SUMS INVOLVING TWO CLASSES OF GIBONACCI POLYNOMIALS REVISITED

THOMAS KOSHY

Abstract. We explore six infinite sums involving gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $2\alpha(x) = x + \Delta$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 3]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} -(-1)^{n+k} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise}; \end{cases} \tag{1.1}$$

$$g_{n+k+1}g_{n-k} - g_{n+k}g_{n-k+1} = \begin{cases} -(-1)^{n+k} f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_{2k}, & \text{otherwise}; \end{cases} \tag{1.2}$$

$$g_{n+k+1}g_{n-k} + g_{n+k}g_{n-k+1} = \begin{cases} \frac{1}{\Delta^2}[2l_{2n+1} - (-1)^{n+k} x l_{2k}], & \text{if } g_n = f_n; \\ 2l_{2n+1} + (-1)^{n+k} x l_{2k}, & \text{otherwise}. \end{cases} \tag{1.3}$$

These properties can be confirmed using the Binet-like formulas.

Identity (1.3) has an interesting dividend: $2l_{2n+1} \equiv (-1)^{n+k} x l_{2k} \pmod{\Delta^2}$. In particular, we have $2L_{2n+1} \equiv (-1)^{n+k} L_{2k} \pmod 5$. Replacing n with $2nk$ and k with $2k$, we get $2L_{4nk+1} \equiv L_{4k} \pmod 5$.

Using the identity $l_{n+1} + l_{n-1} = \Delta^2 f_n$ ([2, p. 57]), this yields

$$\begin{aligned} 2L_{4nk+1} + L_{4k+2} &\equiv L_{4k+2} + L_{4k} \pmod 5 \\ &\equiv 5F_{4k+1} \pmod 5 \\ &\equiv 0 \pmod 5. \end{aligned}$$

It follows by identities (1.2) and (1.3) that

$$g_{n+k+1}^2 g_{n-k}^2 - g_{n+k}^2 g_{n-k+1}^2 = \begin{cases} -\dfrac{(-1)^{n+k}}{\Delta^2}[2l_{2n+1} - (-1)^{n+k} x l_{2k}] f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 [2l_{2n+1} + (-1)^{n+k} x l_{2k}] f_{2k}, & \text{otherwise.} \end{cases} \tag{1.4}$$

2. Telescoping Gibonacci Sums

We established the following telescoping gibonacci sums in [3], where k and λ are positive integers; for convenience, we call them lemmas.

Lemma 2.1.

$$\sum_{n=1}^{\infty} \left[\frac{g_{(2n-1)k+1}^{\lambda}}{g_{(2n-1)k}^{\lambda}} - \frac{g_{(2n+1)k+1}^{\lambda}}{g_{(2n+1)k}^{\lambda}} \right] = \frac{g_{k+1}^{\lambda}}{g_k^{\lambda}} - \alpha^{\lambda}.$$

Lemma 2.2.

$$\sum_{n=1}^{\infty} \left[\frac{g_{2nk+1}^{\lambda}}{g_{2nk}^{\lambda}} - \frac{g_{(2n+2)k+1}^{\lambda}}{g_{(2n+2)k}^{\lambda}} \right] = \frac{g_{2k+1}^{\lambda}}{g_{2k}^{\lambda}} - \alpha^{\lambda}.$$

Lemma 2.3.

$$\sum_{n=1}^{\infty} \left[\frac{g_{(4n-3)k+1}^{\lambda}}{g_{(4n-3)k}^{\lambda}} - \frac{g_{(4n+1)k+1}^{\lambda}}{g_{(4n+1)k}^{\lambda}} \right] = \frac{g_{k+1}^{\lambda}}{g_k^{\lambda}} - \alpha^{\lambda}.$$

Lemma 2.4.

$$\sum_{n=1}^{\infty} \left[\frac{g_{(4n-2)k+1}^{\lambda}}{g_{(4n-2)k}^{\lambda}} - \frac{g_{(4n+2)k+1}^{\lambda}}{g_{(4n+2)k}^{\lambda}} \right] = \frac{g_{2k+1}^{\lambda}}{g_{2k}^{\lambda}} - \alpha^{\lambda}.$$

Lemma 2.5.

