ON PYTHAGOREAN TRIPLE PRESERVING MATRICES THAT CONTAIN FIBONACCI NUMBERS

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ABSTRACT. In this paper, we show how to use Pythagorean triples whose entries can be expressed using Fibonacci numbers to construct Pythagorean triple preserving matrices with entries that can also be expressed using Fibonacci numbers. We conclude with another Pythagorean triple preserving matrix whose powers contain Fibonacci numbers.

1. Introduction and Preliminaries

Many interesting connections exist between Pythagorean triples and Fibonacci numbers, as well as between triples and generalized Fibonacci sequences [2, 4, 7, 8, 9]. In this paper, we present new connections between Fibonacci numbers and Pythagorean triples, specifically to Pythagorean triple preserving matrices. We need a few definitions, which are given below with some additional details.

Definition 1.1. *A Pythagorean triple (PT) is an ordered triple of positive integers,* (a, b, c)*, such that* $a^2 + b^2 = c^2$*. A PT is called primitive provided* $\gcd(a, b, c) = 1$.

Two known formulations of PTs that we will need are $(m^2 - n^2, 2mn, m^2 + n^2)$ for positive integers m and n with $m > n$ (see, e.g., [5, p. 248]) and $(r + t, s + t, r + s + t)$, where r, s, and t are positive integers with $t^2 = 2rs$ ([6, p. 169]).

Definition 1.2. *A generalized Pythagorean triple (gPT) is an ordered triple of real numbers,* (x, y, z) *such that* $x^2 + y^2 = z^2$.

Definition 1.3. *A Pythagorean triple preserving matrix (PTPM) is a* 3×3 *matrix that transforms any given PT into another.*

Palmer and colleagues [10, 11] explored PTPMs at length and provided the general form of a PTPM. As an example, consider the following:

$$\begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & 4 \\ 4 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 55 \\ 48 \\ 73 \end{bmatrix}.$$

The given matrix is a PTPM, and not only transforms the PT in the example to another PT, but does so for any PT given as a column vector. This may be checked using the general form for PTPMs that can be found in [1], [10], and [11].

Definition 1.4. *A generalized Pythagorean triple preserving matrix (gPTPM) is a* 3×3 *matrix that transforms any given gPT into another.*

It is worth noting that a gPTPM may not transform every PT into a PT and therefore may not be a PTPM.

We will first show how to construct families of PTPMs that contain Fibonacci numbers, then how these families can be used to build a gPTPM that transforms a given Fibonacci-generated PT into another. We conclude with another PTPM whose powers contain Fibonacci numbers.

2. Families of PTPMs Containing Fibonacci Numbers

It is already known that certain PTPMs and their powers contain Fibonacci numbers or terms from generalized Fibonacci sequences [2]. There are several known forms of Fibonacci-generated PTs: i.e., PTs with m and n values that can be expressed in terms of F_k [4]. We will use these PTs to construct PTPMs that also contain Fibonacci numbers.

For any PT $(m^2 - n^2, 2mn, m^2 + n^2)$, we can set $m^2 - n^2 = r + t$, $2mn = s + t$, and $m^2 + n^2 = r + s + t$ to obtain $r = (m - n)^2$, $s = 2n^2$, and $t = 2n(m - n)$. The r, s, and t values from the PTs given in [4] specifically will be used to construct our matrices and are shown in Table 1. The values of k for which each form produces a PT are also shown in the table.

r	s	t	k
F_{k-1}^2	$2F_k^2$	$2F_kF_{k-1}$	$k \geq 2$
F_k^2	$2F_{k-1}^2$	$2F_kF_{k-1}$	$k \geq 2$
$(F_k - 1)^2$	2	$2(F_k - 1)$	$k \geq 3$
$\dfrac{(F_{6k} - 2)^2}{4}$	2	$F_{6k} - 2$	$k \geq 1$
1	$\dfrac{(F_{3k+1} - 1)^2}{2}$	$F_{3k+1} - 1$	$k \geq 1$
1	$\dfrac{(F_{3k-1} - 1)^2}{2}$	$F_{3k-1} - 1$	$k \geq 2$

TABLE 1. r, s, and t values for Fibonacci-generated PTs.

