A TRIPLE FIBONACCI SUM AND SOME HISTORICAL NOTES

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Abstract. We study a parametric sum over a triple product of Fibonacci numbers, derive a closed-form expression, show how this allows for a unified solution of some recent elementary problems in *The Fibonacci Quarterly*, and mention their relation to identities that date back to 1953 by K. Subba Rao.

1. Introduction

In this note, we study a parametric sum of a triple product of Fibonacci numbers and show how this sum relates to results published in 1953 by K. Subba Rao, one of the early contributors to the [then small] set of identities for Fibonacci numbers. The author's motivation to study these sums came from a problem that was posed recently by Davenport in the *Elementary Problems* column of this journal [2].

2. Parametric Sum

Let a and n be nonnegative integers, t an arbitrary real or complex number, and define

$$S_a(t,n)=\sum_{i=0}^{n} t^i F_{i-a}F_iF_{i+a}. \tag{2.1}$$

Note that the Fibonacci numbers are defined at negative indices by extending their recurrence relation backward and that they satisfy the relation $F_{-n}=(-1)^{n+1}F_n$. Thus, there is no need to curtail the range of summation to keep the indices of the Fibonacci numbers nonnegative.

Applying Catalan's identity,[1] given by $F_i^2-F_{i-a}F_{i+a}=(-1)^{i-a}F_a^2$, to the summand in (2.1), we obtain

$$S_a(t,n)=S_0(t,n)-(-1)^aF_a^2\sum_{i=0}^{n}(-t)^iF_i. \tag{2.2}$$

Now use the identity $3xy(x+y)=(x+y)^3-x^3-y^3$, and take x and y as consecutive Fibonacci numbers, to give $3F_{i-1}F_iF_{i+1}=F_{i+1}^3-F_{i-1}^3-F_i^3$, valid for all integers i. Multiply both sides by t^i, sum over i from zero to n, and simplify the result to give

$$3S_1(t,n)=t^nF_{n+1}^3+t^{n+1}F_n^3-1+\left(t^{-1}-t-1\right)S_0(t,n).$$

Combining this with (2.2) for $a=1$ gives two identities in $S_0(n)$ and $S_1(n)$, and we can solve for $S_0(n)$. Substitution of the result in (2.2) gives

$$\left(4-t^{-1}+t\right)S_a(t,n)=t^nF_{n+1}^3+t^{n+1}F_n^3-1-\left(3+\left(4-t^{-1}+t\right)(-1)^aF_a^2\right)\sum_{i=0}^{n}(-t)^iF_i. \tag{2.3}$$

[1]Some historical notes on Catalan's identity can be found in a recent paper by the current author [12].

This can be turned into a closed-form expression by means of the identity

$$\left(t^2+t-1\right)\sum_{i=0}^{n} t^i F_i = F_n t^{n+2} + F_{n+1}t^{n+1} - t, \tag{2.4}$$

that in turn is easily proven by induction.

3. Applications

By taking $a=1$ in (2.2), we obtain the difference $S_1 - S_0 = \sum_{i=0}^{n}(-t)^i F_i$. This allows us to eliminate the summation in (2.2) and derive the principal result

$$S_a = S_0 - (-1)^a F_a^2 \left(S_1 - S_0\right), \tag{3.1}$$

where we suppressed the parameters t and n to more clearly exhibit the structure. This shows that any of the sums $S_a(t,n)$ can be written as an affine combination of $S_0(t,n)$ and $S_1(t,n)$. To generalize this result, observe that $\left(S_c - S_b\right)S_a + \left(S_a - S_c\right)S_b + \left(S_b - S_a\right)S_c = 0$. Simplify the expressions in brackets using (3.1), divide by $S_1 - S_0$, and arrive at

$$w(b,c)S_a + w(c,a)S_b + w(a,b)S_c = 0, \tag{3.2}$$

where we defined the weights $w(p,q) = (-1)^p F_p^2 - (-1)^q F_q^2$. Note that the sum of the three weights in (3.2) is zero, and that, when a, b, and c are distinct, nonnegative integers, none of the weights can be zero. Identity (3.2) shows that, for any given nonnegative integer a, we can represent $S_a(t,n)$ as an affine combination of $S_b(t,n)$ and $S_c(t,n)$, where b and c are arbitrary nonnegative integers, such that a, b, and c are distinct. We may choose the particular sums $S_b(t,n)$ and $S_c(t,n)$ as the simplest or most conveniently determined sums, or to make use of known results. This representation principle may well hold and be put to good use for other parametric sums involving the Fibonacci numbers, or, for that matter, other number sequences generated by a recurrence.

3.1. **Elementary Problems.**

Let us return to the setting that inspired this note. For $t=\pm 1$, the expressions (2.3) and (2.4) simplify considerably. In particular, they yield the quartet

$$4\sum_{i=0}^{n} F_i^3 = F_{n+1}^3 + F_n^3 - 3(-1)^n F_{n-1} + 2, \tag{3.3a}$$

$$4\sum_{i=0}^{n} (-1)^i F_i^3 = (-1)^n\left(F_{n+1}^3 - F_n^3\right) - 3F_{n+2} + 2, \tag{3.3b}$$

$$4\sum_{i=0}^{n} F_{i-1}F_iF_{i+1} = F_{n+1}^3 + F_n^3 + (-1)^n F_{n-1} - 2, \tag{3.3c}$$

$$4\sum_{i=0}^{n} (-1)^i F_{i-1}F_iF_{i+1} = (-1)^n\left(F_{n+1}^3 - F_n^3\right) + F_{n+2} - 2. \tag{3.3d}$$

In identity (3.3a), we have an alternative solution to elementary problem B-1211 by Ohtsuka [5]. Identity (3.3b) provides an alternative closed-form expression for the one given by Zeitlin [13, Eqn. (XXI)]. In identities (3.3c) and (3.3d), we have alternative solutions to elementary problems B-1235 and B-1318 by Davenport [1, 2], respectively. For reference, the problems B–1211, B-1235, and B-1318, and Zeitlin's closed-form expression are given in Appendix B. The sequences in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [4], associated with the quartet (3.3), can be found in entries A005968, A119284, A215037, and A363753, respectively.

3.2. **General Sums.** Now that we have obtained expressions for the particular sums S_0 and S_1, for $t = \pm 1$, we can use (3.1) to derive expressions for the general sums. This gives the pair

$$\begin{aligned} 4\sum_{i=0}^{n} F_{i-a}F_iF_{i+a} &= F_{n+1}^3 + F_n^3 - \left[3 + 4(-1)^a F_a^2\right](-1)^n F_{n-1} + 2 + 4(-1)^a F_a^2, \\ 4\sum_{i=0}^{n} (-1)^i F_{i-a}F_iF_{i+a} &= (-1)^n\left(F_{n+1}^3 - F_n^3\right) - \left[3 + 4(-1)^a F_a^2\right] F_{n+2} + 2 + 4(-1)^a F_a^2. \end{aligned}$$

If we were traditionalists and prefer to use previously published results, we would use (B.1a) and (B.1c) as the particular sums to derive an expression for $S_a(1, n)$. We can also combine the new and old, and use (3.3b) and (B.1d) as the particular sums to derive an expression for $S_a(-1, n)$.

3.3. **Lacunary Sums.** By adding and subtracting identities (3.3a) and (3.3b), we can easily derive closed-form expressions for the even and odd indexed sums of cubed Fibonacci numbers. We can do the same [adding and subtracting] with identities (3.3c) and (3.3d). This yields the quartet

$$4\sum_{i=0}^{n} F_{2i}^3 = F_{2n+1}^3 - 3F_{2n+1} + 2, \tag{3.4a}$$

$$4\sum_{i=0}^{n} F_{2i+1}^3 = F_{2n+2}^3 + 3F_{2n+2}, \tag{3.4b}$$

$$4\sum_{i=0}^{n} F_{2i-1}F_{2i}F_{2i+1} = F_{2n+1}^3 + F_{2n+1} - 2, \tag{3.4c}$$

$$4\sum_{i=0}^{n} F_{2i}F_{2i+1}F_{2i+2} = F_{2n+2}^3 - F_{2n+2}. \tag{3.4d}$$

The sequences in the OEIS, associated with this quartet of identities, can be found in entries A163198, A163200, A363754, and A256178, respectively.

3.4. **General Lacunary Sums.** The principle that we used to construct the general sums from the quartet (3.3) also works for the quartet (3.4), although we have to invoke the more general Theorem A.1. Using identity (A.2), first combining (3.4a) with (3.4c), and then combining (3.4b) with (3.4d), yields the pair

$$\begin{aligned} 4\sum_{i=0}^{n} F_{2i-a}F_{2i}F_{2i+a} &= F_{2n+1}^3 - \left[3 + 4(-1)^a F_a^2\right] F_{2n+1} + 2 + 4(-1)^a F_a^2, \\ 4\sum_{i=0}^{n} F_{2i+1-a}F_{2i+1}F_{2i+1+a} &= F_{2n+2}^3 + \left[3 + 4(-1)^a F_a^2\right] F_{2n+2}. \end{aligned}$$

It is interesting to note that, for all values of a, these sums are [depressed] cubic polynomials in F_{2n+1} and F_{2n+2}, respectively.

4. Historical Notes

Identity (3.3a) and the quartet of identities in (3.4) go back to 1953 and appear in the *American Mathematical Monthly* in an article by K. Subba Rao [8, p. 682]. The identities (3.3b), (3.3c), and (3.3d) are implicit in [and easily derived from] the five identities that Subba Rao gives. Subba Rao was only a tiny step away from deriving the complete octet (3.3) and (3.4), and it is curious that he only gave five out of the eight identities that were clearly within his reach. For the reader wanting to dig deeper into his 1953 article, note that he uses the symbol U_n, corresponding to an older notation and convention for the Fibonacci sequence to start with $U_0 = 1$ and $U_1 = 1$, so that $U_n = F_{n+1}$ in today's notation.

Little is known about the Indian mathematician Karri Subba Rao, his background, and how he became interested in the Fibonacci numbers. In the 1950s,[2] he published five articles on the Fibonacci numbers and their properties [7–11], listing his affiliation as Maharajah's College in Vizianagaram, one of the oldest colleges (established in 1879) in India. He also served as head of the mathematics department of the College. The current author would welcome any additional information about Subba Rao, his life, and works.

5. Epilogue

As indicated in Section 3, we naturally wonder if results similar to (3.1) hold for other sequences of numbers, such as the Lucas numbers. To see that it does, let a and n be nonnegative integers, t an arbitrary real or complex number, and define $\bar{S}_a(t,n) = \sum_{i=0}^{n} t^i L_{i-a} L_i L_{i+a}$. The Lucas numbers are well defined for negative indices and satisfy $L_{-n} = (-1)^n L_n$. The equivalent of Catalan's identity for the Lucas numbers is $L_{i-a}L_{i+a} = L_i^2 + ((-1)^a L_a^2 - 4)(-1)^i$, and is easily proven by the Binet formula for the Lucas numbers. Using suppressed notation, this gives $\bar{S}_a = \bar{S}_0 - \frac{1}{5}\left((-1)^a L_a^2 - 4\right)\left(\bar{S}_1 - \bar{S}_0\right)$, and shows that $\bar{S}_a$ is an affine combination of $\bar{S}_0$ and $\bar{S}_1$. We can apply the same analysis that was done for the parametric sum $S_a(t,n)$ to the parametric sum $\bar{S}_a(t,n)$ and derive similar sets of summation identities for the Lucas numbers.

The same ideas can also be applied to sums over the product of four [or more] Fibonacci numbers. For instance, consider the sum $S_{a,b}(t,n) = \sum_{i=0}^{n} t^i F_{i-b}F_{i-a}F_{i+a}F_{i+b}$, where a, b, and n are nonnegative integers, and t an arbitrary real or complex number. The key for this particular sum is the identity[3]

$$F_{i-b}F_{i-a}F_{i+a}F_{i+b} = F_i^4 - (-1)^i\left[(-1)^a F_a^2 + (-1)^b F_b^2\right] F_i^2 + (-1)^{b-a} F_a^2 F_b^2 \tag{5.1}$$

that readily follows from a double application of Catalan's identity. We can leverage identity (5.1) to generalize the results by David Zeitlin (1924–1996), who published new identities in 1963 for various products of four Fibonacci numbers and closed-form expressions for their associated sums [13]. Zeitlin was an active contributor to the *Elementary Problems* column and *The Fibonacci Quarterly* since its inception in 1963 [6].

6. Acknowledgment

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[2]There are some number-theoretical articles published in the 1930s by a K. Subba Rao from Madras, India, but I have not been able to definitively attribute them to the "Fibonacci" Subba Rao. If these articles are by his hand, they are from an earlier stage in his career and might be from his student days.

[3]Identity (5.1) can be seen as a generalization of the Gelin-Cesàro identity that relates five consecutive Fibonacci numbers in the form $F_{i-2}F_{i-1}F_{i+1}F_{i+2} = F_i^4 - 1$, valid for all integers i. The identity dates to 1880 and was stated by the Belgian abbott Emile Gelin (1851–1921) [3]. A proof was provided in the same year by the Italian mathematician Ernesto Cesàro (1859–1906).

Appendix A. Generalizing the Index

The parametric sum that we studied in Section 2 has summands that are indexed over consecutive integers. Many variations of the sum of a triple product of Fibonacci numbers involve lacunary sequences, where the index follows an arithmetic progression, such as the sum of the cube of the even or odd indexed Fibonacci numbers. It turns out that the principle that we enunciated in Section 3 holds more generally.

Theorem A.1. *Let a and n be nonnegative integers, t an arbitrary real or complex number, $\{\mu_i\}$ a sequence of integers, and define*

$$S_a(t,n;\mu) = \sum_{i=0}^{n} t^i F_{\mu_i - a} F_{\mu_i} F_{\mu_i + a}. \tag{A.1}$$

Then,

$$S_a = S_0 - (-1)^a F_a^2 \left(S_1 - S_0\right), \tag{A.2}$$

where we suppressed the parameters t and n, and the index μ_i to more clearly exhibit the structure.

The proof is easy and does not need to be spelled out; it directly follows the proof of (2.2) and (3.1), but using Catalan's identity with μ_i instead of i. Likewise, using the suppressed notation, identity (3.2) also holds for $S_a(t,n;\mu)$.

Appendix B. Published Identities

For ease of reference, we provide the previously published identities for the sums in (3.3). The second identity is from 1963, a decade after Subba Rao published his identities [8], and given as identity (XXI) by Zeitlin, who mentions that it is readily established by the method of mathematical induction [13]. The other identities appear, well over a semicentury after Zeitlin, in the proposals of the elementary problems B-1211, B-1235, and B-1318 in *The Fibonacci Quarterly* [1, 2, 5].

$$\sum_{i=1}^{n} F_i^3 = \frac{F_{3n-1}+1}{2} - F_{n-1}^3, \tag{B.1a}$$

$$2\sum_{i=1}^{n} (-1)^i F_i^3 = (-1)^n F_{n-1} F_{n+1}^2 - F_{n+3} + 1, \tag{B.1b}$$

$$3\sum_{i=1}^{n} F_{i-1} F_i F_{i+1} = F_{n-1}^3 + F_n^3 + F_{n+1}^3 - \frac{F_{3n-1}+3}{2}, \tag{B.1c}$$

$$\sum_{i=1}^{n} (-1)^i F_{i-1} F_i F_{i+1} = \frac{(-1)^n}{12}\left(-4F_n^3 - F_{n+1}^3 + 2F_{n+2}^3 + 4F_{n+3}^3 - F_{n+4}^3\right) - \frac{1}{2}. \tag{B.1d}$$

These identities all have the condition $n \geq 1$, to avoid negatively indexed Fibonacci numbers, and necessitates their summations to start at the index $i = 1$. This is not strictly necessary. As $F_{-1} = 1$, these identities remain valid when we take n to be a nonnegative integer and start the summation at index $i = 0$.

The identities, appearing in the proposals of the elementary problems B-1211, B-1235, and B-1318, are equivalent to their counterparts in (3.3). This can be established by linearizing the cube of the Fibonacci numbers, by means of the identity $5F_n^3 = F_{3n} - 3(-1)^n F_n$, and simplifying the resulting expression.

References

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