^{k}\frac{1}{l_{2r-1}^4}$. Replacing $x$ with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{4r-2}$, we get

$$\begin{aligned} B &= \frac{x}{D^2}\sum_{r=1}^{k}\frac{1}{l_{2r-1}^4}\\ &= \frac{x}{D^2}\sum_{r=1}^{k}\frac{x^{4r-2}}{\left[x^{(2r-1)/2}l_{2r-1}\right]^4};\\ \text{RHS} &= \frac{x}{D^2}\sum_{r=1}^{k}\frac{x^{4r-2}}{j_{2r-1}^4}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Using equation (16) with $k$ even, this gives

$$\sum_{\substack{n=k/2\\k\geq 1,\,\text{even}}}^{\infty} \frac{x^{4n-2k-1}\left(J_{4k}J_{8n+4}-4x^{2n+k+1}J_{2k}J_{4n+2}\right)}{\left(j_{2n+1}^2-D^2x^{2n-k+1}J_k^2\right)^4} = \frac{x}{D^2}\sum_{r=1}^{k}\frac{x^{4r-2}}{j_{2r-1}^4}.$$

Merging this with equation (17), we get the desired Jacobsthal version:

$$\sum_{n=M}^{\infty} \frac{x^{4n-2k-1}\left[J_{4k}J_{8n+4}-4(-1)^k x^{2n+k+1}J_{2k}J_{4n+2}\right]}{\left[j_{2n+1}^2-(-1)^k D^2x^{2n-k+1}J_k^2\right]^4} = \frac{1}{D^2}\sum_{r=1}^{k}\frac{x^{2t}}{j_t^4}. \tag{18}$$

□

In particular, we then have

$$\begin{aligned} \sum_{n=M}^{\infty}\frac{F_{4k}F_{8n+4}-4(-1)^kF_{2k}F_{4n+2}}{\left[L_{2n+1}^2-5(-1)^kF_k^2\right]^4} &= \frac{1}{5}\sum_{r=1}^{k}\frac{1}{L_t^4};\\ \sum_{n=M}^{\infty}\frac{4^{2n-k-1}[J_{4k}J_{8n+4}-(-1)^k2^{2n+k+1}J_{2k}J_{4n+2}]}{\left[j_{2n+1}^2-9(-1)^k2^{2n-k+1}J_k^2\right]^4} &= \frac{1}{9}\sum_{r=1}^{k}\frac{4^t}{j_t^4}. \end{aligned}$$

They yield [6]

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{3F_{8n+4}+4F_{4n+2}}{\left(L_{2n+1}^2+5\right)^4} &= \frac{1}{405}; & \sum_{n=1}^{\infty}\frac{7F_{8n+4}-4F_{4n+2}}{\left(L_{2n+1}^2-5\right)^4} &= \frac{257}{3,840};\\ \sum_{n=1}^{\infty}\frac{16^n(5J_{8n+4}+4^{n+1}J_{4n+2})}{\left(j_{2n+1}^2+9\cdot 4^n\right)^4} &= \frac{256}{225}; & \sum_{n=1}^{\infty}\frac{16^n(17J_{8n+4}-2^{2n+3}J_{4n+2})}{\left(j_{2n+1}^2-9\cdot 2^{2n-1}\right)^4} &= \frac{618,752}{108,045}. \end{aligned}$$

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