MORE SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES

THOMAS KOSHY

Abstract. We explore eight infinite sums involving a family of gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, $2\alpha = x + \Delta$, and $2\beta = x - \Delta$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 4]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise}; \end{cases} \tag{1.1}$$

$$g_{n+k+2}g_{n-k} - g_{n+k}g_{n-k+2} = \begin{cases} (-1)^{n+k+1} x f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 x f_{2k}, & \text{otherwise}. \end{cases} \tag{1.2}$$

These properties can be confirmed using the Binet-like formulas. Identity (1.2) is a gibonacci polynomial extension of *d'Ocagne identity* [2].

2. Telescoping Gibonacci Sums

With recursion, we will now study four telescoping sums involving a special class of gibonacci polynomials.

Lemma 2.1. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^\lambda_{(4n-3)k+2}}{g^\lambda_{(4n-3)k}} - \frac{g^\lambda_{(4n+1)k+2}}{g^\lambda_{(4n+1)k}} \right] = \frac{g^\lambda_{k+2}}{g^\lambda_k} - \alpha^{2\lambda}. \tag{2.1}$$

Proof. Using recursion [2], we will first establish that

$$\sum_{n=1}^{m} \left[\frac{g^\lambda_{(4n-3)k+2}}{g^\lambda_{(4n-3)k}} - \frac{g^\lambda_{(4n+1)k+2}}{g^\lambda_{(4n+1)k}} \right] = \frac{g^\lambda_{k+2}}{g^\lambda_k} - \frac{g^\lambda_{(4m+1)k+2}}{g^\lambda_{(4m+1)k}}. \tag{2.2}$$

Letting A_m denote the left-hand side (LHS) of this equation and B_m its right-hand side (RHS), we get

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

With recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

confirming the validity of equation (2.2).

Because $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$, equation (2.2) yields the given result. □

The next three lemmas can be confirmed using similar steps. In the interest of brevity and clarity, we omit their proofs.

Lemma 2.2. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(4n-2)k+2}}{g^{\lambda}_{(4n-2)k}} - \frac{g^{\lambda}_{(4n+2)k+2}}{g^{\lambda}_{(4n+2)k}} \right] = \frac{g^{\lambda}_{2k+2}}{g^{\lambda}_{2k}} - \alpha^{2\lambda}. \tag{2.3}$$

Lemma 2.3. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(4n-1)k+2}}{g^{\lambda}_{(4n-1)k}} - \frac{g^{\lambda}_{(4n+3)k+2}}{g^{\lambda}_{(4n+3)k}} \right] = \frac{g^{\lambda}_{3k+2}}{g^{\lambda}_{3k}} - \alpha^{2\lambda}. \tag{2.4}$$

Lemma 2.4. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{4nk+2}}{g^{\lambda}_{4nk}} - \frac{g^{\lambda}_{(4n+4)k+2}}{g^{\lambda}_{(4n+4)k}} \right] = \frac{g^{\lambda}_{4k+2}}{g^{\lambda}_{4k}} - \alpha^{2\lambda}. \tag{2.5}$$

The following four lemmas also present telescoping gibonacci sums. They are directly related to the above results. Their proofs follow by those of the above sums [3]. In the interest of brevity and clarity, we omit them and still call them lemmas.

Lemma 2.5. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(4n-3)k}}{g^{\lambda}_{(4n-3)k+2}} - \frac{g^{\lambda}_{(4n+1)k}}{g^{\lambda}_{(4n+1)k+2}} \right] = \frac{g^{\lambda}_{k}}{g^{\lambda}_{k+2}} - \beta^{2\lambda}. \tag{2.6}$$

Lemma 2.6. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(4n-2)k}}{g^{\lambda}_{(4n-2)k+2}} - \frac{g^{\lambda}_{(4n+2)k}}{g^{\lambda}_{(4n+2)k+2}} \right] = \frac{g^{\lambda}_{2k}}{g^{\lambda}_{2k+2}} - \beta^{2\lambda}. \tag{2.7}$$

Lemma 2.7. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(4n-1)k}}{g^{\lambda}_{(4n-1)k+2}} - \frac{g^{\lambda}_{(4n+3)k}}{g^{\lambda}_{(4n+3)k+2}} \right] = \frac{g^{\lambda}_{3k}}{g^{\lambda}_{3k+2}} - \beta^{2\lambda}. \tag{2.8}$$

Lemma 2.8. *Let k and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{4nk}}{g^{\lambda}_{4nk+2}} - \frac{g^{\lambda}_{(4n+4)k}}{g^{\lambda}_{(4n+4)k+2}} \right] = \frac{g^{\lambda}_{4k}}{g^{\lambda}_{4k+2}} - \beta^{2\lambda}. \tag{2.9}$$

3. Gibonacci Sums

The above lemmas with $\lambda = 1$, coupled with identities (1.1) and (1.2), play a pivotal role in our explorations. In the interest of brevity, we now let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \nu = \begin{cases} -1, & \text{if } g_n = f_n; \\ 1, & \text{otherwise;} \end{cases}$$

and $\nu^* = -\nu$.

Theorem 3.1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu^* x f_{4k}}{g_{(4n-1)k}^2 + (-1)^k \mu\nu f_{2k}^2} = \frac{g_{k+2}}{g_k} - \alpha^2. \tag{3.1}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.2), Lemma 2.1 then yields

$$\begin{aligned} \frac{(-1)^k x f_{4k}}{f_{(4n-1)k}^2 - (-1)^k f_{2k}^2} &= \frac{f_{(4n+1)k} f_{(4n-3)k+2} - f_{(4n+1)k+2} f_{(4n-3)k}}{f_{(4n+1)k} f_{(4n-3)k}}; \\ \sum_{n=1}^{\infty} \frac{(-1)^k x f_{4k}}{f_{(4n-1)k}^2 - (-1)^k f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(4n-3)k+2}}{f_{(4n-3)k}} - \frac{f_{(4n+1)k+2}}{f_{(4n+1)k}} \right] \\ &= \frac{f_{k+2}}{f_k} - \alpha^2. \end{aligned}$$

On the other hand, let $g_n = l_n$. With the same identities and Lemma 2.1, we get

$$\begin{aligned} \frac{(-1)^{k+1} \Delta^2 x f_{4k}}{l_{(4n-1)k}^2 + (-1)^k \Delta^2 f_{2k}^2} &= \frac{l_{(4n+1)k} l_{(4n-3)k+2} - l_{(4n+1)k+2} l_{(4n-3)k}}{l_{(4n+1)k} l_{(4n-3)k}}; \\ \sum_{n=1}^{\infty} \frac{(-1)^{k+1} \Delta^2 x f_{4k}}{l_{(4n-1)k}^2 + (-1)^k \Delta^2 f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(4n-3)k+2}}{l_{(4n-3)k}} - \frac{l_{(4n+1)k+2}}{l_{(4n+1)k}} \right] \\ &= \frac{l_{k+2}}{l_k} - \alpha^2. \end{aligned}$$

By combining the two cases, we get equation (3.1), as desired. □

In particular, we get [4]

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{F_{4n-1}^2 + 1} &= -\frac{1}{6} + \frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty} \frac{1}{L_{4n-1}^2 - 5} &= \frac{1}{6} - \frac{\sqrt{5}}{30}; \\ \sum_{n=1}^{\infty} \frac{1}{F_{8n-2}^2 - 9} &= \frac{1}{14} - \frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty} \frac{1}{L_{8n-2}^2 + 45} &= -\frac{1}{126} + \frac{\sqrt{5}}{210}; \\ \sum_{n=1}^{\infty} \frac{1}{F_{12n-3}^2 + 64} &= -\frac{1}{144} + \frac{\sqrt{5}}{288}; & \sum_{n=1}^{\infty} \frac{1}{L_{12n-3}^2 - 320} &= \frac{1}{576} - \frac{\sqrt{5}}{1,440}. \end{aligned}$$

Theorem 3.2. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\nu^* x f_{4k}}{g_{4nk}^2 + \mu\nu f_{2k}^2} = \frac{g_{2k+2}}{g_{2k}} - \alpha^2. \tag{3.2}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.2), Lemma 2.2 then yields

$$\begin{aligned}
\frac{xf_{4k}}{f_{4nk}^2 - f_{2k}^2} &= \frac{f_{(4n+2)k}f_{(4n-2)k+2} - f_{(4n+2)k+2}f_{(4n-2)k}}{f_{(4n+2)k}f_{(4n-2)k}};\\
\sum_{n=1}^{\infty} \frac{xf_{4k}}{f_{4nk}^2 - f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(4n-2)k+2}}{f_{(4n-2)k}} - \frac{f_{(4n+2)k+2}}{f_{(4n+2)k}}\right]\\
&= \frac{f_{2k+2}}{f_{2k}} - \alpha^2. \qquad (3.3)
\end{aligned}$$

