A REMARK ON DEDEKIND SUMS AND PALINDROMIC CONTINUED FRACTIONS

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Abstract. Using a classical theorem of Serret and a well-known property of Dedekind sums, we show that a Dedekind sum vanishes if and only if the quotient of its arguments possesses a palindromic continued fraction expansion of a particular form.

1. Introduction

The Dedekind sum $s(p, q)$ for integral $p \geq 0$, $q > 0$ may be defined as

$$s(p,q) = \sum_{\mu \bmod q} \left(\left(\frac{\mu}{q}\right)\right)\left(\left(\frac{\mu p}{q}\right)\right), \tag{1.1}$$

where we have set

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}; \end{cases} \tag{1.2}$$

with $\lfloor x \rfloor$ denoting the greatest integer less than or equal to x. A number of results related to special values of Dedekind sums are known, see for instance Apostol's textbook [1, pg. 73]. Interestingly, many explicit evaluations of Dedekind sums arise in connection with recurrence sequences [3, 4, 6, 8]. As a particular example, the following theorem appears as an exercise in Apostol [1, pg. 72].

Theorem 1. *Let F_n be the nth Fibonacci number. Then*

$$s(F_{2n}, F_{2n+1}) = 0. \tag{1.3}$$

The above identity was generalized by two independent sets of authors in the same issue of the *Fibonacci Quarterly* [6, 8]. To state their result, we must first define the Lucas sequence $(U_n(a,b))_{n\geq 0}$, which satisfies the recurrence

$$U_0(a,b) = 0, \quad U_1(a,b) = 1, \quad U_{n+1}(a,b) = aU_n(a,b) - bU_{n-1}(a,b) \quad (n \geq 1), \tag{1.4}$$

where $ab \neq 0$ and $a^2 - 4b > 0$. In particular, when $a = 1$ and $b = -1$, $U_n(1,-1) = F_n$ are the Fibonacci numbers. The generalization of Theorem 1 is then given by

Theorem 2 (Zhao and Wang, Robbins)**.** *Let $U_n := U_n(a,-1)$. Then*

$$s(U_{2n}, U_{2n+1}) = 0. \tag{1.5}$$

The purpose of this note is to provide a characterization of those pairs of positive coprime integers (p, q) such that $s(p, q) = 0$. This characterization, given in terms of the continued fraction expansion of p/q, allows us to easily deduce Theorems 1 and 2 as special cases. Specifically, our main result is the following theorem, whose terminology shall be made clear in the next section.

Theorem 3. *Let p, q be positive coprime integers. Then*

$$s(p,q) = 0 \tag{1.6}$$

if and only if p/q possesses a continued fraction expansion whose fractional part is palindromic of even length.

2. Definitions

For our purposes, a continued fraction $\langle a_0, a_1, \ldots, a_n \rangle$ is a fraction of the form

$$\langle a_0, a_1, \ldots, a_n \rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}, \tag{2.1}$$

where the sequence of partial quotients $(a_k)_{k=0}^{n}$ are all positive integers except for a_0, which may be 0. The continued fraction expansion of a positive rational number p/q is unique up to parity. That is, if

$$\frac{p}{q} = \langle a_0, a_1, \ldots, a_n \rangle, \tag{2.2}$$

then p/q also possesses the expansion

$$\frac{p}{q} = \begin{cases} \langle a_0, a_1, \ldots, a_{n-1} + 1 \rangle, & \text{if } a_n = 1; \\ \langle a_0, a_1, \ldots, a_n - 1, 1 \rangle, & \text{if } a_n > 1. \end{cases} \tag{2.3}$$

Consequently, a positive rational number possesses two continued fraction expansions: one with an even number of partial quotients and the other with an odd number.

A string of N partial quotients is said to be *palindromic* of length N if it is identical whether read left-to-right or right-to-left, that is

$$(a_k, a_{k+1}, \ldots, a_{k+N-1}) = (a_{k+N-1}, a_{k+N-2}, \ldots, a_k). \tag{2.4}$$

If the string $(a_0, a_1, \ldots, a_n)$ is palindromic, then the continued fraction expansion $\langle a_0, a_1, \ldots, a_n \rangle$ is said to be palindromic. Lastly, we say that the *fractional part* of a continued fraction expansion $\langle a_0, a_1, \ldots, a_n \rangle$ is the string of partial quotients $(a_1, a_2, \ldots, a_n)$.

3. Proof of Theorem 3

Our proof of Theorem 3 follows almost immediately from the next two theorems.

Theorem 4. *The Dedekind sum $s(p,q) = 0$ if and only if $p^2 + 1 \equiv 0 \pmod{q}$.*

Theorem 5 (Serret)**.** *Let x, y be coprime integers such that $1 < y < x$. Then, x/y possesses a palindromic continued fraction expansion of even (respectively odd) length if and only if $y^2 + 1$ (respectively $y^2 - 1$) is divisible by x.*

Theorem 4 is a well-known property of Dedekind sums and can be found in [1, pg. 65]. Theorem 5 is an old result of Serret [7], and can be found in Perron's treatise on continued fractions [5, p. 33], written in German. A recent proof of Serret's theorem in English is given in [2].

Proof of Theorem 3. Let us first note that, without loss of generality, we may assume that $1 < p < q$. Indeed, if $p > q$, then we may instead consider the pair (p', q), where p' is taken to be the least positive residue of p mod q. This follows because if $p \equiv p' \pmod{q}$, then $s(p, q) = s(p', q)$, which can be established directly from definition (1.1). Moreover, if p'/q possesses a continued fraction expansion whose fractional part is palindromic of even length, then so does p/q because $p/q = k + p'/q$ for some integer $k > 0$.

To begin our proof, let us assume that $s(p, q) = 0$ for positive, coprime integers p, q with $1 < p < q$. By Theorem 4, we know that $p^2 + 1 \equiv 0 \pmod{q}$. As a result, Serret's theorem guarantees that the fraction q/p possesses a palindromic continued fraction expansion of even length. That is,

$$\frac{q}{p} = \langle a_0, \ldots, a_n, a_n, \ldots, a_0 \rangle. \tag{3.1}$$

By the construction of continued fractions, we find that the reciprocal is given by

$$\frac{p}{q} = \langle 0, a_0, \ldots, a_n, a_n, \ldots, a_0 \rangle. \tag{3.2}$$

Therefore, p/q does indeed possess a continued fraction expansion whose fractional part is palindromic of even length. To prove the opposite direction, let us assume that p/q possesses a continued fraction expansion whose fractional part is palindromic of even length. Once again, using our assumption that $1 < p < q$, we may write

$$\frac{p}{q} = \langle 0, a_0, \ldots, a_n, a_n, \ldots, a_0 \rangle, \tag{3.3}$$

and therefore,

$$\frac{q}{p} = \langle a_0, \ldots, a_n, a_n, \ldots, a_0 \rangle \tag{3.4}$$

is completely palindromic of even length. Thus, by Theorem 5, we know that $p^2 + 1 \equiv 0 \pmod{q}$, which in turn implies that $s(p, q) = 0$ by Theorem 4. This completes our proof of Theorem 3. □

4. Closing Remarks

Because the Lucas sequences $U_n := U_n(a, -1)$ are defined by the recurrence

$$U_{n+1} = aU_n + U_{n-1}, \tag{4.1}$$

the ratio of consecutive terms satisfies the identity

$$\frac{U_n}{U_{n+1}} = \cfrac{1}{a + \cfrac{U_{n-1}}{U_n}}. \tag{4.2}$$

Noting that $U_0/U_1 = 0$ leads to the continued fraction expansion

$$\frac{U_n}{U_{n+1}} = \langle 0, \overbrace{a, \ldots, a}^{n} \rangle. \tag{4.3}$$

Thus, when n is even, the expansion has a palindromic fractional part of even length, which shows that Theorem 2 is indeed a special case of Theorem 3. A case that does not fall under Theorem 2, but is covered under Theorem 3, is the Fibonacci quotient F_{2n+1}/F_{2n+3}. It can be shown to possess, for $n \geq 1$, the continued fraction expansion

$$\frac{F_{2n+1}}{F_{2n+3}} = \langle 0, \overbrace{2, 1, \ldots, 1, 2}^{2n} \rangle. \tag{4.4}$$

Moreover, the reciprocal can be written as

$$\frac{F_{2n+3}}{F_{2n+1}} = \langle 2, \overbrace{1, \ldots, 1, 2}^{2n-1} \rangle = \langle 2, \overbrace{1, \ldots, 1, 1, 1}^{2n} \rangle. \tag{4.5}$$

Therefore, F_{2n+1}/F_{2n+3} and F_{2n+3}/F_{2n+1} possess continued fraction expansions with palindromic fractional parts of even length. Indeed, it can be shown that $\{F_{2n+1}, F_{2n+3}\}$ are the only such pairs of positive integers $\{p, q\}$ such that p/q and q/p both possess such expansions. By Theorem 3, the pairs $\{F_{2n+1}, F_{2n+3}\}$ are then the only such positive integers $\{p, q\}$ such that $s(p, q) = 0$ and $s(q, p) = 0$. Because $s(p, q) = s(q, p)$ implies $s(p, q) = 0$ [4, Thm. 2], one can give a new proof of a result of Meyer [4, Thm. 4], which states that $s(p, q) = s(q, p)$ if and only if $p = F_{2n+1}$ and $q = F_{2n+3}$.

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