Austin and Austin [1] showed how the rst-form of a PT can be used to construct a PTPM and proved the following theorem.

Theorem 2.1. *Let $(r + t, s + t, r + s + t)$ be a PT. Then,* $\begin{bmatrix} r & s & t \\ s & r & t \\ t & t & r + s \end{bmatrix}$ *is a PTPM.*

This theorem can be used to build PTPMs containing Fibonacci numbers. For example, because F_{3k} is even for any positive integer k, we can choose $t = F_{3k}$ with appropriately chosen r and s. For example, suppose $t = 8$ (i.e., F_6). By choosing $r = 16$, $s = 2$ or $r = 1$, $s = 32$, we obtain the PTPMs

$$\begin{bmatrix} 16 & 2 & 8 \\ 2 & 16 & 8 \\ 8 & 8 & 18 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 32 & 8 \\ 32 & 1 & 8 \\ 8 & 8 & 33 \end{bmatrix}.$$

For another example, choose $t = 34$ (i.e., F_9); we can then choose $r = 289$, $s = 2$ or $r = 1$, $s = 578$ to obtain

$$\begin{bmatrix} 289 & 2 & 34 \\ 2 & 289 & 34 \\ 34 & 34 & 291 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 578 & 34 \\ 578 & 1 & 34 \\ 34 & 34 & 579 \end{bmatrix}.$$

We can also choose our r, s, and t values from Table 1. As two examples, we use row 1 with $k = 2$ and $k = 6$ to obtain

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 25 & 128 & 80 \\ 128 & 25 & 80 \\ 80 & 80 & 153 \end{bmatrix}.$$

The following example does not provide an exhaustive list but illustrates how one can more generally apply Theorem 2.1 to the values in Table 1.

Example 2.2. *For any positive integer k, each of the following is a PTPM.*

$$(1)\quad \begin{bmatrix} 1 & \frac{F_{3k}^2}{2} & F_{3k} \\ \frac{F_{3k}^2}{2} & 1 & F_{3k} \\ F_{3k} & F_{3k} & \frac{F_{3k}^2}{2}+1 \end{bmatrix}$$

$$(2)\quad \begin{bmatrix} F_{k-1}^2 & 2F_k^2 & 2F_kF_{k-1} \\ 2F_k^2 & F_{k-1}^2 & 2F_kF_{k-1} \\ 2F_kF_{k-1} & 2F_kF_{k-1} & F_{k-1}^2+2F_k^2 \end{bmatrix}$$

$$(3)\quad \begin{bmatrix} (F_k-1)^2 & 2 & 2(F_k-1) \\ 2 & (F_k-1)^2 & 2(F_k-1) \\ 2(F_k-1) & 2(F_k-1) & (F_k-1)^2+2 \end{bmatrix}$$

$$(4)\quad \begin{bmatrix} \frac{(F_{6k}-2)^2}{4} & 2 & F_{6k}-2 \\ 2 & \frac{(F_{6k}-2)^2}{4} & F_{6k}-2 \\ F_{6k}-2 & F_{6k}-2 & \frac{(F_{6k}-2)^2}{4}+2 \end{bmatrix}$$

$$(5)\quad \begin{bmatrix} 1 & \frac{(F_{3k+1}-1)^2}{2} & F_{3k+1}-1 \\ \frac{(F_{3k+1}-1)^2}{2} & 1 & F_{3k+1}-1 \\ F_{3k+1}-1 & F_{3k+1}-1 & \frac{(F_{3k+1}-1)^2}{2}+1 \end{bmatrix}$$

The reader may notice that Example 2.2 does not strictly follow the restrictions on k in Table 1, but it is easy to verify that it still holds for the few additional cases. One may also note that a single PTPM might appear in several of the listed families. For example, the PTPM

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \tag{2.1}$$

appears in multiple families, including (1) and (5) when $k = 1$.