$$\sum_{n=1}^{\infty} \left[\frac{g_{(4n-1)k+1}^{\lambda}}{g_{(4n-1)k}^{\lambda}} - \frac{g_{(4n+3)k+1}^{\lambda}}{g_{(4n+3)k}^{\lambda}} \right] = \frac{g_{3k+1}^{\lambda}}{g_{3k}^{\lambda}} - \alpha^{\lambda}.$$

Lemma 2.6.

$$\sum_{n=1}^{\infty} \left[\frac{g_{4nk+1}^{\lambda}}{g_{4nk}^{\lambda}} - \frac{g_{(4n+4)k+1}^{\lambda}}{g_{(4n+4)k}^{\lambda}} \right] = \frac{g_{4k+1}^{\lambda}}{g_{4k}^{\lambda}} - \alpha^{\lambda}.$$

These lemmas, coupled with identities (1.1) and (1.4), play a pivotal role in our explorations.

3. Gibonacci Sums

Using the above lemmas with $\lambda = 2$, and identities (1.1) and (1.4) at our disposal, we are now ready for further explorations. In the interest of brevity, we let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \nu = \begin{cases} -1, & \text{if } g_n = f_n; \\ 1, & \text{otherwise;} \end{cases}$$

$$\mu^* = \begin{cases} \dfrac{1}{\Delta^2}, & \text{if } g_n = f_n; \\ 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu^* = -\nu.$$

The first result is an application of Lemma 2.1.

Theorem 3.1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\mu^*\nu^*[2l_{2nk+1} + (-1)^k \nu x l_{2k}] f_{2k}}{[g_{2nk}^2 + (-1)^k \mu\nu f_k^2]^2} = \frac{g_{k+1}^2}{g_k^2} - \alpha^2. \tag{3.1}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.4), Lemma 2.1 yields

$$\begin{aligned}
\frac{(-1)^k[2l_{2nk+1} - (-1)^k x l_{2k}]f_{2k}}{\Delta^2[f_{2nk}^2 - (-1)^k f_k^2]^2} &= \frac{f_{(2n+1)k}^2 f_{(2n-1)k+1}^2 - f_{(2n+1)k+1}^2 f_{(2n-1)k}^2}{f_{(2n+1)k}^2 f_{(2n-1)k}^2}; \\
\sum_{n=1}^{\infty} \frac{(-1)^k[2l_{2nk+1} - (-1)^k x l_{2k}]f_{2k}}{\Delta^2[f_{2nk}^2 - (-1)^k f_k^2]^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(2n-1)k+1}^2}{f_{(2n-1)k}^2} - \frac{f_{(2n+1)k+1}^2}{f_{(2n+1)k}^2}\right] \\
&= \frac{f_{k+1}^2}{f_k^2} - \alpha^2. \qquad (3.2)
\end{aligned}$$

On the other hand, let $g_n = l_n$. With identities (1.1) and (1.4), Lemma 2.1 yields

$$\begin{aligned}
\frac{(-1)^{k+1}\Delta^2[2l_{2nk+1} + (-1)^k x l_{2k}]f_{2k}}{[l_{2nk}^2 + (-1)^k \Delta^2 f_k^2]^2} &= \frac{l_{(2n+1)k}^2 l_{(2n-1)k+1}^2 - l_{(2n+1)k+1}^2 l_{(2n-1)k}^2}{l_{(2n+1)k}^2 l_{(2n-1)k}^2}; \\
\sum_{n=1}^{\infty} \frac{(-1)^{k+1}\Delta^2[2l_{2nk+1} + (-1)^k x l_{2k}]f_{2k}}{[l_{2nk}^2 + (-1)^k \Delta^2 f_k^2]^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(2n-1)k+1}^2}{l_{(2n-1)k}^2} - \frac{l_{(2n+1)k+1}^2}{l_{(2n+1)k}^2}\right] \\
&= \frac{l_{k+1}^2}{l_k^2} - \alpha^2. \qquad (3.3)
\end{aligned}$$

Combining equations (3.2) and (3.3), we get the desired result. □

In particular, we have

$$\sum_{n=1}^{\infty} \frac{2L_{2n+1} + 3}{(F_{2n}^2 + 1)^2} = \frac{5}{2} + \frac{5\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{2L_{2n+1} - 3}{(L_{2n}^2 - 5)^2} = \frac{3}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2L_{4n+1} - 7}{(F_{4n}^2 - 1)^2} = \frac{25}{6} - \frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{4n+1} + 7}{(L_{4n}^2 + 5)^2} = -\frac{1}{54} + \frac{\sqrt{5}}{30}.$$

The next result invokes Lemma 2.2.

