MORE SUMS INVOLVING GIBONACCI POLYNOMIAL SQUARES REVISITED

THOMAS KOSHY

Abstract. We explore generalizations of two infinite sums involving a special class of gibonacci polynomial squares, and their implications.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, $2\alpha = x + \Delta$, and $2\beta = x - \Delta$.

It follows by the Binet-like formulas that $\lim\limits_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$ and $\lim\limits_{m\to\infty} \dfrac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 3, 4, 5]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n, \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1.1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n, \\ (-1)^{n+k}\Delta^2 f_r f_{2k}, & \text{otherwise;} \end{cases} \tag{1.2}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas. Identity (1.2) is a gibonacci polynomial extension of *d'Ocagne identity* [2].

2. Telescoping Gibonacci Sums

Using recursion, we will now explore two telescoping gibonacci sums.

Lemma 2.1. *Let k, r, and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(2n-1)k+r}}{g^{\lambda}_{(2n-1)k}} - \frac{g^{\lambda}_{(2n+1)k+r}}{g^{\lambda}_{(2n+1)k}} \right] = \frac{g^{\lambda}_{k+r}}{g^{\lambda}_{k}} - \alpha^{r\lambda}. \tag{2.1}$$

Proof. Using recursion [2, 4], we will first confirm that

$$\sum_{n=1}^{m} \left[\frac{g^{\lambda}_{(2n-1)k+r}}{g^{\lambda}_{(2n-1)k}} - \frac{g^{\lambda}_{(2n+1)k+r}}{g^{\lambda}_{(2n+1)k}} \right] = \frac{g^{\lambda}_{k+r}}{g^{\lambda}_{k}} - \frac{g^{\lambda}_{(2m+1)k+r}}{g^{\lambda}_{(2m+1)k}}. \tag{2.2}$$

Letting A_m denote the left side of this equation and B_m its right side, we get

$$\begin{aligned} B_m - B_{m-1} &= \frac{g^{\lambda}_{(2m-1)k+r}}{g^{\lambda}_{(2m-1)k}} - \frac{g^{\lambda}_{(2m+1)k+r}}{g^{\lambda}_{(2m+1)k}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

confirming the validity of equation (2.2).

Because $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$, equation (2.2) yields the desired result. □

The following result is a byproduct of this lemma.

Lemma 2.2. *Let k, r, and λ be positive integers. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(2n-1)k}}{g^{\lambda}_{(2n-1)k+r}} - \frac{g^{\lambda}_{(2n+1)k}}{g^{\lambda}_{(2n+1)k+r}} \right] = \frac{g^{\lambda}_{k}}{g^{\lambda}_{k+r}} - (-\beta)^{r\lambda}. \tag{2.3}$$

Proof. It follows by the proof of Lemma 1 that

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(2n-1)k}}{g^{\lambda}_{(2n-1)k+r}} - \frac{g^{\lambda}_{(2n+1)k}}{g^{\lambda}_{(2n+1)k+r}} \right] &= \frac{g^{\lambda}_{k}}{g^{\lambda}_{k+r}} - \frac{1}{\alpha^{r\lambda}} \\ &= \frac{g^{\lambda}_{k}}{g^{\lambda}_{k+r}} - (-\beta)^{r\lambda}, \end{aligned}$$

as expected. □

3. Gibonacci Sums

The above lemmas with $\lambda = 1$, coupled with identities (1.1) and (1.2), play a pivotal role in our discourse. In the interest of brevity, we let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n, \\ \Delta^2, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} -1, & \text{if } g_n = f_n, \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.1. *Let k and r be positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu f_r f_{2k}}{g^2_{2nk} + (-1)^k \mu\nu f^2_k} = \frac{g_{k+r}}{g_k} - \alpha^r. \tag{3.1}$$

Proof. Suppose $g_n = f_n$. With identities (1.1) and (1.2), Lemma 2.1 then yields

$$\begin{aligned} \frac{(-1)^k f_r f_{2k}}{f^2_{2nk} - (-1)^k f^2_k} &= \frac{f_{(2n+1)k} f_{(2n-1)k+r} - f_{(2nk+1)k+r} f_{(2n-1)k}}{f_{(2n+1)k} f_{(2n-1)k}}, \\ \sum_{n=1}^{\infty} \frac{(-1)^k f_r f_{2k}}{f^2_{2nk} - (-1)^k f^2_k} &= \sum_{n=1}^{\infty} \left[\frac{f_{(2n-1)k+r}}{f_{(2n-1)k}} - \frac{f_{(2n+1)k+r}}{f_{(2n+1)k}} \right] \\ &= \frac{f_{k+r}}{f_k} - \alpha^r. \end{aligned} \tag{3.2}$$

Now, let $g_n = l_n$. Using the two identities cited above and Lemma 2.1, we get

$$\begin{aligned}
\frac{(-1)^{k+1}\Delta^2 f_r f_{2k}}{l_{2nk}^2 + (-1)^k \Delta^2 f_k^2} &= \frac{l_{(2n+1)k} l_{(2n-1)k+r} - l_{(2nk+1)k+r} l_{(2n-1)k}}{l_{(2n+1)k} l_{(2n-1)k}}, \\
\sum_{n=1}^{\infty} \frac{(-1)^{k+1}\Delta^2 f_r f_{2k}}{l_{2nk}^2 + (-1)^k \Delta^2 f_k^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(2n-1)k+r}}{l_{(2n-1)k}} - \frac{l_{(2n+1)k+r}}{l_{(2n+1)k}} \right] \\
&= \frac{l_{k+r}}{l_k} - \alpha^r.
\end{aligned}$$

By combining this result with equation (3.2), we get the desired result. □

With $r = 1$, this theorem implies [3, 5]

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2 + 1} &= -\frac{1}{2} + \frac{\sqrt{5}}{2}; & \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2 - 5} &= \frac{1}{2} - \frac{\sqrt{5}}{10}; \\
\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} &= \frac{1}{2} - \frac{\sqrt{5}}{6}; & \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} &= -\frac{1}{18} + \frac{\sqrt{5}}{30}; \\
\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} &= -\frac{1}{8} + \frac{\sqrt{5}}{16}; & \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} &= \frac{1}{32} - \frac{\sqrt{5}}{80}; \\
\sum_{n=1}^{\infty} \frac{1}{F_{8n}^2 - 9} &= \frac{1}{18} - \frac{\sqrt{5}}{42}; & \sum_{n=1}^{\infty} \frac{1}{L_{8n}^2 + 45} &= -\frac{1}{98} + \frac{\sqrt{5}}{210}.
\end{aligned}$$

The next theorem invokes Lemma 2.2 with $\lambda = 1$.

Theorem 3.2. *Let k and r be positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^k \mu\nu f_r f_{2k}}{g_{2nk+r}^2 + (-1)^{r+k} \mu\nu f_k^2} = \frac{g_k}{g_{k+r}} - (-\beta)^r. \tag{3.3}$$

Proof. Let $g_n = f_n$. With identities (1.1) and (1.2), Lemma 2.2 yields

$$\begin{aligned}
\frac{(-1)^{k+1} f_r f_{2k}}{f_{2nk+r}^2 - (-1)^{r+k} f_k^2} &= \frac{f_{(2n+1)k+r} f_{(2n-1)k} - f_{(2n+1)k} f_{(2n-1)k+r}}{f_{(2n+1)k+r} f_{(2n-1)k+r}}, \\
\sum_{n=1}^{\infty} \frac{(-1)^{k+1} f_r f_{2k}}{f_{2nk+r}^2 - (-1)^{r+k} f_k^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(2n-1)k}}{f_{(2n-1)k+r}} - \frac{f_{(2n+1)k}}{f_{(2n+1)k+r}} \right] \\
&= \frac{f_k}{f_{k+r}} - (-\beta)^r.
\end{aligned}$$

On the other hand, suppose $g_n = l_n$. Using the two above identities and Lemma 2.2, we get

$$\begin{aligned}
\frac{(-1)^k \Delta^2 f_r f_{2k}}{l_{2nk+r}^2 + (-1)^{r+k} \Delta^2 f_k^2} &= \frac{l_{(2n+1)k+r} l_{(2n-1)k} - l_{(2n+1)k} l_{(2n-1)k+r}}{l_{(2n+1)k+r} l_{(2n-1)k+r}}, \\
\sum_{n=1}^{\infty} \frac{(-1)^k \Delta^2 f_r f_{2k}}{l_{2nk+r}^2 + (-1)^{r+k} \Delta^2 f_k^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(2n-1)k}}{l_{(2n-1)k+r}} - \frac{l_{(2n+1)k}}{l_{(2n+1)k+r}} \right] \\
&= \frac{l_k}{l_{k+r}} - (-\beta)^r.
\end{aligned}$$

Combining the two cases, we get equation (3.3), as desired. □

In particular, we have the following results.
With $r = 1$, we get [3, 5]:

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n+1}^2-1} = \frac{3}{2}-\frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{2n+1}^2+5} = -\frac{1}{6}+\frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{4n+1}^2+1} = -\frac{1}{3}+\frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{4n+1}^2-5} = \frac{1}{12}-\frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{6n+1}^2-4} = \frac{7}{48}-\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{6n+1}^2+20} = -\frac{3}{112}+\frac{\sqrt{5}}{80};$$

when $r=2$, the theorem yields [3, 5]:

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n+2}^2+1} = -1+\frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{2n+2}^2-5} = \frac{1}{4}-\frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{4n+2}^2-1} = \frac{7}{18}-\frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{4n+2}^2+5} = -\frac{1}{14}+\frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{6n+2}^2+4} = -\frac{11}{80}+\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{6n+2}^2-20} = \frac{5}{176}-\frac{\sqrt{5}}{80};$$

and when $r=3$, we get [3]:

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n+3}^2-1} = \frac{5}{12}-\frac{\sqrt{5}}{4}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{2n+3}^2+5} = -\frac{3}{14}+\frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{4n+3}^2+1} = -\frac{11}{30}+\frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{4n+3}^2-5} = \frac{5}{66}-\frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{6n+3}^2-4} = \frac{9}{64}-\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L_{6n+3}^2+20} = -\frac{1}{36}+\frac{\sqrt{5}}{80}.$$

3.1. **Gibonacci Delights.** Using some of the above results, we can compute additional sums [3, 5]:

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{1}{F_{2n+1}^2+1} &= \sum_{n=1}^{\infty}\frac{1}{F_{4n+1}^2+1}+\sum_{n=1}^{\infty}\frac{1}{F_{4n+3}^2+1}\\
&= -\frac{7}{10}+\frac{\sqrt{5}}{3};\\
\sum_{n=2}^{\infty}\frac{1}{F_{2n}^2-1} &= \sum_{n=1}^{\infty}\frac{1}{F_{4n}^2-1}+\sum_{n=1}^{\infty}\frac{1}{F_{4n+2}^2-1}\\
&= \frac{8}{9}-\frac{\sqrt{5}}{3};\\
\sum_{n=2}^{\infty}\frac{1}{L_{2n+1}^2-5} &= \sum_{n=1}^{\infty}\frac{1}{L_{4n+1}^2-5}+\sum_{n=1}^{\infty}\frac{1}{L_{4n+3}^2-5}\\
&= \frac{7}{44}-\frac{\sqrt{5}}{15};\\
\sum_{n=2}^{\infty}\frac{1}{L_{2n}^2+5} &= \sum_{n=1}^{\infty}\frac{1}{L_{4n}^2+5}+\sum_{n=1}^{\infty}\frac{1}{L_{4n+2}^2+5}\\
&= -\frac{8}{63}+\frac{\sqrt{5}}{15}.
\end{aligned}$$

4. Acknowledgment

The author is grateful to the reviewer for a careful reading of the article, and for the encouraging words and constructive suggestions.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[3] T. Koshy, *Sums involving two classes of gibonacci polynomials*, The Fibonacci Quarterly, **61.4** (2023), 312–320.
[4] T. Koshy, *Additional sums involving gibonacci polynomials*, The Fibonacci Quarterly, **61.1** (2023), 12–20.
[5] T. Koshy, *More sums involving gibonacci polynomial squares*, The Fibonacci Quarterly, **61.4** (2023), 346–356.

MSC2020: Primary 11B37, 11B39, 11C08.

Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu