SUMS INVOLVING A CLASS OF GIBONACCI POLYNOMIAL SQUARES: GENERALIZATIONS

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Abstract. We explore an infinite sum involving a special class of gibonacci polynomial squares and its consequences.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2+4}$, and $2\alpha = x + \Delta$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 3, 4, 5]

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise,} \end{cases} \tag{1.1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_r f_{2k}, & \text{otherwise,} \end{cases} \tag{1.2}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas. Identity (1.2) is a gibonacci polynomial extension of *d'Ocagne identity* [2].

2. A Telescoping Gibonacci Sum

Using recursion, we will now explore a telescoping gibonacci sum.

Lemma 2.1. *Let k, r, t, and λ be positive integers, where $t \leq 6$. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g^{\lambda}_{(6n+t-6)k+r}}{g^{\lambda}_{(6n+t-6)k}} - \frac{g^{\lambda}_{(6n+t)k+r}}{g^{\lambda}_{(6n+t)k}} \right] = \frac{g^{\lambda}_{tk+r}}{g^{\lambda}_{tk}} - \alpha^{\lambda r}. \tag{2.1}$$

Proof. With recursion [2, 5], we will first establish that

$$\sum_{n=1}^{m} \left[\frac{g^{\lambda}_{(6n+t-6)k+r}}{g^{\lambda}_{(6n+t-6)k}} - \frac{g^{\lambda}_{(6n+t)k+r}}{g^{\lambda}_{6ntk}} \right] = \frac{g^{\lambda}_{tk+r}}{g^{\lambda}_{tk}} - \frac{g^{\lambda}_{(6m+t)k+r}}{g^{\lambda}_{6mtk}}. \tag{2.2}$$

Letting A_m denote the left side of this equation and B_m its right side, we get

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

Recursively, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

establishing the validity of equation (2.2).

Because $\lim_{m\to\infty} \dfrac{g_{m+r}}{g_m} = \alpha^r$, equation (2.2) yields the desired result. □

3. Gibonacci Sums

It follows by identities (1.1) and (1.2) that

$$g_{(6n+t)k}g_{(6n+t-6)k} - g^2_{(6n+t-3)k} = \begin{cases} (-1)^{tk+1}f^2_{3k}, & \text{if } g_n = f_n; \\ (-1)^{tk}\Delta^2 f^2_{3k}, & \text{otherwise,} \end{cases} \tag{3.1}$$

$$g_{(6n+t)k+r}g_{(6n+t-6)k} - g_{(6n+t)k}g_{(6n+t-6)k+r} = \begin{cases} (-1)^{tk+1}f_r f_{6k}, & \text{if } g_n = f_n; \\ (-1)^{tk}\Delta^2 f_r f_{6k}, & \text{otherwise,} \end{cases} \tag{3.2}$$

respectively.

Coupled with these two identities, Lemma 2.1 with $\lambda = 1$ plays a major role in our explorations. To this end, in the interest of brevity, we let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise,} \end{cases} \quad \text{and} \quad \nu^* = \begin{cases} 1, & \text{if } g_n = f_n; \\ -1, & \text{otherwise.} \end{cases}$$

Some of the numeric examples following Theorem 3.1 involve large numbers. So, again in the interest of brevity and convenience, we let $A = 104,006$; $B = 372,096$; $C = 788,120$; $D = 3,524,576$; and $E = 6,677,056$.

With these tools at our disposal, we now begin our discourse with the next result.

Theorem 3.1. *Let k, r, and t be positive integers, where $t \le 6$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu\nu^* f_r f_{6k}}{g^2_{(6n+t-3)k} - (-1)^{tk}\mu\nu^* f^2_{3k}} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{3.3}$$

Proof. Suppose $g_n = f_n$. Lemma 2.1, coupled with identities (3.1) and (3.2) yields

$$\begin{aligned} \frac{(-1)^{tk} f_r f_{6k}}{f^2_{(6n+t-3)k} - (-1)^{tk} f^2_{3k}} &= \frac{f_{(6n+t)k}f_{(6n+t-6)k+r} - f_{(6n+t)k+r}f_{(6n+t-6)k}}{f_{(6n+t)k}f_{(6n+t-6)k}} \\ \sum_{n=1}^{\infty} \frac{(-1)^{tk} f_r f_{6k}}{f^2_{(6n+t-3)k} - (-1)^{tk} f^2_{3k}} &= \sum_{n=1}^{\infty} \left[\frac{f_{(6n+t-6)k+r}}{f_{(6n+t-6)k}} - \frac{f_{(6n+t)k+r}}{f_{(6n+t)k}} \right] \\ &= \frac{f_{tk+r}}{f_{tk}} - \alpha^r. \end{aligned}$$

