INFINITE SUMS INVOLVING GIBONACCI POLYNOMIALS REVISITED: GENERALIZATIONS

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Abstract. We explore the generalizations of four infinite sums involving gibonacci polynomials, and their alternate and Pell versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial.* They can also be defined by *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, and $E = \sqrt{x^2+1}$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$.

1.1. Some Fundamental Identities.

Gibonacci polynomials g_n satisfy the following fundamental properties [4]:

$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n; \text{ [p. 36]} \tag{1}$$

$$l_{n+k} l_{n-k} = l_n^2 + (-1)^{n+k}\Delta^2 f_k^2; \text{ [p. 13]} \tag{2}$$

$$l_{n+k}^2 - l_{n-k}^2 = \Delta^2 f_{2n} f_{2k}. \text{ [p. 57]} \tag{3}$$

These three identities play a major role in our discourse.

2. Gibonacci Polynomial Sums

With the above background, we now begin our explorations of gibonacci sums.

Theorem 1. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+1)k}}{[l_{(2n+1)k}^2 + \Delta^2 f_k^2]^2} = \frac{1}{l_{2k}^2}. \tag{4}$$

Proof. Using recursion [4, 5], we will first establish that

$$\sum_{n=1}^{m} \frac{\Delta^2 f_{2k} f_{2(2n+1)k}}{[l_{(2n+1)k}^2 + \Delta^2 f_k^2]^2} = \frac{1}{l_{2k}^2} - \frac{1}{l_{(2m+2)k}^2}. \tag{5}$$

To achieve this goal, we let A_m = LHS of this equation and B_m its RHS. It then follows by identities (2) and (3) that

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{2mk}^2} - \frac{1}{l_{(2m+2)k}^2} \\ &= \frac{l_{(2m+2)k}^2 - l_{2mk}^2}{l_{(2m+2)k}^2 l_{2mk}^2} \\ &= \frac{\Delta^2 f_{2k} f_{2(2m+1)k}}{\left[l_{(2m+1)k}^2 + \Delta^2 f_k^2\right]^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= \frac{\Delta^2 f_{2k} f_{6k}}{(l_{3k}^2 + \Delta^2 f_k^2)^2} - \left(\frac{1}{l_{2k}^2} - \frac{1}{l_{4k}^2}\right) \\ &= \frac{l_{4k}^2 - l_{2k}^2}{l_{4k}^2 l_{2k}^2} - \left(\frac{1}{l_{2k}^2} - \frac{1}{l_{4k}^2}\right) \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, as claimed.

Because $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$, equation (5) yields the desired result. □

It follows from equation (4) that

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(L_{2n+1}^2 + 5)^2} = \frac{1}{45}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[L_{2(2n+1)}^2 + 5]^2} = \frac{1}{735};$$

$$\sum_{n=1}^{\infty} \frac{F_{6(2n+1)}}{[L_{3(2n+1)}^2 + 20]^2} = \frac{1}{12,960}; \qquad \sum_{n=1}^{\infty} \frac{F_{8(2n+1)}}{[L_{4(2n+1)}^2 + 45]^2} = \frac{1}{231,945}.$$

An Alternate Version. With identity (1), we can rewrite equation (4) in terms of Fibonacci polynomials alone:

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+1)k}}{\{\Delta^2[f_{(2n+1)k}^2 + f_k^2] - 4(-1)^k\}^2} = \frac{1}{\Delta^2 f_{2k}^2 + 4}. \tag{6}$$

This implies

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(5F_{2n+1}^2 + 9)^2} = \frac{1}{45}; \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[5F_{2(2n+1)}^2 + 1]^2} = \frac{1}{735};$$

$$\sum_{n=1}^{\infty} \frac{F_{6(2n+1)}}{[5F_{3(2n+1)}^2 + 24]^2} = \frac{1}{12,960}; \qquad \sum_{n=1}^{\infty} \frac{F_{8(2n+1)}}{[5F_{4(2n+1)}^2 + 41]^2} = \frac{1}{231,945}.$$

The following result involves even-numbered gibonacci polynomials.

Theorem 2. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+2)k}}{[l_{(2n+2)k}^2 + \Delta^2 f_{2k}^2]^2} = \frac{1}{l_{2k}^2} + \frac{1}{l_{4k}^2}. \tag{7}$$

Proof. With recursion [4, 5], and identities (2) and (3), we will first confirm that

$$\sum_{n=1}^{m} \frac{\Delta^2 f_{4k} f_{2(2n+2)k}}{[l_{(2n+2)k}^2 + \Delta^2 f_{2k}^2]^2} = \frac{1}{l_{2k}^2} + \frac{1}{l_{4k}^2} - \frac{1}{l_{(2m+2)k}^2} - \frac{1}{l_{(2m+4)k}^2}. \tag{8}$$

Letting A_m = LHS and B_m = RHS of this equation, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{2mk}^2} - \frac{1}{l_{(2m+4)k}^2} \\ &= \frac{l_{(2m+4)k}^2 - l_{2mk}^2}{l_{(2m+4)k}^2 l_{2mk}^2} \\ &= \frac{\Delta^2 f_{4k} f_{2(2m+2)k}}{[l_{(2m+2)k}^2 + \Delta^2 f_{2k}^2]^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= \frac{\Delta^2 f_{4k} f_{8k}}{(l_{4k}^2 + \Delta^2 f_{2k}^2)^2} - \left[\left(\frac{1}{l_{2k}^2} + \frac{1}{l_{4k}^2}\right) - \left(\frac{1}{l_{4k}^2} + \frac{1}{l_6^2}\right)\right] \\ &= \frac{l_{6k}^2 - l_{2k}^2}{l_{6k}^2 l_{2k}^2} - \left(\frac{1}{l_{2k}^2} - \frac{1}{l_{6k}^2}\right) \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, as desired.

Now letting $m \to \infty$ in equation (8), we get the given result. □

Equation (7) yields

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(L_{2n+2}^2 + 5)^2} = \frac{58}{6,615}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{[L_{2(2n+2)}^2 + 45]^2} = \frac{2,258}{11,365,305};$$

$$\sum_{n=1}^{\infty} \frac{F_{6(2n+2)}}{[L_{3(2n+2)}^2 + 320]^2} = \frac{13,001}{3,023,425,440}; \qquad \sum_{n=1}^{\infty} \frac{F_{8(2n+2)}}{[L_{4(2n+2)}^2 + 2,205]^2} = \frac{4,873,058}{103,729^2}.$$

An Alternate Version. Using identity (1), we can rewrite equation (7) with Fibonacci polynomials alone:

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+2)k}}{\{\Delta^2[f_{(2n+2)k}^2 + f_k^2] + 4\}^2} = \frac{1}{\Delta^2 f_{2k}^2 + 4} + \frac{1}{\Delta^2 f_{4k}^2 + 4}. \tag{9}$$

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(5F_{2n+2}^2 + 9)^2} = \frac{58}{6,615}; \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{[5F_{2(2n+1)}^2 + 49]^2} = \frac{2,258}{11,365,305}.$$

A Gibonacci Delight: Equations (4) and (7) yield

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{(L_n^2+5)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(L_{2n+1}^2+5)^2} + \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(L_{2n+2}^2+5)^2} \\ &= \frac{41}{1,323}; \\ \sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2+5)^2} &= \frac{2}{27}.\end{aligned}$$

The next result is interesting in its own right.

Theorem 3. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+1)k}}{[l_{(2n+1)k}^2 - \Delta^2 f_{2k}^2]^2} = \frac{1}{l_k^2} + \frac{1}{l_{3k}^2}. \tag{10}$$

Proof. By invoking recursion [4, 5], we will first prove that

$$\sum_{n=1}^{m} \frac{\Delta^2 f_{4k} f_{2(2n+1)k}}{[l_{(2n+1)k}^2 - \Delta^2 f_{2k}^2]^2} = \frac{1}{l_k^2} + \frac{1}{l_{3k}^2} - \frac{1}{l_{(2m+1)k}^2} - \frac{1}{l_{(2m+3)k}^2}. \tag{11}$$

With A_m = LHS and B_m = RHS of this equation, it follows by identities (2) and (3) that

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l_{(2m-1)k}^2} - \frac{1}{l_{(2m+3)k}^2} \\ &= \frac{l_{(2m+3)k}^2 - l_{(2m-1)k}^2}{l_{(2m+3)k}^2 l_{(2m-1)k}^2} \\ &= \frac{\Delta^2 f_{4k} f_{2(2m+1)k}}{[l_{(2m+1)k}^2 - \Delta^2 f_{2k}^2]^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

As before, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \ldots = A_1 - B_1 \\ &= \frac{\Delta^2 f_{4k} f_{6k}}{(l_{3k}^2 - \Delta^2 f_{2k}^2)^2} - \left[\left(\frac{1}{l_k^2} - \frac{1}{l_{3k}^2}\right) - \left(\frac{1}{l_{3k}^2} - \frac{1}{l_{5k}^2}\right)\right] \\ &= \frac{l_{5k}^2 - l_k^2}{l_{5k}^2 l_k^2} - \left(\frac{1}{l_k^2} - \frac{1}{l_{5k}^2}\right) \\ &= 0, \end{aligned}$$

confirming the validity of formula (11).

Equation (10) follows from this result, as desired. □

Equation (10) yields

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(L_{2n+1}^2-5)^2} = \frac{17}{240}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[L_{2(2n+1)}^2-45]^2} = \frac{37}{34,020};$$

$$\sum_{n=1}^{\infty} \frac{F_{6(2n+1)}}{[L_{3(2n+1)}^2-320]^2} = \frac{181}{2,079,360}; \qquad \sum_{n=1}^{\infty} \frac{F_{8(2n+1)}}{[L_{4(2n+1)}^2-2,205]^2} = \frac{2,117}{511,680,540}.$$

An Alternate Version. With identity (1), we can express equation (10) with Fibonacci polynomials alone:

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{4k} f_{2(2n+1)k}}{\{\Delta^2[f^2_{(2n+1)k} - f^2_{2k}] + 4(-1)^k\}^2} = \frac{1}{\Delta^2 f_k^2 + 4(-1)^k} + \frac{1}{\Delta^2 f_{3k}^2 + 4(-1)^k}. \tag{12}$$

This implies

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(5F^2_{2n+1} - 9)^2} = \frac{17}{240}; \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+1)}}{[5F^2_{2(2n+1)} - 41]^2} = \frac{37}{34,020}.$$

Finally, we explore one more gibonacci sum.

Theorem 4. *Let k be a positive integer. Then*

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+2)k}}{[l^2_{(2n+2)k} + (-1)^k \Delta^2 f_k^2]^2} = \frac{1}{l^2_{3k}}. \tag{13}$$

Proof. Using recursion [4, 5], and identities (2) and (3), we will first confirm that

$$\sum_{n=1}^{m} \frac{\Delta^2 f_{2k} f_{2(2n+2)k}}{[l^2_{(2n+2)k} + (-1)^k \Delta^2 f_k^2]^2} = \frac{1}{l^2_{3k}} - \frac{1}{l^2_{(2m+3)k}}. \tag{14}$$

With A_m = LHS and B_m = RHS of this equation, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{1}{l^2_{(2m+1)k}} - \frac{1}{l^2_{(2m+3)k}} \\ &= \frac{l^2_{(2m+3)k} - l^2_{(2m+1)k}}{l^2_{(2m+3)k} l^2_{(2m+1)k}} \\ &= \frac{\Delta^2 f_{2k} f_{2(2m+2)k}}{[l^2_{(2m+2)k} + (-1)^k \Delta^2 f_k^2]^2} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= \frac{\Delta^2 f_{2k} f_{8k}}{\left[l^2_{4k} + (-1)^k \Delta^2 f_k^2\right]^2} - \left(\frac{1}{l^2_{3k}} - \frac{1}{l^2_{5k}}\right) \\ &= \frac{l^2_{5k} - l^2_{3k}}{l^2_{5k} l^2_{3k}} - \left(\frac{1}{l^2_{3k}} - \frac{1}{l^2_{5k}}\right) \\ &= 0. \end{aligned}$$

Consequently, $A_m = B_m$, establishing the validity of equation (14).

The given result now follows from equation (14), as desired. □

It follows from equation (13) that

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(L^2_{2n+2} - 5)^2} = \frac{1}{80}; \text{ [5]} \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{[L^2_{2(2n+2)} + 5]^2} = \frac{1}{4,860};$$

$$\sum_{n=1}^{\infty} \frac{F_{6(2n+2)}}{[L^2_{3(2n+2)} - 20]^2} = \frac{1}{231,040}; \qquad \sum_{n=1}^{\infty} \frac{F_{8(2n+2)}}{[L^2_{4(2n+2)} + 45]^2} = \frac{1}{10,886,820}.$$

A Gibonacci Delight: Equation (10), coupled with equation (13), yields

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{F_{2n}}{(L_n^2-5)^2} &= \sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{(L_{2n+1}^2-5)^2} + \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(L_{2n+2}^2-5)^2} \\ &= \frac{1}{12}; \\ \sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2-5)^2} &= \frac{1}{3}. \text{ [5, 6]}\end{aligned}$$

An Alternate Version. With identity (1), we can find a slightly different form of equation (13) with Fibonacci polynomials alone:

$$\sum_{n=1}^{\infty} \frac{\Delta^2 f_{2k} f_{2(2n+2)k}}{\{\Delta^2[f_{(2n+2)k}^2 + (-1)^k f_k^2] + 4\}^2} = \frac{1}{\Delta^2 f_{3k}^2 + 4(-1)^k}. \tag{15}$$

This yields

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{(5F_{2n+2}^2-1)^2} = \frac{1}{80}; \qquad \sum_{n=1}^{\infty} \frac{F_{4(2n+2)}}{[5F_{2(2n+2)}^2+9]^2} = \frac{1}{4,860}.$$

Finally, we extract the Pell implications of the sums in Theorems 1 through 4.

3. Pell Consequences

Using the relationship $b_n(x) = g_n(2x)$, equations (4), (7), (10), and (13) yield the following sums:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{4E^2 p_{2k} p_{2(2n+1)k}}{[q_{(2n+1)k}^2 + 4E^2 p_k^2]^2} &= \frac{1}{q_{2k}^2}; & \sum_{n=1}^{\infty} \frac{4E^2 p_{4k} p_{2(2n+2)k}}{[q_{(2n+2)k}^2 + 4E^2 p_{2k}^2]^2} &= \frac{1}{q_{2k}^2} + \frac{1}{q_{4k}^2}; \\ \sum_{n=1}^{\infty} \frac{4E^2 p_{4k} p_{2(2n+1)k}}{[q_{(2n+1)k}^2 - 4E^2 p_{2k}^2]^2} &= \frac{1}{q_k^2} + \frac{1}{q_{3k}^2}; & \sum_{n=1}^{\infty} \frac{4E^2 p_{2k} p_{2(2n+2)}}{[q_{(2n+2)k}^2 + 4(-1)^k E^2 p_k^2]^2} &= \frac{1}{q_{3k}^2},\end{aligned}$$

respectively.

In particular, we then get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{(Q_{2n+1}^2+2)^2} &= \frac{1}{36}; & \sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{(Q_{2n+2}^2+8)^2} &= \frac{149}{7,803}; \\ \sum_{n=1}^{\infty} \frac{P_{2(2n+1)}}{(Q_{2n+1}^2-8)^2} &= \frac{25}{588}; & \sum_{n=1}^{\infty} \frac{P_{2(2n+2)}}{(Q_{2n+2}^2-2)^2} &= \frac{1}{196},\end{aligned}$$

again respectively.

Additionally, we can find the Pell versions of equations (6), (9), (12), and (15). In the interest of brevity, we omit them and encourage gibonacci enthusiasts to pursue them.

4. Chebyshev and Vieta Implications

Finally, we add that Chebyshev polynomials T_n and U_n, Vieta polynomials V_n and v_n, and gibonacci polynomials g_n are linked by the relationships $V_n(x) = i^{n-1} f_n(-ix)$, $v_n(x) = i^n l_n(-ix)$, $V_n(x) = U_{n-1}(x/2)$, and $v_n(x) = 2T_n(x/2)$ [2, 3, 4], where $i = \sqrt{-1}$. They can be employed to find the Chebyshev and Vieta consequences of equations (4), (7), (10), and (13), and their alternate versions. Again, in the interest of brevity, we omit them and encourage gibonacci enthusiasts to explore them.

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