Theorem 3.2. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\mu^*\nu^*[2l_{2(2n+1)k+1} + \nu x l_{2k}]f_{2k}}{[g_{(2n+1)k}^2 + \mu\nu f_k^2]^2} = \frac{g_{2k+1}^2}{g_{2k}^2} - \alpha^2. \qquad (3.4)$$

Proof. Let $g_n = f_n$. With identities (1.1) and (1.4), it then follows by Lemma 2.2 that

$$\begin{aligned}
\frac{[2l_{2(2n+1)k+1} - x l_{2k}]f_{2k}}{\Delta^2[f_{(2n+1)k}^2 - f_k^2]^2} &= \frac{f_{(2n+2)k}^2 f_{2nk+1}^2 - f_{(2n+2)k+1}^2 f_{2nk}^2}{f_{(2n+2)k}^2 f_{2nk}^2}; \\
\sum_{n=1}^{\infty} \frac{[2l_{2(2n+1)k+1} - x l_{2k}]f_{2k}}{\Delta^2[f_{(2n+1)k}^2 - f_k^2]^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{2nk+1}^2}{f_{2nk}^2} - \frac{f_{(2n+2)k+1}^2}{f_{(2n+2)k}^2}\right] \\
&= \frac{f_{2k+1}^2}{f_{2k}^2} - \alpha^2. \qquad (3.5)
\end{aligned}$$

On the other hand, let $g_n = l_n$. Using identities (1.1) and (1.4), Lemma 2.2 yields

$$\begin{aligned}
\frac{-\Delta^2[2l_{2(2n+1)k+1} + xl_{2k}]f_{2k}}{[l^2_{(2n+1)k} + \Delta^2 f_k^2]^2} &= \frac{l^2_{(2n+2)k}l^2_{2nk+1} - l^2_{(2n+2)k+1}l^2_{2nk}}{l^2_{(2n+2)k}l^2_{2nk}};\\
\sum_{n=1}^{\infty} \frac{-\Delta^2[2l_{2(2n+1)k+1} + xl_{2k}]f_{2k}}{[l^2_{(2n+1)k} + \Delta^2 f_k^2]^2} &= \sum_{n=1}^{\infty}\left[\frac{l^2_{2nk+1}}{l^2_{2nk}} - \frac{l^2_{(2n+2)k+1}}{l^2_{(2n+2)k}}\right]\\
&= \frac{l^2_{2k+1}}{l^2_{2k}} - \alpha^2. \qquad (3.6)
\end{aligned}$$

Combining the two cases, we get the desired result. □

It follows by this theorem that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2L_{4n+3} - 3}{(F^2_{2n+1} - 1)^2} &= \frac{25}{2} - \frac{5\sqrt{5}}{2}; & \sum_{n=1}^{\infty} \frac{2L_{4n+3} + 3}{(L^2_{2n+1} + 5)^2} &= -\frac{1}{18} + \frac{\sqrt{5}}{10};\\
\sum_{n=1}^{\infty} \frac{2L_{8n+5} - 7}{(F^2_{4n+2} - 1)^2} &= \frac{115}{54} - \frac{5\sqrt{5}}{6}; & \sum_{n=1}^{\infty} \frac{2L_{8n+5} + 7}{(L^2_{4n+2} + 5)^2} &= -\frac{19}{294} + \frac{\sqrt{5}}{30}.
\end{aligned}$$

Next, we investigate sums involving squares of gibonacci polynomials of another special class. The following result invokes Lemma 2.3.

The following theorem is an application of Lemma 2.3.