On the other hand, let $g_n = l_n$. Using the same two identities and Lemma 2.1, we get

$$\begin{aligned}
\frac{(-1)^{tk+1}\Delta^2 f_r f_{6k}}{l^2_{(6n+t-3)k} + (-1)^{tk}\Delta^2 f^2_{3k}} &= \frac{l_{(6n+t)k}l_{(6n+t-6)k+r} - l_{(6n+t)k+r}l_{(6n+t-6)k}}{l_{(6n+t)k}l_{(6n+t-6)k}}\\
\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}\Delta^2 f_r f_{6k}}{l^2_{(6n+t-3)k} + (-1)^{tk}\Delta^2 f^2_{3k}} &= \sum_{n=1}^{\infty}\left[\frac{l_{(6n+t-6)k+r}}{l_{(6n+t-6)k}} - \frac{l_{(6n+t)k+r}}{l_{(6n+t)k}}\right]\\
&= \frac{l_{tk+r}}{l_{tk}} - \alpha^r.
\end{aligned}$$

By combining the two cases, we get the desired result. □

Next, we compute sum (3.3) for six values of t, with the restrictions $r = 1$ and $1 \le k \le 3$ for simplicity.

When $t = 1$, the theorem yields

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{6n-2}+4} = -\frac{1}{16}+\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{6n-2}-20} = \frac{1}{16}-\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{2(6n-2)}-64} = \frac{1}{96}-\frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{2(6n-2)}+320} = -\frac{1}{864}+\frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{3(6n-2)}+1,156} = -\frac{1}{2,584}+\frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{3(6n-2)}-5,780} = \frac{1}{10,336}-\frac{\sqrt{5}}{25,840}.$$

When $t = 2$, we get

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{6n-1}-4} = \frac{3}{16}-\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{6n-1}+20} = -\frac{1}{48}+\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{2(6n-1)}-64} = \frac{7}{864}-\frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{2(6n-1)}+320} = -\frac{1}{672}+\frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{3(6n-1)}-1,156} = \frac{9}{20,672}-\frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{3(6n-1)}+5,780} = -\frac{1}{11,628}+\frac{\sqrt{5}}{25,840}.$$

When $t = 3$, this theorem implies

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{6n}+4} = -\frac{1}{8}+\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{6n}-20} = \frac{1}{32}-\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{12n}-64} = \frac{1}{128}-\frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{12n}+320} = -\frac{1}{648}+\frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{18n}+1,156} = -\frac{1}{2,312}+\frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{18n}-5,780} = \frac{1}{11,552}-\frac{\sqrt{5}}{25,840}.$$

With $t = 4$, we have

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{6n+1}-4} = \frac{7}{48}-\frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{6n+1}+20} = -\frac{3}{112}+\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{2(6n+1)}-64} = \frac{47}{6,048}-\frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{2(6n+1)}+320} = -\frac{7}{4,512}+\frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty}\frac{1}{F^2_{3(6n+1)}-1,156} = \frac{161}{B}-\frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty}\frac{1}{L^2_{3(6n+1)}+5,780} = -\frac{9}{A}+\frac{\sqrt{5}}{25,840}.$$

Using $t = 5$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+2}^2 + 4} = -\frac{11}{80} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+2}^2 - 20} = \frac{5}{176} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n+2)}^2 - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+2)}^2 + 320} = -\frac{55}{35,424} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(6n+2)}^2 + 1,156} = -\frac{341}{C} + \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(6n+2)}^2 - 5,780} = \frac{305}{D} - \frac{\sqrt{5}}{25,840}.$$

When $t = 6$, the theorem yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+3}^2 - 4} = \frac{9}{64} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+3}^2 + 20} = -\frac{1}{36} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n+3)}^2 - 64} = \frac{161}{20,736} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+3)}^2 + 320} = -\frac{1}{644} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(6n+3)}^2 - 1,156} = \frac{2,889}{E} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(6n+3)}^2 + 5,780} = -\frac{1}{11,556} + \frac{\sqrt{5}}{25,840}.$$

Similarly, we can compute the close counterparts of the above sums for $r \geq 2$. For example, with $t = 2$ and $r = 3$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n-1}^2 - 4} = \frac{3}{16} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n-1}^2 + 20} = -\frac{1}{48} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2(6n-1)}^2 - 64} = \frac{7}{864} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(6n-1)}^2 + 320} = -\frac{1}{672} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(6n-1)}^2 - 1,156} = \frac{9}{20,672} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(6n-1)}^2 + 5,780} = -\frac{1}{11,628} + \frac{\sqrt{5}}{25,840};$$

as found earlier.

3.1. **Gibonacci Delights.** Using $\{a_{3n-1}\}_{n\geq 1} \cup \{a_{3n}\}_{n\geq 1} \cup \{a_{3n+1}\}_{n\geq 1} = \{a_n\}_{n\geq 2}$, we can employ the above results to extract additional dividends. Using the cases $t = 1$, 3, and 5, we get

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{2n}^2 + 4} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(3n-1)}^2 + 4} + \sum_{n=1}^{\infty} \frac{1}{F_{2(3n)}^2 + 4} + \sum_{n=1}^{\infty} \frac{1}{F_{2(3n+1)}^2 + 4} \\
&= -\frac{13}{40} + \frac{3\sqrt{5}}{16}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{4(3n-1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{4(3n)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{4(3n+1)}^2 - 64} \\
&= \frac{183}{7,040} - \frac{\sqrt{5}}{96}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{6n}^2 + 1,156} &= \sum_{n=1}^{\infty} \frac{1}{F_{6(3n-1)}^2 + 1,156} + \sum_{n=1}^{\infty} \frac{1}{F_{6(3n)}^2 + 1,156} + \sum_{n=1}^{\infty} \frac{1}{F_{6(3n+1)}^2 + 1,156} \\
&= -\frac{883}{705,1640} + \frac{3\sqrt{5}}{5,168};
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 - 20} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(3n-1)}^2 - 20} + \sum_{n=1}^{\infty} \frac{1}{L_{2(3n)}^2 - 20} + \sum_{n=1}^{\infty} \frac{1}{L_{2(3n+1)}^2 - 20} \\
&= \frac{43}{352} - \frac{3\sqrt{5}}{80}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{4n}^2 + 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{4(3n-1)}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{4(3n)}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{4(3n+1)}^2 + 320} \\
&= -\frac{113}{26,568} + \frac{\sqrt{5}}{480}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{6n}^2 - 5,780} &= \sum_{n=1}^{\infty} \frac{1}{L_{6(3n-1)}^2 - 5,780} + \sum_{n=1}^{\infty} \frac{1}{L_{6(3n)}^2 - 5,780} + \sum_{n=1}^{\infty} \frac{1}{L_{6(3n+1)}^2 - 5,780} \\
&= \frac{1,063}{3,939,232} - \frac{3\sqrt{5}}{25,840}.
\end{aligned}$$

Likewise, when $t = 2$, 4, and 6, the theorem yields

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{2n+1}^2 - 4} &= \sum_{n=1}^{\infty} \frac{1}{F_{6n-1}^2 - 4} + \sum_{n=1}^{\infty} \frac{1}{F_{6n+1}^2 - 4} + \sum_{n=1}^{\infty} \frac{1}{F_{6n+3}^2 - 4} \\
&= \frac{91}{192} - \frac{3\sqrt{5}}{16}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(6n-1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{2(6n+1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{2(6n+3)}^2 - 64} \\
&= \frac{3,431}{145,152} - \frac{\sqrt{5}}{96}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{3(2n+1)}^2 - 1,156} &= \sum_{n=1}^{\infty} \frac{1}{F_{3(6n-1)}^2 - 1,156} + \sum_{n=1}^{\infty} \frac{1}{F_{3(6n+1)}^2 - 1,156} + \sum_{n=1}^{\infty} \frac{1}{F_{3(6n+3)}^2 - 1,156} \\
&= \frac{156,331}{120,187,008} - \frac{3\sqrt{5}}{5,168}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{2n+1}^2 + 20} &= \sum_{n=1}^{\infty} \frac{1}{L_{6n-1}^2 + 20} + \sum_{n=1}^{\infty} \frac{1}{L_{6n+1}^2 + 20} + \sum_{n=1}^{\infty} \frac{1}{L_{6n+3}^2 + 20} \\
&= -\frac{29}{504} + \frac{3\sqrt{5}}{80}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(6n-1)}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+1)}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+3)}^2 + 320} \\
&= -\frac{139}{30,268} + \frac{\sqrt{5}}{480};
\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{L_{3(2n+1)}^2+5,780} &= \sum_{n=1}^{\infty}\frac{1}{L_{3(6n-1)}^2+5,780}+\sum_{n=1}^{\infty}\frac{1}{L_{3(6n+1)}^2+5,780}+\sum_{n=1}^{\infty}\frac{1}{L_{3(6n+3)}^2+5,780}\\ &= -\frac{241}{930,258}+\frac{3\sqrt{5}}{25,840}.\end{aligned}$$

3.2. **Gibonacci Pleasantries.** Using four of the above sums, we can compute two additional sums

$$\begin{aligned}\sum_{n=4}^{\infty}\frac{1}{F_{2n}^2-64} &= \sum_{n=2}^{\infty}\frac{1}{F_{4n}^2-64}+\sum_{n=2}^{\infty}\frac{1}{F_{2(2n+1)}^2-64}\\ &= \frac{396,227}{7,983,360}-\frac{\sqrt{5}}{48};\end{aligned}$$

$$\begin{aligned}\sum_{n=4}^{\infty}\frac{1}{L_{2n}^2+320} &= \sum_{n=2}^{\infty}\frac{1}{L_{4n}^2+320}+\sum_{n=2}^{\infty}\frac{1}{L_{2(2n+1)}^2+320}\\ &= -\frac{1,778,309}{201,040,056}+\frac{\sqrt{5}}{240}.\end{aligned}$$

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