BINOMIAL SUMS INVOLVING SECOND-ORDER LINEARLY RECURRENT SEQUENCES

JOHN M. CAMPBELL AND EMRAH KILIÇ

Abstract. Consider the sequences $(U_n : n \in \mathbb{N}_0)$ and $(V_n : n \in \mathbb{N})$ satisfying the second-order linear recurrences $U_n = pU_{n-1} + U_{n-2}$ and $V_n = pV_{n-1} + V_{n-2}$ with the initial conditions $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = p$. We explore the problem of evaluating binomial sums involving products consisting of entries in the U- and V-sequences. We apply a hypergeometric approach, inspired by Dilcher's work on hypergeometric identities for Fibonacci numbers, to obtain many new identities for sums involving U and V and products of binomial coefficients, including a non-hypergeometric analogue of Dixon's binomial identity.

1. Introduction

The recursion

$$F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

satisfied by the Fibonacci sequence $(F_n : n \in \mathbb{N}_0)$ is among the most famous and the most basic recurrences in all of mathematics. Second-order linear recurrences generalizing (1.1) form a main object of study in number-theoretic areas of research concerning the Fibonacci sequence. In this regard, past research concerning such recurrences, as in [3, 6, 7, 8, 9, 10, 11, 13, 14], motivates further explorations based on new properties of integer sequences generalizing the Fibonacci sequence and the Lucas sequence $(L_n : n \in \mathbb{N}_0)$. In this article, we introduce identities and techniques for summing products of entries of second-order linearly recurrent sequences and binomial coefficients.

We define $(U_n : n \in \mathbb{N}_0)$ and $(V_n : n \in \mathbb{N}_0)$ according to the second-order linear recurrences given as follows for a parameter p, and we enforce the initial conditions such that $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = p$:

$$U_n = pU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} + V_{n-2}. \tag{1.2}$$

If $p = 1$, then $U_n = F_n$ and $V_n = L_n$. The Binet formulas for U and V are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.3}$$

where

$$\alpha = \frac{p + \sqrt{\triangle}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{\triangle}}{2}, \tag{1.4}$$

with $\triangle = p^2 + 4$. This article is mainly concerned with terminating sums containing products of binomial coefficients with U- or V-entries. For the case whereby there is only one binomial factor, then such sums typically may be evaluated using (1.3), as we illustrate below. In contrast, the situation becomes much more difficult if there are multiple binomial factors, as we later consider.

Throughout this paper, $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$. Experimentally, we have discovered that

$$\sum_{k=0}^{n} \binom{n}{k} U_{k+2r-1} U_{k+2r+1} = \frac{1}{\Delta} \sum_{k=0}^{n} \binom{n}{k} V_{k+2r-1} V_{k+2r+1} \tag{1.5}$$

$$= \Delta^{\lfloor \frac{n-1}{2} \rfloor} \begin{cases} V_{n+4r}, & \text{if } n \text{ is even;} \\ U_{n+4r}, & \text{if } n \text{ is odd;} \end{cases} \tag{1.6}$$

for $n \in \mathbb{N}_0$, where Δ is defined previously. We claim that the above indicated identities may be proved in a direct way using the Binet-type formulas in (1.3), in contrast to our new formulas involving multiple binomial factors. This is illustrated below.

For the even case, the desired evaluation for the left side sum is

$$\sum_{k=0}^{2n} \binom{2n}{k} U_{k+2r-1} U_{k+2r+1} = \left(p^2+4\right)^{n-1} V_{2n+4r}.$$

So, it remains to prove that

$$\sum_{k=0}^{2n} \frac{\binom{2n}{k} \left(\alpha^{k+2r-1} - \beta^{k+2r-1}\right) \left(\alpha^{k+2r+1} - \beta^{k+2r+1}\right)}{(\alpha-\beta)(\alpha-\beta)} = \tag{1.7}$$

$$\left(p^2+4\right)^{n-1} \left(\alpha^{2n+4r} + \beta^{2n+4r}\right). \tag{1.8}$$

By the binomial theorem, we may evaluate the finite sum given above as

$$\frac{-\left(\alpha^2+\beta^2\right) \alpha^{2r} \beta^{2r} \left((\alpha\beta+1)^2\right)^n + \beta \left(\left(\alpha^2+1\right)^2\right)^n \alpha^{4r+1} + \alpha \left(\left(\beta^2+1\right)^2\right)^n \beta^{4r+1}}{\alpha\beta(\alpha-\beta)^2},$$

so that it remains a matter of routine to verify that the above expression may be reduced to (1.8). A similar approach may be applied to the remaining cases for the equalities in (1.5)–(1.6). In contrast, we introduce explicit evaluations for summations for which the binomial theorem cannot be applied directly, as in the terminating sum

$$\sum_{k=0}^{2n} \binom{2n}{k}^2 (-1)^k U_{2n-k} V_k \tag{1.9}$$

considered in the following section. Finite sums such as $\sum_{k=0}^{2n} \binom{2n}{k}^2 x^k$ and $\sum_{k=0}^{2n} \binom{2n}{k}^3 x^k$ do not admit closed forms for a free variable x, so Binet-type formulas cannot be applied directly to sums as in (1.9), which motivates the hypergeometric approaches we employ in Section 2 below. In Section 3, we conclude by introducing summation identities with single binomial factors in the vein of (1.5)–(1.6).

1.1. **Fibonacci numbers and hypergeometric series.** Dilcher's work on hypergeometric functions and Fibonacci number identities [5], with related work as in [1, 15], motivates our application of classic hypergeometric identities toward the problem of proving new, experimentally discovered identities as in (1.9).

The famous Γ-function is such that $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du$ for $\Re(x) > 0$ [12, Section 2]. This leads us to define the Pochhammer symbol so that $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, and we adopt the notational shorthands

$$\begin{bmatrix} \alpha, \beta, \ldots, \gamma \\ A, B, \ldots, C \end{bmatrix}_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}$$

and

$$\Gamma\begin{bmatrix}\alpha,\beta,\ldots,\gamma\\A,B,\ldots,C\end{bmatrix}=\frac{\Gamma(\alpha)\,\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\,\Gamma(B)\cdots\Gamma(C)}.$$

Generalized hypergeometric series [2] are denoted by

$${}_pF_q\left[\begin{matrix}a_1,a_2,\ldots,a_p\\b_1,b_2,\ldots,b_q\end{matrix}\,\middle|\,x\right]=\sum_{n=0}^{\infty}\begin{bmatrix}a_1,a_2,\ldots,a_p\\b_1,b_2,\ldots,b_q\end{bmatrix}_n\frac{x^n}{n!}. \tag{1.10}$$

The work of Dilcher [5] on the Fibonacci sequence and hypergeometric series mainly concerned the expression of Fibonacci numbers with (1.10), as in the identities

$$F_n=\frac{n}{2^{n-1}}\,{}_2F_1\left[\begin{matrix}\frac{1-n}{2},\frac{2-n}{2}\\\frac{3}{2}\end{matrix}\,\middle|\,5\right]$$

and

$$F_{2n+1}=(-1)^n(2n+1)\,{}_2F_1\left[\begin{matrix}-n,n+1\\\frac{3}{2}\end{matrix}\,\middle|\,\frac{5}{4}\right]$$

given in [5]. Instead of expressing Fibonacci sequence entries with ${}_pF_q$-series, we explore the use of ${}_pF_q$-identities to evaluate summations as in (1.9) with summands given by products of hypergeometric expressions and entries of the U- and V-sequences satisfying the second-order recurrences in (1.2).

To prove our new evaluation for (1.9), we make use of Kummer's hypergeometric theorem, which is

$${}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix}\,\middle|\,-1\right]=\Gamma\begin{bmatrix}1+\frac{a}{2},1+a-b\\1+a,1+\frac{a}{2}-b\end{bmatrix}, \tag{1.11}$$

if $\Re(b)<\frac{1}{2}$ and $a-b\neq-1,-2,\ldots$ [2, Section 2.3]. A terminating version of Dixon's hypergeometric summation identity [2, Section 3.1] is

$${}_3F_2\left[\begin{matrix}-n,x,y\\1-x-n,1-y-n\end{matrix}\,\middle|\,1\right]=\frac{(1+\ell)_\ell(x+y+\ell)_\ell}{(x+\ell)_\ell(y+\ell)_\ell}\chi(n=2\ell).$$

As below, *trinomial* coefficients are

$$\binom{n}{a,b,c}=\frac{n!}{a!b!c!}.$$

The following well known binomial sum identity is a special case of the above identity.

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k=(-1)^n\binom{3n}{n,n,n}. \tag{1.12}$$

This is to be applied in our article, in which an analogue of (1.12) involving generalizations of Fibonacci/Lucas numbers is introduced and proved.

2. New Binomial Sums

In view of the Binet formulas in (1.3) and the finite sum in (1.9), we are to make use of the equivalence

$$\sum_{k=0}^{2n}\binom{2n}{k}^2(-1)^kx^k={}_2F_1\left[\begin{matrix}-2n,-2n\\1\end{matrix}\,\middle|\,-x\right] \tag{2.1}$$

for $n\in\mathbb{N}_0$.

Theorem 2.1. *The identity*

$$\sum_{k=0}^{2n} \binom{2n}{k}^2 (-1)^k U_{2n-k} V_k = (-1)^n \binom{2n}{n} U_{2n}$$

holds for $n \in \mathbb{N}_0$.

Proof. The $x = 1$ case of (2.1), is equivalent, by the Kummer identity in (1.11), to

$$\sum_{k=0}^{2n} \binom{2n}{k}^2 (-1)^k = (-1)^n \binom{2n}{n}. \tag{2.2}$$

According to the Binet formulas in (1.3), it remains to evaluate, in closed form

$$\sum_{k=0}^{2n} \frac{\binom{2n}{k}^2 (-1)^k \left(\alpha^{2n-k} - \beta^{2n-k}\right)\left(\alpha^k + \beta^k\right)}{\alpha - \beta} \tag{2.3}$$

for α and β as in (1.4). According to the equivalence in (2.1) with the special case of Kummer's theorem shown in (2.2), we may rewrite (2.3) as

$$\frac{1}{\alpha-\beta}\left(\frac{4^n \Gamma\left(n+\frac{1}{2}\right)\left(\left(-\alpha^2\right)^n - \left(-\beta^2\right)^n\right)}{\sqrt{\pi}\Gamma(n+1)} - \beta^{2n} {}_2F_1\left[\begin{matrix} -2n, -2n \\ 1 \end{matrix} \,\middle|\, -\frac{\alpha}{\beta}\right] + \alpha^{2n} {}_2F_1\left[\begin{matrix} -2n, -2n \\ 1 \end{matrix} \,\middle|\, -\frac{\beta}{\alpha}\right]\right). \tag{2.4}$$

By symmetry, we have that

$${}_2F_1\left[\begin{matrix} -2n, -2n \\ 1 \end{matrix} \,\middle|\, x\right] = x^{2n} {}_2F_1\left[\begin{matrix} -2n, -2n \\ 1 \end{matrix} \,\middle|\, \frac{1}{x}\right] \tag{2.5}$$

for $x \neq 0$, i.e., by rewriting the ${}_2F_1$-sums in (2.5) as finite sums, applying a reindexing argument, and making use of the symmetry of binomial coefficients. This gives us that (2.4) reduces to the right side of

$$\sum_{k=0}^{2n} \binom{2n}{k}^2 (-1)^k U_{2n-k} V_k = \frac{4^n \Gamma\left(n+\frac{1}{2}\right)\left(\left(-\alpha^2\right)^n - \left(-\beta^2\right)^n\right)}{\sqrt{\pi}(\alpha-\beta)\Gamma(n+1)}.$$

From the initial Binet formula in (1.3) with the Legendre duplication formula, this is equivalent to the desired result. □

A similar approach, relative to the above proof, may be used to obtain a Fibonacci-type sum analogue of Dixon's binomial sum. The interest in Dixon's binomial sum motivates the following analogue involving generalizations of $(F_n : n \in \mathbb{N}_0)$ and $(L_n : n \in \mathbb{N}_0)$ and our proof of the following result.

Theorem 2.2. *The identity*

$$\sum_{k=0}^{2n} \binom{2n}{k}^3 (-1)^k U_{2n-k} V_k = (-1)^n \binom{3n}{n, n, n} U_{2n}$$

holds for $n \in \mathbb{N}_0$.

Proof. According to the Binet formulas in (1.3), it remains to evaluate

$$\sum_{k=0}^{2n} \frac{\binom{2n}{k}^3 (-1)^k \left(\alpha^{2n-k} - \beta^{2n-k}\right)\left(\alpha^k + \beta^k\right)}{\alpha - \beta}. \tag{2.6}$$

Using the Dixon identity in (1.12), we find that (2.6) may be rewritten as

$$\frac{1}{\alpha-\beta}\Bigg(\frac{\Gamma(3n+1)\left(\left(-\alpha^2\right)^n - \left(-\beta^2\right)^n\right)}{\Gamma^3(n+1)} - \beta^{2n} {}_3F_2\left[\begin{matrix} -2n, -2n, -2n \\ 1, 1 \end{matrix} \,\middle|\, \frac{\alpha}{\beta}\right] + \alpha^{2n} {}_3F_2\left[\begin{matrix} -2n, -2n, -2n \\ 1, 1 \end{matrix} \,\middle|\, \frac{\beta}{\alpha}\right] \Bigg).$$

A symmetric argument, as in our proof of Theorem 2.1, may be used to complete the proof. □

2.1. **Further Results.** We may mimic the above proofs to obtain many similar results, as in the propositions listed below. In the upcoming Conclusion section, we briefly consider the problem as to how results in the following propositions may be generalized in a systematic way.

Proposition 2.3. *The identity*

$$\sum_{k=0}^{n} \binom{n}{k}^2 U_{n-k} V_k = \binom{2n}{n} U_n$$

holds for $n \in \mathbb{N}_0$.

Letting $C_n = \frac{\binom{2n}{n}}{n+1}$ denote the nth Catalan number, our method, as applied in the proofs for Theorems 2.1 and 2.2, may be applied to obtain new sums involving products of Catalan numbers and expressions of the form $U_k V_{n-k}$.

Proposition 2.4. *The identity*

$$\sum_{k=0}^{n} C_{2n-2k} C_{2k} U_k V_{n-k} = 2^{2n-1} \frac{n+2}{2n+1} C_{n+1} U_n$$

holds for $n \in \mathbb{N}_0$.

Proposition 2.5. *The identity*

$$\sum_{k=0}^{n} C_{n-k+m} C_{k+m} U_k V_{n-k} = 2\binom{2m-1}{m} \frac{(n+m+2)(n+1)}{(n+2m+1)(n+2m+2)} C_{n+m+1} U_n$$

holds for $n, m \in \mathbb{N}_0$.

Proposition 2.6. *The identity*

$$\sum_{k=0}^{n} \binom{2n}{k}\binom{2n}{n+k} U_k V_{n-k} = \binom{4n}{n} U_n$$

holds for $n \in \mathbb{N}_0$.

Proposition 2.7. *The identity*

$$\sum_{k=0}^{2n} \binom{2n}{k}\binom{4n}{2k} (-1)^k U_k V_{2n-k} = (-1)^n \binom{6n}{2n}\binom{2n}{n}\binom{3n}{n}^{-1} U_{2n}$$

holds for $n \in \mathbb{N}_0$.

Proposition 2.8. *The identity*

$$\sum_{k=0}^{2n}\binom{2n}{k}\binom{4n}{k}\binom{4n}{2n+k}(-1)^k U_k V_{2n-k} = (-1)^n\binom{2n}{n}\binom{5n}{2n}U_{2n}$$

holds for $n \in \mathbb{N}_0$.

Proposition 2.9. *The identity*

$$\sum_{k=0}^{2n}\binom{2n}{k}\binom{4n}{k}^{-1}\binom{4n}{2n+k}^{-1}(-1)^k U_k V_{2n-k} = \frac{4n+1}{3n+1}\binom{4n}{2n}^{-1}U_{2n}$$

holds for $n \in \mathbb{N}_0$.

Proposition 2.10. *The identity*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{n+k}\binom{n+k}{k}U_k V_{n-k} = \binom{2n}{n}^2 U_n$$

holds for $n \in \mathbb{N}_0$.

Our method also allows us to obtain Fibonacci-type sum analogues of the main results from [4]. To begin, Theorem 1 from [4] is such that $\sum_{k=0}^{n}\binom{2n}{2k}C_k C_{n-k} = C_n C_{n+1}$, and we can mimic the proofs of Theorems 2.1 and 2.2, with the proof of Theorem 1 from [4], to prove the following proposition.

Proposition 2.11. *The identity*

$$\sum_{k=0}^{n}\binom{2n}{2k}C_k C_{n-k} U_{n-k} V_k = C_n C_{n+1} U_n$$

holds for $n \in \mathbb{N}_0$.

Part (a) of Theorem 2 from [4] gives us that

$$\sum_{k=0}^{\ell}(-1)^k\binom{2\ell}{2k}C_k C_{\ell-k} = (-1)^{\binom{\ell}{2}}C_\ell C_{\ell/2},$$

if ℓ is even and that the above sum vanishes otherwise. The hypergeometric proof of this from [4] can be used, via an analogue of our proofs of Theorems 2.1 and 2.2, to prove the following proposition.

Proposition 2.12. *The identity*

$$\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}C_k C_{2n-k} U_{2n-k} V_k = (-1)^n C_n C_{2n} U_{2n}$$

holds for $n \in \mathbb{N}_0$.

A similar approach may be used to obtain the following proposition, using the proof of Theorem 3 from [4].

Proposition 2.13. *The identity*

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n}{2k}C_k C_{2n-k} U_{2n-k} V_k = (-1)^n\binom{3n+1}{n}C_n C_{2n} U_{2n}$$

holds for $n \in \mathbb{N}_0$.

3. Conclusion

The results given in Section 2 lead us to consider the conditions under which a finite sum identity of the form

$$\sum_{k=0}^{n} F(n,k) = G(n) \tag{3.1}$$

would imply

$$\sum_{k=0}^{n} F(n,k) U_{n-k} V_k = G(n) U_n, \tag{3.2}$$

and similarly for sums like $\sum_{k=0}^{2n} F(n,k)$. For example, if we set $F(n,k) = \binom{n}{k}^2$ and $G(n) = \binom{2n}{n}$, then we find that the desired implication holds, in view of Proposition 2.3. However, the implication suggested by (3.1) and (3.2) does not, in general, hold. For example, if we set $F(n,k) = k$, then (3.1) would not imply (3.2). Similarly, if we set $F(n,k) = \binom{n}{k}k$, then (3.1) would again not imply (3.2). To be able to apply the method given by our proofs of Theorems 2.1 and 2.2, we need to mimic the symmetry-based argument involved in these proofs. We leave it to a separate project to formalize and further explore this notion.

As described above, it is generally much harder to evaluate Fibonacci-type sums involving products of binomial coefficients, as opposed to single binomial factors. However, the single binomial factor case may be worthy of further attention, as suggested by the following results.

For the following results, we let $X \in \{U, V\}$, with $\bar{X} = \{U, V\} \setminus X$, and we recall that $\Delta = p^2 + 4$.

Proposition 3.1. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{4n+1} \binom{4n+1}{k} (-1)^{\binom{k}{2}} U_k V_{4n+1-k}$$
$$= \sum_{k=0}^{4n+1} \binom{4n+1}{k} (-1)^{\binom{k}{2}} V_k U_{4n+1-k} = (-1)^n \, 2^{2n+1} U_{4n+1}.$$

Now, we present the second group sums identities.

Proposition 3.2. *For $n > 0$ and any integers m, r,*

$$\sum_{k=-n}^{n} \binom{2n}{n+k} X_{k+m} X_{k+r} = \Delta^{n-1} V_{m+r} \begin{cases} 1, & \text{if } X = U; \\ \Delta, & \text{if } X = V. \end{cases}$$

and

$$\sum_{k=-n}^{n} \binom{2n}{n+k} X_{k+m} \bar{X}_{k+r} = \Delta^{n} U_{m+r}.$$

Proposition 3.3. *For $n > 0$ and any integers m, r,*

$$\sum_{k=-n}^{n} \binom{2n}{n+k} X_{k+m} X_{k+r} k = np\Delta^{n-1} U_{m+r} \begin{cases} 1, & \text{if } X = U; \\ \Delta, & \text{if } X = V; \end{cases}$$

and

$$\sum_{k=-n}^{n} \binom{2n}{n+k} X_{k+m} \bar{X}_{k+r} k = np\Delta^{n-1} V_{m+r}.$$

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Department of Mathematics, Toronto Metropolitan University, Toronto, Ontario, M3J 1P3, Canada

Email address: jmaxwellcampbell@gmail.com

Department of Mathematics, TOBB University of Economics and Technology, Ankara, 06560, Turkey

Email address: ekilic@etu.edu.tr