Now let $g_n = l_n$. Using the same identities and Lemma 2.2, we get

$$\begin{aligned}
\frac{-\Delta^2 xf_{4k}}{l_{4nk}^2 + \Delta^2 f_{2k}^2} &= \frac{l_{(4n+2)k}l_{(4n-2)k+2} - l_{(4n+2)k+2}l_{(4n-2)k}}{l_{(4n+2)k}l_{(4n-2)k}};\\
\sum_{n=1}^{\infty} \frac{-\Delta^2 xf_{4k}}{l_{4nk}^2 + \Delta^2 f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(4n-2)k+2}}{l_{(4n-2)k}} - \frac{l_{(4n+2)k+2}}{l_{(4n+2)k}}\right]\\
&= \frac{l_{2k+2}}{l_{2k}} - \alpha^2.
\end{aligned}$$

This result, with equation (3.3), yields the desired result. □

It follows by this theorem that [4]

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} &= \frac{1}{2} - \frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} &= -\frac{1}{18} + \frac{\sqrt{5}}{30};\\
\sum_{n=1}^{\infty} \frac{1}{F_{8n}^2 - 9} &= \frac{1}{18} - \frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty} \frac{1}{L_{8n}^2 + 45} &= -\frac{1}{98} + \frac{\sqrt{5}}{210};\\
\sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} &= \frac{1}{128} - \frac{\sqrt{5}}{288}; & \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} &= -\frac{1}{648} + \frac{\sqrt{5}}{1,440}.
\end{aligned}$$

The next result invokes Lemma 2.3.

Theorem 3.3. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu^* xf_{4k}}{g_{(4n+1)k}^2 + (-1)^k \mu\nu f_{2k}^2} = \frac{g_{3k+2}}{g_{3k}} - \alpha^2. \qquad (3.4)$$

Proof. Let $g_n = f_n$. With identities (1.1) and (1.2), Lemma 2.3 then yields

$$\begin{aligned}
\frac{(-1)^k xf_{4k}}{f_{(4n+1)k}^2 - (-1)^k f_{2k}^2} &= \frac{f_{(4n+3)k}f_{(4n-1)k+2} - f_{(4n+3)k+2}f_{(4n-1)k}}{f_{(4n+3)k}f_{(4n-1)k}};\\
\sum_{n=1}^{\infty} \frac{(-1)^k xf_{4k}}{f_{(4n+1)k}^2 - (-1)^k f_{2k}^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(4n-1)k+2}}{f_{(4n-1)k}} - \frac{f_{(4n+3)k+2}}{f_{(4n+3)k}}\right]\\
&= \frac{f_{3k+2}}{f_{3k}} - \alpha^2. \qquad (3.5)
\end{aligned}$$

Suppose $g_n = l_n$. Using the same two identities and Lemma 2.3, we get

$$\begin{aligned}\frac{(-1)^{k+1}\Delta^2 x f_{4k}}{l^2_{(4n+1)k} + (-1)^k\Delta^2 f^2_{2k}} &= \frac{l_{(4n+3)k}l_{(4n-1)k+2} - l_{(4n+3)k+2}l_{(4n-1)k}}{l_{(4n+3)k}l_{(4n-1)k}};\\ \sum_{n=1}^{\infty}\frac{(-1)^{k+1}\Delta^2 x f_{4k}}{l^2_{(4n+1)k} + (-1)^k\Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{l_{(4n-1)k+2}}{l_{(4n-1)k}} - \frac{l_{(4n+3)k+2}}{l_{(4n+3)k}}\right]\\ &= \frac{l_{3k+2}}{l_{3k}} - \alpha^2.\end{aligned}$$

This result, with equation (3.5), yields the given result. □

This theorem yields [4]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{F^2_{4n+1}+1} &= -\frac{1}{3}+\frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{4n+1}-5} &= \frac{1}{12}-\frac{\sqrt{5}}{30};\\ \sum_{n=1}^{\infty}\frac{1}{F^2_{8n+2}-9} &= \frac{3}{56}-\frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{8n+2}+45} &= -\frac{2}{189}+\frac{\sqrt{5}}{210};\\ \sum_{n=1}^{\infty}\frac{1}{F^2_{12n+3}+64} &= -\frac{19}{2,448}+\frac{\sqrt{5}}{288}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{12n+3}-320} &= \frac{17}{10,944}-\frac{\sqrt{5}}{1,440}.\end{aligned}$$

Finally, we present an application of Lemma 2.4.

Theorem 3.4. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{(-1)^k\mu\nu^* x f_{4k}}{g^2_{(4n+2)k} + \mu\nu f^2_{2k}} = \frac{g_{4k+2}}{g_{4k}} - \alpha^2. \tag{3.6}$$

Proof. Suppose $g_n = f_n$. Using identities (1.1) and (1.2), Lemma 2.4 yields

$$\begin{aligned}\frac{x f_{4k}}{f^2_{(4n+2)k} - f^2_{2k}} &= \frac{f_{(4n+4)k}f_{4nk+2} - f_{(4n+4)k+2}f_{4nk}}{f_{(4n+4)k}f_{4nk}};\\ \sum_{n=1}^{\infty}\frac{x f_{4k}}{f^2_{(4n+2)k} - f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{f_{4nk+2}}{f_{4nk}} - \frac{f_{(4n+4)k+2}}{f_{(4n+4)k}}\right]\\ &= \frac{f_{4k+2}}{f_{4k}} - \alpha^2. \qquad (3.7)\end{aligned}$$

On the other hand, let $g_n = l_n$. Using the same two identities and Lemma 2.4, we get

$$\begin{aligned}\frac{-\Delta^2 x f_{4k}}{l^2_{(4n+2)k} + \Delta^2 f^2_{2k}} &= \frac{l_{(4n+4)k}l_{4nk+2} - l_{(4n+4)k+2}l_{4nk}}{l_{(4n+4)k}l_{4nk}};\\ \sum_{n=1}^{\infty}\frac{-\Delta^2 x f_{4k}}{l^2_{(4n+2)k} + \Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{l_{4nk+2}}{l_{4nk}} - \frac{l_{(4n+4)k+2}}{l_{(4n+4)k}}\right]\\ &= \frac{l_{4k+2}}{l_{4k}} - \alpha^2.\end{aligned}$$

Combining the two cases, we get the desired result. □

This theorem implies [4]

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+2}^2 - 1} = \frac{7}{18} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+2}^2 + 5} = -\frac{1}{14} + \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+4}^2 - 9} = \frac{47}{882} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+4}^2 + 45} = -\frac{1}{94} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+6}^2 - 64} = \frac{161}{20,736} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+6}^2 + 320} = -\frac{1}{644} + \frac{\sqrt{5}}{1,440}.$$

3.1. **Gibonacci Delights.** Using the theorems, we can extract interesting byproducts. With Theorems 3.1 and 3.3, we get [4]

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 + 1} &= \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n-1}^2 + 1} \\
&= -\frac{1}{2} + \frac{\sqrt{5}}{3}; \\
\sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(4n-1)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{2(4n+1)}^2 - 9} \\
&= \frac{1}{8} - \frac{\sqrt{5}}{21}; \qquad (3.8) \\
\sum_{n=1}^{\infty} \frac{1}{F_{3(2n+1)}^2 + 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{12n-3}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+3}^2 + 64} \\
&= -\frac{1}{68} + \frac{\sqrt{5}}{144}; \\
\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 5} &= \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n-1}^2 - 5} \\
&= \frac{1}{4} - \frac{\sqrt{5}}{15}; \\
\sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(4n-1)}^2 + 45} + \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+1)}^2 + 45} \\
&= -\frac{1}{54} + \frac{\sqrt{5}}{105}; \qquad (3.9) \\
\sum_{n=1}^{\infty} \frac{1}{L_{3(2n+1)}^2 - 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{12n-3}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+3}^2 - 320} \\
&= \frac{1}{304} - \frac{\sqrt{5}}{720}.
\end{aligned}$$

Likewise, Theorems 3.2 and 3.4 yield [4]

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{2n}^2 - 1} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(2n)}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 1} \\
&= \frac{8}{9} - \frac{\sqrt{5}}{3};
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{4(2n)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{4(2n+1)}^2 - 9} \\
&= \frac{16}{147} - \frac{\sqrt{5}}{21}; \qquad (3.10) \\
\sum_{n=2}^{\infty} \frac{1}{F_{6n}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+6}^2 - 64} \\
&= \frac{323}{20,736} - \frac{\sqrt{5}}{144}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 + 5} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(2n)}^2 + 5} + \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 5} \\
&= -\frac{8}{63} + \frac{\sqrt{5}}{15}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{4n}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{4(2n)}^2 + 45} + \sum_{n=1}^{\infty} \frac{1}{L_{4(2n+1)}^2 + 45} \\
&= -\frac{48}{2,303} + \frac{\sqrt{5}}{105}; \qquad (3.11) \\
\sum_{n=2}^{\infty} \frac{1}{L_{6n}^2 + 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+6}^2 + 320} \\
&= -\frac{323}{104,328} + \frac{\sqrt{5}}{720}.
\end{aligned}$$

3.2. **Gibonacci Delectables.** With equations (3.8)–(3.11), we can get additional dividends [4]

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{F_{2n}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 9} + \sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 9} \\
&= \frac{275}{1,176} - \frac{2\sqrt{5}}{21}; \\
\sum_{n=3}^{\infty} \frac{1}{L_{2n}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 45} + \sum_{n=2}^{\infty} \frac{1}{L_{4n}^2 + 45} \\
&= -\frac{4,895}{124,362} + \frac{2\sqrt{5}}{105}.
\end{aligned}$$

4. Additional Gibonacci Sums

Next, we explore four additional gibonacci sums using Lemmas 2.5–2.8, again with $\lambda = 1$. The first result employs Lemma 2.5.

Theorem 4.1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu x f_{4k}}{g_{(4n-1)k+2}^2 + (-1)^k \mu\nu f_{2k}^2} = \frac{g_k}{g_{k+2}} - \beta^2. \qquad (4.1)$$

Proof. Suppose $g_n = f_n$. With identities (1.1) and (1.2), Lemma 2.5 yields

$$\begin{aligned}\frac{(-1)^{k+1}xf_{4k}}{f^2_{(4n-1)k+2} - (-1)^k f^2_{2k}} &= \frac{f_{(4n+1)k+2}f_{(4n-3)k} - f_{(4n+1)k}f_{(4n-3)k+2}}{f_{(4n+1)k+2}f_{(4n-3)k+2}};\\ \sum_{n=1}^{\infty}\frac{(-1)^{k+1}xf_{4k}}{f^2_{(4n-1)k+2} - (-1)^k f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{f_{(4n-3)k}}{f_{(4n-3)k+2}} - \frac{f_{(4n+1)k}}{f_{(4n+1)k+2}}\right]\\ &= \frac{f_k}{f_{k+2}} - \beta^2.\end{aligned}$$

On the other hand, let $g_n = l_n$. Using the same identities and Lemma 2.5, we get

$$\begin{aligned}\frac{(-1)^k\Delta^2xf_{4k}}{l^2_{(4n-1)k+2} + (-1)^k\Delta^2 f^2_{2k}} &= \frac{l_{(4n+1)k+2}l_{(4n-3)k} - l_{(4n+1)k}l_{(4n-3)k+2}}{l_{(4n+1)k+2}l_{(4n-3)k+2}};\\ \sum_{n=1}^{\infty}\frac{(-1)^k\Delta^2xf_{4k}}{l^2_{(4n-1)k+2} + (-1)^k\Delta^2 f^2_{2k}} &= \sum_{n=1}^{\infty}\left[\frac{l_{(2n-3)k}}{l_{(4n-3)k+2}} - \frac{l_{(4n+1)k}}{l_{(4n+1)k+2}}\right]\\ &= \frac{l_k}{l_{k+2}} - \beta^2.\end{aligned}$$

By combining the two cases, we get the desired result. □

In particular, we then have [3]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{F^2_{4n+1}+1} &= -\frac{1}{3}+\frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{4n+1}-5} &= \frac{1}{12}-\frac{\sqrt{5}}{30};\\ \sum_{n=1}^{\infty}\frac{1}{F^2_{8n}-9} &= \frac{1}{18}-\frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{8n}+45} &= -\frac{1}{98}+\frac{\sqrt{5}}{210}.\end{aligned}$$

The next three theorems can be established similarly using Lemmas 2.6–2.8, respectively. In the interest of conciseness, we omit them; but we encourage gibonacci enthusiasts to confirm them.

Theorem 4.2. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{\mu\nu xf_{4k}}{g^2_{4nk+2} + \mu\nu f^2_{2k}} = \frac{g_{2k}}{g_{2k+2}} - \beta^2. \tag{4.2}$$

This theorem implies [3]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{F^2_{4n+2}-1} &= \frac{7}{18}-\frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{4n+2}+5} &= -\frac{1}{14}+\frac{\sqrt{5}}{30};\\ \sum_{n=1}^{\infty}\frac{1}{F^2_{8n+2}-9} &= \frac{3}{56}-\frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty}\frac{1}{L^2_{8n+2}+45} &= -\frac{2}{189}+\frac{\sqrt{5}}{210}.\end{aligned}$$

The following theorem invokes Lemma 2.7.

Theorem 4.3. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{(-1)^k\mu\nu f_{4k}}{g^2_{(4n+1)k+2} + (-1)^k\mu\nu f^2_{2k}} = \frac{g_{3k}}{g_{3k+2}} - \beta^2. \tag{4.3}$$

In particular, we get [3]

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{11}{30} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{5}{66} - \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+4}^2 - 9} = \frac{47}{882} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+4}^2 + 45} = -\frac{1}{94} + \frac{\sqrt{5}}{210}.$$

The following theorem is an application of Lemma 2.8.

Theorem 4.4. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\mu\nu f_{4k}}{g_{(4n+2)k+2}^2 + \mu\nu f_{2k}^2} = \frac{g_{4k}}{g_{4k+2}} - \beta^2. \tag{4.4}$$

It follows by this theorem that

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+4}^2 - 1} = \frac{3}{8} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+4}^2 + 5} = -\frac{2}{27} + \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+6}^2 - 9} = \frac{41}{770} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+6}^2 + 45} = -\frac{55}{5,166} + \frac{\sqrt{5}}{210}.$$

4.1. **Additional Gibonacci Delights.** Theorems 4.1–4.4 can be used to compute additional gibonacci sums. Using Theorems 4.1 and 4.3, we get

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{2n+1}^2 + 1} &= \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} \\
&= -\frac{7}{10} + \frac{\sqrt{5}}{3}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{4(2n)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{4(2n+1)}^2 - 9} \\
&= \frac{16}{147} - \frac{\sqrt{5}}{21}; \qquad (4.5) \\
\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 5} &= \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} \\
&= \frac{7}{44} - \frac{2\sqrt{5}}{15}; \\
\sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{4(2n)}^2 + 45} + \sum_{n=1}^{\infty} \frac{1}{L_{4(2n+1)}^2 + 45} \\
&= -\frac{48}{2,303} + \frac{\sqrt{5}}{105}. \qquad (4.6)
\end{aligned}$$

Likewise, Theorems 4.2 and 4.4 yield

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{F_{2n}^2 - 1} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{F_{2(2n+2)}^2 - 1} \\
&= \frac{55}{72} - \frac{\sqrt{5}}{3}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 9} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(4n+1)}^2 - 9} + \sum_{n=1}^{\infty} \frac{1}{F_{2(4n+3)}^2 - 9} \\
&= \frac{47}{440} - \frac{\sqrt{5}}{21}; \qquad (4.7) \\
\sum_{n=3}^{\infty} \frac{1}{L_{2n}^2 + 5} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 5} + \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+2)}^2 + 5} \\
&= -\frac{55}{378} + \frac{\sqrt{5}}{15}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+1)}^2 + 45} + \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+3)}^2 + 45} \\
&= -\frac{47}{2,214} + \frac{\sqrt{5}}{105}. \qquad (4.8)
\end{aligned}$$

4.2. **Additional Gibonacci Delectables.** Using four of the above sums, we can evaluate two additional sums. With sums (4.5) and (4.7), we get

$$\begin{aligned}
\sum_{n=4}^{\infty} \frac{1}{F_{2n}^2 - 9} &= \sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 9} + \sum_{n=2}^{\infty} \frac{1}{F_{4n+2}^2 - 9} \\
&= \frac{13,949}{64,680} - \frac{2\sqrt{5}}{21}.
\end{aligned}$$

Likewise, equations (4.6) and (4.8) yield

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 + 45} &= \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 45} + \sum_{n=2}^{\infty} \frac{1}{L_{4n+2}^2 + 45} \\
&= -\frac{214,513}{5,098,842} + \frac{2\sqrt{5}}{105}.
\end{aligned}$$

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Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu