INFINITE SUMS INVOLVING JACOBSTHAL POLYNOMIALS: GENERALIZATIONS

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Abstract. We explore the Jacobsthal versions of four infinite sums involving gibonacci polynomials.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 4].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2+4}$, and $D = \sqrt{4x+1}$.

2. Gibonacci Polynomial Sums

We studied the following sums involving gibonacci polynomials in Theorems 1–4 of [5]:

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+1)k}}{\left[l^2_{(2n+1)k} + \Delta^2 f^2_k\right]^2} = \frac{1}{l^2_{2k}}; \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+2)k}}{\left[l^2_{(2n+2)k} + \Delta^2 f^2_{2k}\right]^2} = \frac{1}{l^2_{2k}} + \frac{1}{l^2_{4k}}; \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+1)k}}{\left[l^2_{(2n+1)k} - \Delta^2 f^2_{2k}\right]^2} = \frac{1}{l^2_{k}} + \frac{1}{l^2_{3k}}; \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+2)k}}{\left[l^2_{(2n+2)k} + (-1)^k \Delta^2 f^2_k\right]^2} = \frac{1}{l^2_{3k}}, \tag{4}$$

where k is a positive integer.

Our objective is to explore the Jacobsthal versions of these four sums.

3. Jacobsthal Polynomial Sums

To accomplish our goal, we will employ the gibonacci-Jacobsthal relationships in Section 1. To this end, in the interest of brevity and clarity, we let A denote the left side of the given gibonacci equation and B its right side, and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobthal equation.

With this brief background, we begin our exploration with sum (1).

3.1. **Jacobsthal Version of Equation** (1).

Proof. We have $\dfrac{\Delta^2 f_{2k} f_{2(2n+1)k}}{\left[l^2_{(2n+1)k} + \Delta^2 f_k^2\right]^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{2(2n+1)k}$, we get

$$\begin{aligned}
A &= \frac{D^2 f_{2k} f_{2(2n+1)k}}{x\left[l^2_{(2n+1)k} + \frac{D^2}{x} f_k^2\right]^2}\\
&= \frac{D^2\left[x^{(2k-1)/2} f_{2k}\right]\left\{x^{[2(2n+1)k-1]/2} f_{2(2n+1)k}\right\}}{\left\{\left[x^{(2n+1)k/2} l_{(2n+1)k}\right]^2 + D^2 x^{2nk}\left[x^{(k-1)/2} f_k\right]^2\right\}^2}\\
&= \frac{D^2 J_{2k} J_{2(2n+1)k}}{\left[j^2_{(2n+1)k} + D^2 x^{2nk} J_k^2\right]^2};\\
\text{LHS} &= \sum_{n=1}^{\infty} \frac{D^2 J_{2k} J_{2(2n+1)k}}{\left[j^2_{(2n+1)k} + D^2 x^{2nk} J_k^2\right]^2},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we turn to $B = \dfrac{1}{l^2_{2k}}$. Replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^{2k}. This yields

$$\begin{aligned}
B &= \frac{x^{2k}}{(x^{2k/2} l_{2k})^2};\\
\text{RHS} &= \frac{x^{2k}}{j^2_{2k}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{D^2 J_{2k} J_{2(2n+1)k}}{\left[j^2_{(2n+1)k} + D^2 x^{2nk} J_k^2\right]^2} = \frac{x^{2k}}{j^2_{2k}}, \tag{5}$$

where $c_n = c_n(x)$. □

This implies

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{L_{2n+1}^2 + 5} = \frac{1}{45}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{J_{2(2n+1)}}{\left(j_{2n+1}^2 + 9 \cdot 4^n\right)^2} = \frac{4}{225}.$$

Next, we pursue the Jacobsthal consequence of sum (2).

3.2. **Jacobsthal Version of Equation** (2).

Proof. We have $\dfrac{\Delta^2 f_{4k} f_{2(2n+2)k}}{\left[l_{(2n+2)k}^2 + \Delta^2 f_{2k}^2\right]^2}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{4(n+1)k}$. This yields

$$\begin{aligned}
A &= \frac{D^2 f_{4k} f_{2(2n+2)k}}{x\left[l_{(2n+2)k}^2 + \dfrac{D^2}{x} f_{2k}^2\right]^2} \\
&= \frac{D^2 x^{2nk}\left[x^{(4k-1)/2} f_{4k}\right]\left\{x^{[2(2n+2)k-1]/2} f_{2(2n+2)k}\right\}}{\left\{\left[x^{(2n+2)k/2} l_{(2n+2)k}\right]^2 + D^2 x^{(2nk-1)/2}\left[x^{(2k-1)/2} f_{2k}\right]^2\right\}^2} \\
&= \frac{D^2 x^{2nk} J_{4k} J_{2(2n+2)k}}{\left[j_{(2n+2)k}^2 + D^2 x^{(2nk-1)/2} J_{2k}^2\right]^2}; \\
\text{LHS} &= \sum_{n=1}^{\infty} \frac{D^2 x^{2nk} J_{4k} J_{2(2n+2)k}}{\left[j_{(2n+2)k}^2 + D^2 x^{(2nk-1)/2} J_{2k}^2\right]^2},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we have $B = \dfrac{1}{l_{2k}^2} + \dfrac{1}{l_{4k}^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with x^{4k} yields

$$\begin{aligned}
B &= \frac{x^{2k}}{(x^{2k/2} l_{2k})^2} + \frac{x^{4k}}{(x^{4k/2} l_{4k})^2}; \\
\text{RHS} &= \frac{x^{2k}}{j_{2k}^2} + \frac{x^{4k}}{j_{4k}^2},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

By combining the two sides, we get the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{D^2 x^{2nk} J_{4k} J_{2(2n+2)k}}{\left[j_{(2n+2)k}^2 + D^2 x^{(2nk-1)/2} J_{2k}^2\right]^2} = \frac{x^{2k}}{j_{2k}^2} + \frac{x^{4k}}{j_{4k}^2}, \tag{6}$$

where $c_n = c_n(x)$. $\square$

This yields

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 + 5\right)^2} = \frac{58}{6,615}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{2^{2n} J_{2(2n+2)}}{\left[j_{2n+2}^2 + 9 \cdot 2^{(2n-1)/2}\right]^2} = \frac{1,556}{325,125}.$$

A Gibonacci Delight. It follows from Subsections 3.1 and 3.2 that [5]

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{\left(L_n^2+5\right)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2+5\right)^2} + \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2+5\right)^2} \\ &= \frac{41}{1,323}; \\ \sum_{n=1}^{\infty} \frac{F_{2n}}{\left(L_n^2+5\right)^2} &= \frac{2}{27}.\end{aligned}$$

Next, we explore the Jacobsthal implication of sum (3).

3.3. **Jacobsthal Version of Equation (3).**

Proof. We have $A = \dfrac{\Delta^2 f_{4k} f_{2(2n+1)k}}{\left[l_{(2n+1)k}^2 - \Delta^2 f_{2k}^2\right]^2}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator with $x^{2(2n+1)k}$ then yields

$$\begin{aligned}A &= \frac{D^2 f_{4k} f_{2(2n+1)k}}{x\left[l_{(2n+1)k}^2 - \dfrac{D^2}{x} f_{2k}^2\right]^2} \\ &= \frac{D^2 x^{(2n-1)k}\left[x^{(4k-1)/2} f_{4k}\right]\left\{x^{[2(2n+1)k-1]/2} f_{2(2n+1)k}\right\}}{\left\{\left[x^{(2n+1)k/2} l_{(2n+1)k}\right]^2 - D^2 x^{(2n-1)k}\left[x^{(2k-1)/2} f_{2k}\right]^2\right\}^2} \\ &= \frac{D^2 x^{(2n-1)k} J_{4k} J_{2(2n+1)k}}{\left[j_{(2n+1)k}^2 - D^2 x^{(2n-1)k} J_{2k}^2\right]^2}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{D^2 x^{(2n-1)k} J_{4k} J_{2(2n+1)k}}{\left[j_{(2n+1)k}^2 - D^2 x^{(2n-1)k} J_{2k}^2\right]^2},\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

With $B = \dfrac{1}{l_k^2} + \dfrac{1}{l_{3k}^2}$, we will now find the corresponding RHS. To this end, we replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^{3k}. We then get

$$\begin{aligned}B &= \frac{x^k}{(x^{k/2} l_{2k})^2} + \frac{x^{3k}}{(x^{3k/2} l_{3k})^2}; \\ \text{RHS} &= \frac{x^k}{j_k^2} + \frac{x^{3k}}{j_{3k}^2},\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides, we get the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{D^2 x^{(2n-1)k} J_{4k} J_{2(2n+1)k}}{\left[j_{(2n+1)k}^2 - D^2 x^{(2n-1)k} J_{2k}^2\right]^2} = \frac{x^k}{j_k^2} + \frac{x^{3k}}{j_{3k}^2}, \tag{7}$$

where $c_n = c_n(x)$. □

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 - 5\right)^2} = \frac{17}{240}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{2^{2n-1} J_{2(2n+1)}}{\left[j_{2n+1}^2 - 9 \cdot 2^{n-1}\right]^2} = \frac{106}{2,205}.$$

Finally, we explore the Jacobsthal implication of sum (4).

3.4. **Jacobsthal Version of Equation** (4)**.**

Proof. We have $A = \dfrac{\Delta^2 f_{2k} f_{2(2n+2)k}}{\left[l_{(2n+2)k}^2 + (-1)^k \Delta^2 f_k^2\right]^2}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with $x^{2(2n+2)k}$. Then,

$$\begin{aligned} A &= \frac{D^2 f_{2k} f_{2(2n+2)k}}{x\left[l_{(2n+2)k}^2 + (-1)^k \frac{D^2}{x} f_k^2\right]^2} \\ &= \frac{D^2 x^{(2n+1)k}\left[x^{(2k-1)/2} f_{2k}\right]\left\{x^{[2(2n+2)k-1]/2} f_{2(2n+2)k}\right\}}{\left\{\left[x^{(2n+2)k/2} l_{(2n+2)k}\right]^2 + (-1)^k D^2 x^{(2n+1)k}\left[x^{(k-1)/2} f_k\right]^2\right\}^2} \\ &= \frac{D^2 x^{(2n+1)k} J_{2k} J_{2(2n+2)k}}{\left[j_{(2n+2)k}^2 + (-1)^k D^2 x^{(2n+1)k} J_k^2\right]^2}; \\ \text{LHS} &= \sum_{n=1}^{\infty} \frac{D^2 x^{(2n+1)k} J_{2k} J_{2(2n+2)k}}{\left[j_{(2n+2)k}^2 + (-1)^k D^2 x^{(2n+1)k} J_k^2\right]^2}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator with x^{3k} in $B = \dfrac{1}{l_{3k}^2}$. We then get

$$\begin{aligned} B &= \frac{x^{3k}}{(x^{3k/2} l_{3k})^2}; \\ \text{RHS} &= \frac{x^{3k}}{j_{3k}^2}, \end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides, we get the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{D^2 x^{(2n+1)k} J_{2k} J_{2(2n+2)k}}{\left[j_{(2n+2)k}^2 + (-1)^k D^2 x^{(2n+1)k} J_k^2\right]^2} = \frac{x^{3k}}{j_{3k}^2}, \tag{8}$$

where $c_n = c_n(x)$. □

In particular, we then get

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 - 5\right)^2} = \frac{1}{80}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{2^{2n+1} J_{2(2n+2)}}{\left(j_{2n+2}^2 - 9 \cdot 2^{2n+1}\right)^2} = \frac{8}{441}.$$

A Gibonacci Delight: Subsection 3.3, coupled with Subsection 3.4, yields

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{(L_n^2-5)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2-5\right)^2} + \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2-5\right)^2}\\ &= \frac{1}{12};\\ \sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2-5)^2} &= \frac{1}{3}. \text{ [5]}\end{aligned}$$

4. Alternate Versions

Using the Jacobsthal counterpart $j_n^2 - D^2 J_n^2 = 4(-x)^n$ [4] of the gibonacci identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [4], we can rewrite equations (5)–(8) in different ways:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{D^2 J_{2k} J_{2(2n+1)k}}{\left\{D^2\left[J_{(2n+1)k}^2 + x^{2nk} J_k^2\right] + 4(-x)^{(2n+1)k}\right\}^2} &= \frac{x^{2k}}{D^2 J_{2k} + 4x^{2k}};\\ \sum_{n=1}^{\infty} \frac{D^2 J_{4k} J_{2(2n+2)k}}{\left\{D^2\left[J_{(2n+2)k}^2 + x^{(2nk-1)/2} J_{2k}^2\right] + 4x^{(2n+2)k}\right\}^2} &= \frac{x^{2k}}{D^2 J_{2k}^2 + 4x^{2k}} + \frac{x^{4k}}{D^2 J_{4k}^2 + 4x^{4k}};\\ \sum_{n=1}^{\infty} \frac{D^2 x^{(2n-1)k} J_{4k} J_{2(2n+1)k}}{\left\{D^2\left[J_{(2n+1)k}^2 - x^{(2n-1)k} J_{2k}^2\right] + 4(-x)^k\right\}^2} &= \frac{x^k}{D^2 J_k^2 + 4(-x)^k} + \frac{x^{3k}}{D^2 J_{3k}^2 + 4(-x)^{3k}};\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{D^2 x^{(2n+1)k} J_{2k} J_{2(2n+2)k}}{\left\{D^2\left[J_{(2n+2)k}^2 + (-1)^k x^{(2n+1)k} J_k^2\right] + 4x^{(2n+2)k}\right\}^2} = \frac{x^{3k}}{D^2 J_{3k}^2 + 4(-x)^{3k}},$$

respectively.

They yield

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(5F_{2n+1}^2+1\right)^2} &= \frac{1}{45}; & \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(5F_{2n+2}^2+9\right)^2} &= \frac{58}{6,615};\\ \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(5F_{2n+1}^2-9\right)^2} &= \frac{17}{240}; & \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(5F_{2n+2}^2-1\right)^2} &= \frac{1}{80},\end{aligned}$$

again respectively.

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