Although these families of PTPMs may be interesting in their own right, we can also use these matrices to construct a gPTPM to transform a given PT generated by Fibonacci numbers into another. Palmer and colleagues presented an algorithm for constructing a gPTPM that transforms a given PT into another [11], but we will use Theorem 2.1 to accomplish this same task differently. For a given Pythagorean triple, (a, b, c), where b is even, let $M_{(a,b,c)}$ denote the corresponding matrix from Theorem 2.1 (where $r = c - b$, $s = c - a$, and $t = a + b - c$). The proof of the following theorem is straightforward so we have omitted it here.

Theorem 2.3. *Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be PTs, where y_1 and y_2 are even. Then,*

(1) $M_{(x_2,y_2,z_2)}M^{-1}_{(x_1,y_1,z_1)}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$.

(2) $M_{(x_2,y_2,z_2)}M^{-1}_{(x_1,y_1,z_1)}$ *is a gPTPM.*

Theorem 2.3 allows us to make use of the families of PTPMs we constructed earlier. As an example, consider the PTs $(21, 20, 29)$, generated by row 2 of Table 1 when $k = 2$, and $(63, 16, 65)$, generated by row 3 of Table 1 when $k = 6$. Then,

$$M_{(63,16,65)}M^{-1}_{(21,20,29)}=\begin{bmatrix}49&2&14\\2&49&14\\14&14&16\end{bmatrix}\begin{bmatrix}9&8&12\\8&9&12\\12&12&17\end{bmatrix}^{-1}=\begin{bmatrix}289&242&-374\\242&289&-374\\-374&-374&531\end{bmatrix}$$

transforms $(21, 20, 29)$ into $(63, 16, 65)$ when these PTs are expressed as column vectors. It should be noted that this matrix is a gPTPM but not a PTPM. For example,

$$\begin{bmatrix}289&242&-374\\242&289&-374\\-374&-374&531\end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix}=\begin{bmatrix}-35\\12\\37\end{bmatrix}.$$

3. Another PTPM That Contains Fibonacci Numbers

Another interesting matrix is

$$B=\begin{bmatrix}6&2&6\\2&3&3\\6&3&7\end{bmatrix}.$$

One may easily verify that B is a PTPM, and therefore, so is each power of B. More Fibonacci numbers, 21, 144, and 987, appear in the first few powers of B;

$$B^2=2\begin{bmatrix}38&18&42\\18&11&21\\42&21&47\end{bmatrix},\ B^3=4\begin{bmatrix}258&128&288\\128&66&144\\288&144&322\end{bmatrix},\ \text{and } B^4=8\begin{bmatrix}1766&882&1974\\882&443&987\\1974&987&2207\end{bmatrix}.$$

These powers of B suggest the following result.

Theorem 3.1. *Let* $B=\begin{bmatrix}6&2&6\\2&3&3\\6&3&7\end{bmatrix}$. *Then, for any positive integer n,*

$$B^n=2^{n-1}\begin{bmatrix}4F_{2n}^2+2&2F_{2n}^2&2F_{4n}\\2F_{2n}^2&F_{2n}^2+2&F_{4n}\\2F_{4n}&F_{4n}&5F_{2n}^2+2\end{bmatrix}.$$

Lemma 3.2. *Let n be a positive integer. Then,* $7F_{2n}^2+3F_{4n}+2=2F_{2n+2}^2$.

Proof. The lemma follows from Catalan's identity and the identity $F_{4n} = F_{2n}(2F_{2n+1} - F_{2n})$ (see [3, p. 20]). □

Lemma 3.3. *Let n be a positive integer. Then,* $15F_{2n}^2+7F_{4n}+6=2F_{4n+4}$.

Proof. The lemma follows from d'Ocagne's identity and the identities $F_{4n+4} = F_{2n+2}(F_{2n+2} + 2F_{2n+1})$ and $F_{4n} = F_{2n}(2F_{2n+1} - F_{2n})$ (see [3, p. 20]). □

Proof of Theorem 3.1. The proof is by induction. The statement is easily verified for $n = 1$. Now, suppose

$$B^n = 2^{n-1}\begin{bmatrix} 4F_{2n}^2+2 & 2F_{2n}^2 & 2F_{4n} \\ 2F_{2n}^2 & F_{2n}^2+2 & F_{4n} \\ 2F_{4n} & F_{4n} & 5F_{2n}^2+2 \end{bmatrix}.$$

Then,

$$B^{n+1} = BB^n = \begin{bmatrix} 6 & 2 & 6 \\ 2 & 3 & 3 \\ 6 & 3 & 7 \end{bmatrix} 2^{n-1}\begin{bmatrix} 4F_{2n}^2+2 & 2F_{2n}^2 & 2F_{4n} \\ 2F_{2n}^2 & F_{2n}^2+2 & F_{4n} \\ 2F_{4n} & F_{4n} & 5F_{2n}^2+2 \end{bmatrix}.$$

The first column of this product simplifies to

$$2^n\begin{bmatrix} 14F_{2n}^2+6F_{4n}+6 \\ 7F_{2n}^2+3F_{4n}+2 \\ 15F_{2n}^2+7F_{4n}+6 \end{bmatrix}.$$

The desired result is obtained by applying Lemmas 3.2 and 3.3. The second and third columns of the product, respectively, simplify to

$$2^{n-1}\begin{bmatrix} 14F_{2n}^2+6F_{4n}+4 \\ 7F_{2n}^2+3F_{4n}+6 \\ 15F_{2n}^2+7F_{4n}+6 \end{bmatrix} \text{ and } 2^{n-1}\begin{bmatrix} 30F_{2n}^2+14F_{4n}+12 \\ 15F_{2n}^2+7F_{4n}+6 \\ 35F_{2n}^2+15F_{4n}+14 \end{bmatrix}.$$

The desired results are again obtained by applying Lemmas 3.2 and 3.3. □

The form of the matrix B may be generalized to

$$\begin{bmatrix} k(k+1) & k & k(k+1) \\ k & k+1 & k+1 \\ k(k+1) & k+1 & k(k+1)+1 \end{bmatrix},$$

where B is the case when $k = 2$. When $k = 1$, if we interchange rows 1 and 2, we obtain the PTPM (2.1) from the previous section; this matrix helps generate Hall's tree of all primitive PTs [7], and entries of the powers of the matrix can be expressed in terms of the Pell sequence [2]. The interested reader may wish to explore whether other generalized Fibonacci sequences appear in such matrices and their powers for larger integer values of k.

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References

[1] H. W. Austin and J. W. Austin, *On a special set of symmetric Pythagorean triple preserving matrices*, Advances and Applications in Mathematical Sciences, **12.2** (2012), 97–104.
[2] J. Austin and L. Schneider, *Generalized Fibonacci numbers in Pythagorean triple preserving matrices*, The Fibonacci Quarterly, **58.4** (2020), 340–350.
[3] S. L. Basin and V. E. Hoggart, *A primer on the Fibonacci sequence part II: A matrix which generates Fibonacci identities*, The Fibonacci Quarterly, **1.2** (1963), 61–68.
[4] M. Bicknell-Johnson, *Pythagorean triples containing Fibonacci numbers: Solutions for $F_n^2 \pm F_k^2 = K^2$*, The Fibonacci Quarterly, **17.1** (1979), 1–12.
[5] D. M. Burton, *Elementary Number Theory*, Seventh Edition, McGraw Hill, New York, 2010.
[6] L. Dickson, *History of the Theory of Numbers: Volume II*, Carnegie Institute of Washington, 1920.
[7] A. Hall, *Genealogy of Pythagorean triads*, The Mathematical Gazette, **54.390** (1970), 377–379.
[8] A. F. Horadam, *Fibonacci Number Triples*, The American Mathematical Monthly, **68.8** (1961), 751–753.
[9] J. V. Leyendekkers and A. G. Shannon, *Primitive Pythagorean triples and generalized Fibonacci sequences*, Notes on Number Theory and Discrete Mathematics, **23.1** (2017), 54–62.
[10] L. Palmer, M. Ahuja, and M. Tikoo, *Constructing Pythagorean triple preserving matrices*, Missouri Journal of Mathematical Sciences, **10.3** (1998), 159–168.
[11] L. Palmer, M. Ahuja, and M. Tikoo, *Finding Pythagorean triple preserving matrices*, Missouri Journal of Mathematical Sciences, **10.2** (1998), 99–105.

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