Theorem 3.3. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k\mu\mu^*\nu^*[2l_{(8n-2)k+1} + (-1)^k\nu x l_{4k}]f_{4k}}{[g^2_{(4n-1)k} + (-1)^k\mu\nu f^2_{2k}]^2} = \frac{g^2_{k+1}}{g^2_k} - \alpha^2. \qquad (3.7)$$

Proof. Suppose $g_n = f_n$. Lemma 2.4, coupled with identities (1.1) and (1.4), yields

$$\begin{aligned}
\frac{(-1)^k[2l_{(8n-2)k+1} - (-1)^k x l_{4k}]f_{4k}}{\Delta^2[f^2_{(4n-1)k} - (-1)^k f^2_{2k}]^2} &= \frac{f^2_{(4n+1)k}f^2_{(4n-3)k+1} - f^2_{(4n+1)k+1}f^2_{(4n-3)k}}{f^2_{(4n+1)k}f^2_{(4n-3)k}};\\
\sum_{n=1}^{\infty} \frac{(-1)^k[2l_{(8n-2)k+1} - (-1)^k x l_{4k}]f_{4k}}{\Delta^2[f^2_{(4n-1)k} - (-1)^k f^2_{2k}]^2} &= \sum_{n=1}^{\infty}\left[\frac{f^2_{(4n-3)k+1}}{f^2_{(4n-3)k}} - \frac{f^2_{(4n+1)k+1}}{f^2_{(4n+1)k}}\right]\\
&= \frac{f^2_{k+1}}{f^2_k} - \alpha^2. \qquad (3.8)
\end{aligned}$$

On the other hand, let $g_n = l_n$. Using identities (1.1) and (1.4), and Lemma 2.4, we get

$$\begin{aligned}
\frac{-(-1)^k\Delta^2[2l_{(8n-2)k+1} + (-1)^k x l_{4k}]f_{4k}}{[l^2_{(4n-1)k} + (-1)^k\Delta^2 f^2_{2k}]^2} &= \frac{l^2_{(4n+1)k}l^2_{(4n-3)k+1} - l^2_{(4n+1)k+1}l^2_{(4n-3)k}}{l^2_{(4n+1)k}l^2_{(4n-3)k}};\\
\sum_{n=1}^{\infty} \frac{-(-1)^k\Delta^2[2l_{(8n-2)k+1} + (-1)^k x l_{4k}]f_{4k}}{[l^2_{(4n-1)k} + (-1)^k\Delta^2 f^2_{2k}]^2} &= \sum_{n=1}^{\infty}\left[\frac{l^2_{(4n-3)k+1}}{l^2_{(4n-3)k}} - \frac{l^2_{(4n+1)k+1}}{l^2_{(4n+1)k}}\right]\\
&= \frac{l^2_{k+1}}{l^2_k} - \alpha^2. \qquad (3.9)
\end{aligned}$$

This, coupled with equation (3.8), yields the given result, as desired. □

This theorem yields

$$\sum_{n=1}^{\infty} \frac{2L_{8n-1}+7}{(F_{4n-1}^2+1)^2} = \frac{5}{6}+\frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n-1}-7}{(L_{4n-1}^2-5)^2} = \frac{1}{2}-\frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{2L_{16n-3}-47}{(F_{8n-2}^2-9)^2} = \frac{25}{42}-\frac{5\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{2L_{16n-3}+47}{(L_{8n-2}^2+45)^2} = -\frac{1}{378}+\frac{\sqrt{5}}{210}.$$

The next theorem employs Lemma 2.4.

Theorem 3.4. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\mu^*\nu^*[2l_{2nk+1}+\nu x l_{4k}]f_{4k}}{(g_{4nk}^2+\mu\nu f_{2k}^2)^2} = \frac{g_{2k+1}^2}{g_{2k}^2}-\alpha^2. \tag{3.10}$$

Proof. Let $g_n = f_n$. It then follows by identities (1.1) and (1.4), and Lemma 2.6 that

$$\begin{aligned}
\frac{(2l_{8nk+1}-xl_{4k})f_{4k}}{\Delta^2(f_{4nk}^2-f_{2k}^2)^2} &= \frac{f_{(4n+2)k}^2 f_{(4n-2)k+1}^2 - f_{(4n+2)k+1}^2 f_{(4n-2)k}^2}{f_{(4n+2)k}^2 f_{(4n-2)k}^2}; \\
\sum_{n=1}^{\infty} \frac{(2l_{8nk+1}-xl_{4k})f_{4k}}{\Delta^2(f_{4nk}^2-f_{2k}^2)^2} &= \sum_{n=1}^{\infty}\left[\frac{f_{(4n-2)k+1}^2}{f_{(4n-2)k}^2}-\frac{f_{(4n+2)k+1}^2}{f_{(4n+2)k}^2}\right] \\
&= \frac{f_{2k+1}^2}{f_{2k}^2}-\alpha^2.
\end{aligned} \tag{3.11}$$

On the other hand, let $g_n = l_n$. Lemma 2.6, coupled with identities (1.1) and (1.4), yields

$$\begin{aligned}
\frac{-\Delta^2[(2l_{8nk+1}+xl_{4k})]f_{4k}}{(l_{4nk}^2+\Delta^2 f_{2k}^2)^2} &= \frac{l_{(4n+2)k}^2 l_{(4n-2)k+1}^2 - l_{(4n+2)k+1}^2 l_{(4n-2)k}^2}{l_{(4n+2)k}^2 l_{(4n-2)k}^2}; \\
\sum_{n=1}^{\infty} \frac{-\Delta^2[(2l_{8nk+1}+xl_{4k})]f_{4k}}{(l_{4nk}^2+\Delta^2 f_{2k}^2)^2} &= \sum_{n=1}^{\infty}\left[\frac{l_{(4n-2)k+1}^2}{l_{(4n-2)k}^2}-\frac{l_{(4n+2)k+1}^2}{l_{(4n+2)k}^2}\right] \\
&= \frac{l_{2k+1}^2}{l_{2k}^2}-\alpha^2.
\end{aligned} \tag{3.12}$$

This, combined with equation (3.11), yields the desired result. □

It follows by formula (3.10) that

$$\sum_{n=1}^{\infty} \frac{2L_{8n+1}-7}{(F_{4n}^2-1)^2} = \frac{25}{6}-\frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n+1}+7}{(L_{4n}^2+5)^2} = -\frac{1}{54}+\frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{2L_{16n+1}-47}{(F_{8n}^2-9)^2} = \frac{115}{378}-\frac{5\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{2L_{16n+1}+47}{(L_{8n}^2+45)^2} = -\frac{19}{2,058}+\frac{\sqrt{5}}{210}.$$

Theorem 3.5. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k\mu\mu^*\nu^*[2l_{(8n+2)k+1}+(-1)^k\nu x l_{4k}]f_{4k}}{[g_{(4n+1)k}^2+(-1)^k\mu\nu f_{2k}^2]^2} = \frac{g_{3k+1}^2}{g_{3k}^4}-\alpha^2. \tag{3.13}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.4), and Lemma 2.5, we get

$$\begin{aligned}
\frac{(-1)^k[2l_{(8n+2)k+1} - (-1)^k x l_{4k}]f_{4k}}{\Delta^2[f^2_{(4n+1)k} - (-1)^k f^2_{2k}]^2} &= \frac{f^2_{(4n+3)k}f^2_{(4n-1)k+1} - f^2_{(4n+3)k+1}f^2_{(4n-1)k}}{f^2_{(4n+3)k}f^2_{(4n-1)k}};\\
\sum_{n=1}^{\infty}\frac{(-1)^k[2l_{(8n+2)k+1} - (-1)^k x l_{4k}]f_{4k}}{\Delta^2[f^2_{(4n+1)k} - (-1)^k f^2_{2k}]^2} &= \sum_{n=1}^{\infty}\left[\frac{f^2_{(4n-1)k+1}}{f^2_{(4n-1)k}} - \frac{f^2_{(4n+3)k+1}}{f^2_{(4n+3)k}}\right]\\
&= \frac{f^2_{3k+1}}{f^2_{3k}} - \alpha^2. \qquad (3.14)
\end{aligned}$$

On the other hand, let $g_n = l_n$. With the same identities and Lemma 2.5, we get

$$\begin{aligned}
\frac{(-1)^{k+1}\Delta^2[2l_{(8n+2)k+1} + (-1)^k x l_{4k}]f_{4k}}{l^2_{(4n+1)k} + (-1)^k\Delta^2 f^2_{2k}} &= \frac{l^2_{(4n+3)k}l^2_{(4n-1)k+1} - l^2_{(4n+3)k+1}l^2_{(4n-1)k}}{l^2_{(4n+3)k}l^2_{(4n-1)k}};\\
\sum_{n=1}^{\infty}\frac{(-1)^{k+1}\Delta^2[2l_{(8n+2)k+1} + (-1)^k x l_{4k}]f_{4k}}{l^2_{(4n+1)k} + (-1)^k\Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{l^2_{(4n-1)k+1}}{l^2_{(4n-1)k}} - \frac{l^2_{(4n+3)k+1}}{l^2_{(4n+3)k}}\right]\\
&= \frac{l^2_{3k+1}}{l^2_{3k}} - \alpha^2. \qquad (3.15)
\end{aligned}$$

By combining equations (3.14) and (3.15), we get the desired result. □

It follows by this theorem that

$$\sum_{n=1}^{\infty}\frac{2L_{8n+3}+7}{(F^2_{4n+1}+1)^2} = -\frac{5}{4} + \frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty}\frac{2L_{8n+3}-7}{(L^2_{4n+1}-5)^2} = \frac{5}{48} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty}\frac{2L_{16n+5}-47}{(F^2_{8n+2}-9)^2} = \frac{365}{1,344} - \frac{5\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty}\frac{2L_{16n+5}+47}{(L^2_{8n+2}+45)^2} = -\frac{71}{6,804} + \frac{\sqrt{5}}{210}.$$

Finally, the next theorem exemplifies an application of Lemma 2.6.

Theorem 3.6. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{\mu\mu^*\nu^*[2l_{(8n+4)k+1} + \nu x l_{4k}]f_{4k}}{[g^2_{(4n+2)k} + \mu\nu f^2_{2k}]^2} = \frac{g^2_{4k+1}}{g^4_{4k}} - \alpha^2. \qquad (3.16)$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.4), and Lemma 2.5, we get

$$\begin{aligned}
\frac{[2l_{(8n+4)k+1} - x l_{4k}]f_{4k}}{\Delta^2[f^2_{(4n+2)k} - f^2_{2k}]^2} &= \frac{f^2_{(4n+4)k}f^2_{4nk+1} - f^2_{(4n+4)k+1}f^2_{4nk}}{f^2_{(4n+4)k}f^2_{4nk}};\\
\sum_{n=1}^{\infty}\frac{[2l_{(8n+4)k+1} - x l_{4k}]f_{4k}}{\Delta^2[f^2_{(4n+2)k} - f^2_{2k}]^2} &= \sum_{n=1}^{\infty}\left[\frac{f^2_{4nk+1}}{f^2_{4nk}} - \frac{f^2_{(4n+4)k+1}}{f^2_{(4n+4)k}}\right]\\
&= \frac{f^2_{4k+1}}{f^2_{4k}} - \alpha^2. \qquad (3.17)
\end{aligned}$$

On the other hand, suppose $g_n = l_n$. With identities (1.1) and (1.4), Lemma 2.5 yields

$$\begin{aligned}
\frac{-\Delta^2[2l_{(8n+4)k+1} + xl_{4k}]f_{4k}}{l^2_{(4n+2)k} + \Delta^2 f^2_{2k}} &= \frac{l^2_{(4n+4)k} l^2_{4nk+1} - l^2_{(4n+4)k+1} l^2_{4nk}}{l^2_{(4n+4)k} l^2_{4nk}}; \\
\sum_{n=1}^{\infty} \frac{-\Delta^2[2l_{(8n+4)k+1} + xl_{4k}]f_{4k}}{l^2_{(4n+2)k} + \Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty} \left[\frac{l^2_{4nk+1}}{l^2_{4nk}} - \frac{l^2_{(4n+4)k+1}}{l^2_{(4n+4)k}} \right] \\
&= \frac{l^2_{4k+1}}{l^2_{4k}} - \alpha^2. \qquad (3.18)
\end{aligned}$$

By combining the two cases, we get the desired result. □

In particular, we get

$$\sum_{n=1}^{\infty} \frac{2L_{8n+5} - 7}{(F^2_{4n+2} - 1)^2} = \frac{115}{54} - \frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n+5} + 7}{(L^2_{4n+2} + 5)^2} = -\frac{19}{294} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{2L_{16n+9} - 47}{(F^2_{8n+4} - 9)^2} = \frac{4,945}{18,522} - \frac{5\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{2L_{16n+9} + 47}{(L^2_{8n+4} + 45)^2} = -\frac{985}{92,778} + \frac{\sqrt{5}}{210}.$$

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Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu