TRIBONACCI NUMBERS THAT ARE PRODUCTS OF TWO FIBONACCI NUMBERS

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Abstract. Let T_m be the mth Tribonacci number and F_n be the nth Fibonacci number. In this paper, we solve the Diophantine equation

$$T_m = F_n F_k$$

in positive integer unknowns m, n, and k.

1. Introduction

The Tribonacci sequence $\{T_m\}_{m\geq 0}$ is given by $T_0 = 0$, $T_1 = T_2 = 1$, and the recurrence

$$T_{m+3} = T_{m+2} + T_{m+1} + T_m \quad \text{for all } \; m \geq 0.$$

Its characteristic polynomial is $X^3 - X^2 - X - 1 = (X - \alpha)(X - \beta)(X - \bar{\beta})$, where

$$\alpha = \frac{1 + r_1 + r_2}{3}, \; \beta = \frac{2 - (r_1 + r_2) + i\sqrt{3}(r_1 - r_2)}{6},$$

with

$$r_1 = \sqrt[3]{19 + 3\sqrt{33}} \qquad \text{and} \qquad r_2 = \sqrt[3]{19 - 3\sqrt{33}}.$$

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for } \; n \geq 0.$$

Its characteristic polynomial is $X^2 - X - 1 = (X - \gamma)(X - \delta)$, where

$$\gamma = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \delta = \frac{1 - \sqrt{5}}{2}.$$

In this paper, we study the Diophantine equation

$$T_m = F_n F_k \tag{1}$$

in positive unknowns m, n, and k.

Theorem 1. *The only nonzero Tribonacci numbers that are products of two Fibonacci numbers are*

$$1, \, 2, \, 4, \, 13, \text{ and } 24.$$

Our method of proof involves the application of Baker's lower bounds for nonzero linear forms in logarithms of algebraic numbers, and the Baker-Davenport reduction procedure. Computations are done with the help of a computer program in Mathematica.

2. Preliminary Results

2.1. **Results on Tribonacci and Fibonacci Numbers.** Here, we recall some well-known formulas and inequalities on Tribonacci and Fibonacci sequences. Binet's formula for the Tribonacci sequence is

$$T_m = a\alpha^m + b\beta^m + \bar{b}\bar{\beta}^m \quad \text{for all } m \geq 0,$$

where

$$a = \frac{5\alpha^2 - 3\alpha - 4}{22}, \quad \text{and} \quad b = \frac{5\beta^2 - 3\beta - 4}{22}.$$

The minimal polynomial of a over integers is $44X^3 - 2X - 1$, with zeros a, b, and $\bar{b}$ with $\max\{|a|, |b|, |\bar{b}|\} < 1$. We have the numerical estimates

$$\begin{aligned} 1.83 < \alpha &< 1.84; \\ 0.73 < |\beta| = \alpha^{-1/2} &< 0.74; \\ 0.33 < |a| &< 0.34; \\ 0.25 < |b| &< 0.27. \end{aligned} \tag{2}$$

For $m \geq 1$, denoting $e(m) := T_m - a\alpha^m$, we have

$$|e(m)| < \alpha^{-\frac{m}{2}}. \tag{3}$$

Furthermore,

$$\alpha^{m-2} \leq T_m \leq \alpha^{m-1} \qquad \text{for all} \quad m \geq 1. \tag{4}$$

Binet's formula for the Fibonacci sequence is

$$F_n = \frac{\gamma^n - \delta^n}{\sqrt{5}} \quad \text{for all} \quad n \geq 0.$$

One has $\gamma\delta = -1$. Furthermore, for $n \geq 1$, one has

$$\gamma^{n-2} \leq F_n \leq \gamma^{n-1}. \tag{5}$$

For a number field $\mathbb{K}$, let $d_{\mathbb{K}}$ be its degree over $\mathbb{Q}$, also usually denoted by $[\mathbb{K} : \mathbb{Q}]$. Because $d_{\mathbb{Q}(\alpha)} = 3$ and $d_{\mathbb{Q}(\gamma)} = 2$, it follows that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\gamma)$. Further, $\mathbb{Q}(\alpha) = \mathbb{Q}(a)$. The numbers α, γ, and a are positive and belong to the real field $\mathbb{K} = \mathbb{Q}(\alpha, \gamma)$ of degree $d_{\mathbb{K}} = 6$.

2.2. **Auxiliary Results on Linear Forms in Logarithms of Algebraic Numbers.** In this subsection, we point out some useful results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. Let $\eta \neq 0$ be an algebraic number of degree d and let

$$a_0(X - \eta^{(1)}) \cdots (X - \eta^{(d)}) \in \mathbb{Z}[X]$$

be the minimal polynomial over $\mathbb{Z}$ of $\eta = \eta^{(1)}$ with positive leading coefficient $a_0 \geq 1$. Then, the absolute logarithmic Weil height is defined by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \max\{0, \log|\eta^{(i)}|\}\right).$$

The height has the following basic properties. For algebraic numbers η, γ, and $s \in \mathbb{Z}$, we have

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{6}$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{7}$$
$$h(\eta^s) = |s|\, h(\eta) \quad (s \in \mathbb{Z}). \tag{8}$$

In the case that η is a rational number, say $\eta = p/q \in \mathbb{Q}$ with coprime integers p, $q \geq 1$, we have $h(p/q) = \max\{\log|p|, \log q\}$.

Let $\mathbb{K}$ be a real number field, $\eta_1, \ldots, \eta_t \in \mathbb{K}$ and $b_1, \ldots, b_t \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_t|\}$ and put

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1.$$

Let A_1, …, A_t be real numbers with

$$A_i \geq \max\{d_{\mathbb{K}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, t.$$

With these basic notations, we have the following version of a lower bound for a nonzero linear form in logarithms from Matveev [6].

Theorem 2. [3, Theorem 9.4] *Assume that* $\Lambda \neq 0$. *Then,*

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_{\mathbb{K}}^2 \cdot (1 + \log d_{\mathbb{K}}) \cdot (1 + \log B) \cdot A_1 \cdots A_t.$$

We also need the following lemma.

Lemma 1. [5, Lemma 7] *If* $l \geq 1$, $H > \left(4l^2\right)^l$, *and* $H > L/(\log L)^l$, *then*

$$L < 2^l H (\log H)^l.$$

Applying these results to some appropriate linear forms in logarithms resulting from equation (1), we end up with a large upper bound on $\max\{k, m, n\}$, which we need to reduce. For that, we use the following result of Dujella and Pethő [4], which is a variant of the reduction method from Baker and Davenport [1].

Lemma 2. *Let* M *be a positive integer,* p/q *be a convergent of the continued fraction expansion of the irrational number* τ *such that* $q > 6M$, *and* A, B, *and* μ *be some real numbers with* $A > 0$ *and* $B > 1$. *If*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\| > 0,$$

then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v, *and* w *with*

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. The Proof of the Main Result

Let (m, n, k) be a solution of Diophantine equation (1). We suppose $2 \leq n \leq k$ and $m \geq 2$ because $F_1 = F_2 = T_1 = T_2 = 1$. We assume that $n \geq 3$, because for $n = 2$, we get $T_m = F_k$ and the only common terms of the Fibonacci and Tribonacci sequence are 1, $2 = F_3 = T_3$, and $13 = F_7 = T_6$ (see, for example, Theorem 1 in [2] for a more general result). From now on, we assume $k \geq n \geq 3$ and $m \geq 4$.

From (4) and (5), we have

$$\alpha^{m-2} < T_m = F_n F_k < \gamma^{n+k-2} \quad \text{and} \quad \gamma^{n+k-4} < \alpha^{m-1}.$$

This implies that

$$\frac{\log\gamma}{\log\alpha}(n+k) - 2.2 < m < \frac{\log\gamma}{\log\alpha}(n+k) + 0.5. \tag{9}$$

Furthermore, from equation (1), we have

$$\begin{aligned}\left|\frac{5a\alpha^m}{\gamma^{n+k}}-1\right| &= \left|-5e(m)\gamma^{-(n+k)}-(-1)^n\delta^{2n}-(-1)^k\delta^{2k}+(-1)^{n+k}\delta^{2(n+k)}\right|\\ &< 5|e(m)|\gamma^{-(n+k)}+|\delta|^{2n}+|\delta|^{2k}+|\delta|^{2(n+k)}\\ &< 5\alpha^{\frac{-m}{2}}\gamma^{-(n+k)}+\gamma^{-2n}+\gamma^{-2k}+\gamma^{-2(n+k)}\\ &< (5\alpha^{-1}+3)\gamma^{-2n},\end{aligned}$$

because $m \geq 4$ and $k \geq n$. We thus obtain

$$\left|\frac{5a\alpha^m}{\gamma^{n+k}}-1\right| < 5.74\gamma^{-2n}. \tag{10}$$

Let $\Lambda_1 := 5a\alpha^m\gamma^{-n-k}-1$. We have $\Lambda_1 \neq 0$. To see this, if $\Lambda_1 = 0$, then

$$5a \cdot \alpha^m \cdot 2^{n+k} = (1+\sqrt{5})^{n+k} \in \mathbb{Q}(\sqrt{5}),$$

which is false ($a\alpha^m$ is a real number larger than 1 that has two complex conjugates of absolute values smaller than 1, which cannot be in the normal real field $\mathbb{Q}(\sqrt{5})$).

We can now apply Theorem 2 to Λ_1 with $t := 3$ and

$$(\eta_1, b_1) := (\alpha, m), \quad (\eta_2, b_2) := (\gamma, -n-k), \quad \text{and} \quad (\eta_3, b_3) := (5a, 1).$$

We compute upper bounds on the absolute logarithmic Weil height of each of the above algebraic numbers and we get

$$\begin{aligned} h(\eta_1) &= h(\alpha) < 0.204,\\ h(\eta_2) &= h(\gamma) < 0.241,\\ h(\eta_3) &= h(5a) \leq \log 5 + h(a) < 2.871.\end{aligned} \tag{11}$$

We have $d_{\mathbb{K}} = 6$ and we can choose

$$A_1 = 1.23, \quad A_2 = 1.45, \quad A_3 = 17.23, \quad \text{and} \quad B = n+k.$$

We then get

$$\log|\Lambda_1| > -8.85 \cdot 10^{14}\log(n+k).$$

In the above, we used that $1+\log(n+k) < 2\log(n+k)$ because $n+k \geq 4$. Combining the above bound with inequality (10), we obtain

$$2n\log\gamma < 8.85\cdot 10^{14}\log(n+k)+\log 5.74 < 8.85\cdot 10^{14}\log(n+k)+1.748,$$

so

$$n\log\gamma < 4.43\cdot 10^{14}\log(n+k). \tag{12}$$

We rewrite equation (1) as

$$a\alpha^m + e(m) = F_n\left(\frac{\gamma^k-\delta^k}{\sqrt{5}}\right).$$

Because $n \geq 3$, we have

$$\left|\frac{a\alpha^m}{F_n} - \frac{\gamma^k}{\sqrt{5}}\right| < \frac{|e(m)|}{F_n} + \frac{|\delta|^k}{\sqrt{5}}$$

$$\left|\frac{\sqrt{5}a\alpha^m}{F_n\gamma^k} - 1\right| < \frac{\sqrt{5}}{\gamma^k}\left(\frac{1}{F_n} + \frac{1}{\sqrt{5}\gamma^k}\right)$$

$$< \left(\frac{\sqrt{5}}{2} + \frac{1}{\gamma^3}\right)\gamma^{-k},$$

because $k \geq 3$ and $F_n \geq 2$. Thus, putting

$$\Lambda_2 := (\sqrt{5}a/F_n)\alpha^m\gamma^{-k} - 1,$$

we obtain

$$|\Lambda_2| < 1.4\gamma^{-k}. \tag{13}$$

Of course, we have $\Lambda_2 \neq 0$ again because $a\alpha^m \notin \mathbb{Q}(\sqrt{5})$. So, we can apply Theorem 2 again to Λ_2 with $t := 3$ and

$$(\eta_1, b_1) := (\alpha, m), \quad (\eta_2, b_2) := (\gamma, -k), \quad \text{and} \quad (\eta_3, b_3) := \left(\frac{\sqrt{5}a}{F_n}, 1\right).$$

We have

$$h\left(\frac{\sqrt{5}a}{F_n}\right) \leq h(\sqrt{5}) + h(a) + \log(F_n) < h(a) + \log(\sqrt{5}) + \log\left(\gamma^{n-1}\right) < 2.5n\log\gamma,$$

where the last inequality above holds because $n \geq 3$. We again have $d_{\mathbb{K}} = 6$ and we can choose

$$A_1 = 1.23, \quad A_2 = 1.45, \quad \text{and} \quad A_3 = 15n\log\gamma,\ B = n + k$$

and get

$$\log|\Lambda_2| > -7.7 \cdot 10^{14} n \log\gamma \log(n + k).$$

Combining this with inequality (13), we obtain

$$k\log\gamma < 7.7 \cdot 10^{14} \cdot n\log\gamma\log(n + k) + \log(1.35)$$

leading to

$$k\log\gamma < 7.71 \cdot 10^{14}(n\log\gamma)\log(n + k). \tag{14}$$

Therefore, considering the upper bound of $n\log\gamma$ from inequality (12), we get

$$k < \left(\frac{(7.71 \cdot 10^{14}) \cdot (4.43 \cdot 10^{14})}{\log\gamma}\right)\log^2(n + k) < 7.1 \cdot 10^{29}\log^2(2k).$$

or

$$2k < 14.2 \cdot 10^{30}\log(2k)^2.$$

Applying Lemma 1 with $l = 2$, $L = 2k$, and $H = 14.2 \cdot 10^{30}$, we obtain

$$k < 1.37 \cdot 10^{34}. \tag{15}$$

We next reduce the above bound by applying Lemma 2. Recall that for a positive real x, if $|x - 1| < 0.5$, then $|\log x| < 1.5\,|x - 1|$ (see [7, Lemma 4]). Hence, we have from inequality (10) (note that because $n \geq 3$ the right side of inequality (10) is indeed smaller than 0.5),

$$0 < |m\log\alpha - (n + k)\log\gamma + \log(5a)| < 8.61\gamma^{-2n},$$

which implies that

$$0 < \left| m\frac{\log\alpha}{\log\gamma} - (n+k) + \frac{\log(5a)}{\log\gamma} \right| < 17.893\gamma^{-2n}. \tag{16}$$

Note that α and γ are multiplicatively independent. Indeed, if $\alpha^x = \gamma^y$ for integers x and y, then this common value lives in $\mathbb{Q}(\alpha)\cap\mathbb{Q}(\gamma) = \mathbb{Q}$ because $\mathbb{Q}(\alpha)$ has degree 3 and $\mathbb{Q}(\gamma)$ has degree 2. Thus, $\alpha^x = \gamma^y$ is a rational unit so it is ± 1, and this is possible only when $x = y = 0$. Thus, $\log\alpha/\log\gamma$ is an irrational. From inequalities (9) and (15), we have

$$m < \frac{\log\gamma}{\log\alpha}\cdot(2\cdot k) + 0.5 < \frac{\log(0.5+0.5\sqrt{5})}{\log 1.83}\cdot(2\cdot 13700+0.5)\cdot 10^{30};$$

i.e.,

$$m < 2.22\cdot 10^{34}. \tag{17}$$

We apply Lemma 2 with $w := 2n$,

$$\tau := \frac{\log\alpha}{\log\gamma}, \quad \mu := \frac{\log(5a)}{\log\gamma}, \quad A := 17.893, \quad B := \gamma, \quad \text{and} \quad M := 2.22\times 10^{34}.$$

With the help of Mathematica, we find that the 71st convergent of τ is

$$\frac{p_{71}}{q_{71}} = \frac{4525445232205414399824110390796611113}{3573641069135323348796366297377338770}.$$

Its denominator satisfies $q_{71} > 6M$ and $\varepsilon > 0.247647 > 0$. Thus, inequality (16) has no solution for

$$2n \geq \frac{\log(17.893 q_{71}/\varepsilon)}{\log\gamma} > 179.01,$$

which implies that

$$n \leq 89.$$

Substituting this upper bound for n into inequality (14), we obtain

$$k < 6.9\cdot 10^{16}\log(n+k) \leq 6.9\cdot 10^{16}\log(2k).$$

We again apply Lemma 1 with $l = 1$, $H = 13.8\cdot 10^{16}$, and $L = 2k$ and get

$$k < 5.49\cdot 10^{18}.$$

From here, because $n \leq k$ and from inequality (9), we have

$$m < 8.75\cdot 10^{18}.$$

We consider Λ_2 and we get from inequality (13)

$$0 < \left| m\log\alpha - k\log\gamma + \log\left(\frac{\sqrt{5}a}{F_n}\right) \right| < 3\gamma^{-k}.$$

This implies that

$$0 < \left| m\frac{\log\alpha}{\log\gamma} - k + \frac{\log\left(\sqrt{5}a/F_n\right)}{\log\gamma} \right| < 7\gamma^{-k}. \tag{18}$$

We then apply Lemma 2 with $w := k$,

$$\tau := \frac{\log\alpha}{\log\gamma}, \quad \mu := \frac{\log\left(\sqrt{5}a/F_n\right)}{\log\gamma}, \quad A := 7, \quad B := \gamma, \quad M := 8.75\cdot 10^{18}.$$

With the help of Mathematica, we find that the 41st convergent of τ is

$$\frac{p_{41}}{q_{41}} = \frac{237161759629456603958}{187281242121494666147}.$$

It satisfies $q_{41} > 6M$ and $\varepsilon > 0.010633 > 0$ for all $n \in [3, 89]$. Hence, inequality (18) has no solution for

$$k \geq \frac{\log(7q_{41}/\varepsilon)}{\log \gamma} > 110.4.$$

Thus, we obtain $k \leq 110$ and consequently $m \leq 175$ by inequality (9). We now check for the solutions of equation (1) for $3 \leq n \leq 89$, $n \leq k \leq 110$, and $4 \leq m \leq 175$. This is done quickly with a small program in Mathematica and we get

$$(m, k, n) \in \{(4,\ 3,\ 3),\ (7,\ 6,\ 4)\}.$$

This yields the additional Tribonacci numbers $T_4 = 4 = F_3^2$ and $T_7 = 24 = F_4 \cdot F_6$, which are products of two Fibonacci numbers from the statement of Theorem 1. This finishes the proof.

Acknowledgment

We thank the anonymous referee for carefully reading our manuscript and making valuable suggestions. The second author worked on this paper during a visit to the School of Maths of Wits University in Johannesburg, South Africa, in August 2022 partially supported by an EMS (European Mathematical Society) collaboration grant. This author thanks the School for its hospitality and support.

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MSC2020: 11D45, 11B37, 11D61

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