SUMS INVOLVING A CLASS OF JACOBSTHAL POLYNOMIAL SQUARES

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Abstract. We explore the Jacobsthal version of an infinite sum involving gibonacci polynomial squares.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 4].

Gibonacci and Jacobsthal polynomials are linked by the relationships
$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4, 5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2+4}$, $2\alpha = x + \Delta$, $D = \sqrt{4x+1}$, and $2w = 1 + D$.

2. Gibonacci Polynomial Sum

Before presenting an interesting gibonacci sum, again in the interest of brevity and expediency, we let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n, \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \nu^* = \begin{cases} 1, & \text{if } g_n = f_n, \\ -1, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad D^* = \begin{cases} 1, & \text{if } c_n = J_n, \\ D^2, & \text{otherwise.} \end{cases}$$

Using these tools, we established the following result in [6].

Theorem 2.1. *Let k, r, and t be positive integers, where $t \leq 6$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu\nu^* f_r f_{6k}}{g^2_{(6n+t-3)k} - (-1)^{tk}\mu\nu^* f^2_{3k}} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{2.1}$$

Our objective is to explore the Jacobsthal counterpart of this sum.

3. Jacobsthal Polynomial Sum

To achieve our goal, we will employ the gibonacci-Jacobsthal relationships in Section 1. To this end, in the interest of brevity and clarity, we let A denote the fractional expression on the left side of the given gibonacci equation and B that on its right side, and the left-hand side (LHS) and right-hand side (RHS) of the corresponding Jacobsthal equation, as in [5].

Notice that $\alpha(1/\sqrt{x}) = \dfrac{1+D}{2\sqrt{x}} = \dfrac{w}{\sqrt{x}}$.

With this brief background, we begin our exploration.

Proof. *Case 1.* Suppose $g_n = f_n$. We have $A = \dfrac{(-1)^{tk} f_r f_{6k}}{f^2_{(6n+t-3)k} - (-1)^{tk} f^2_{3k}}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{(6n+t)k-2+r/2}$, we get

$$\begin{aligned}
A &= \frac{(-1)^{tk} x^{(6n+t-3)k-1}\left[x^{(r-1)/2} f_r\right]\left[x^{(6k-1)/2} f_{6k}\right]}{x^{3k-1+r/2}\left\{x^{[(6n+t-3)k-1]/2} f_{(6n+t-3)k}\right\}^2 - (-1)^{tk} x^{(6n+t-3)k-1+r/2}[x^{(3k-1)/2} f_{3k}]^2}\\
&= \frac{(-1)^{tk} x^{(6n+t-3)k-1} J_r J_{6k}}{x^{3k-1+r/2} J^2_{(6n+t-3)k} - (-1)^{tk} x^{(6n+t-3)k-1+r/2} J^2_{3k}};\\
\text{LHS} &= \sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(6n+t-6)k-r/2} J_r J_{6k}}{J^2_{(6n+t-3)k} - (-1)^{tk} x^{(6n+t-6)k} J^2_{3k}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we turn to $B = \dfrac{f_{tk+r}}{f_{tk}} - \alpha^r$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{(tk+r-1)/2}$. This yields

$$\begin{aligned}
B &= \frac{x^{(tk+r-1)/2} f_{tk+r}}{x^{r/2}[x^{(tk-1)/2} f_{tk}]} - \frac{w^r}{x^{r/2}};\\
\text{RHS} &= \frac{J_{tk+r}}{x^{r/2} J_{tk}} - \frac{w^r}{x^{r/2}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the Jacobsthal version of equation (2.1):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(6n+t-6)k} J_r J_{6k}}{J^2_{(6n+t-3)k} - (-1)^{tk} x^{(6n+t-6)k} J^2_{3k}} = \frac{J_{tk+r}}{J_{tk}} - w^r, \qquad (3.1)$$

where $c_n = c_n(x)$.

Next, we explore the Jacobsthal-Lucas version of Theorem 2.1.

Case 2. Let $g_n = l_n$. Then $A = \dfrac{(-1)^{tk+1} \Delta^2 f_r f_{6k}}{l^2_{(6n+t-3)k} + (-1)^{tk} \Delta^2 f^2_{3k}}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{(6n+t-3)k}$, we get

$$A = \frac{(-1)^{tk+1} \frac{D^2}{x} f_r f_{6k}}{l^2_{(6n+t-3)k} + (-1)^{tk} \frac{D^2}{x} f^2_{3k}}$$

$$= \frac{(-1)^{tk+1} D^2 x^{(6n+t-6)k-r/2} \left[x^{(r-1)/2} f_r\right] \left[x^{(6k-1)/2} f_{6k}\right]}{\left[x^{[(6n+t-3)k]/2} l_{(6n+t-3)k}\right]^2 + (-1)^{tk} D^2 x^{(6n+t-6)k} [x^{(3k-1)/2} f_{3k}]^2};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk+1} D^2 x^{(6n+t-6)k-r/2} J_r J_{6k}}{j^2_{(6n+t-3)k} + (-1)^{tk} D^2 x^{(6n+t-6)k} J^2_{3k}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

We now have $B = \dfrac{l_{tk+r}}{l_{tk}} - \alpha^r$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{(tk+r)/2}$. This yields

$$B = \frac{x^{(tk+r)/2} l_{tk+r}}{x^{r/2}[x^{tk/2} l_{tk}]} - \frac{w^r}{x^{r/2}};$$

$$\text{RHS} = \frac{j_{tk+r}}{x^{r/2} j_{tk}} - \frac{w^r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the desired Jacobsthal-Lucas version:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} D^2 x^{(6n+t-6)k} J_r J_{6k}}{j^2_{(6n+t-3)k} + (-1)^{tk} D^2 x^{(6n+t-6)k} J^2_{3k}} = \frac{j_{tk+r}}{j_{tk}} - w^r, \tag{3.2}$$

where $c_n = c_n(x)$.

Combining equations (3.1) and (3.2), we get the Jacobsthal version of Theorem 2.1, as the following theorem features. □

Theorem 3.1. *Let k, r, and t be positive integers, where $t \le 6$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu^* x^{(6n+t-6)k} J_r J_{6k}}{c^2_{(6n+t-3)k} - (-1)^{tk} D^* \nu^* x^{(6n+t-6)k} J^2_{3k}} = \frac{c_{tk+r}}{c_{tk}} - w^r. \tag{3.3}$$

Finally, we feature a compact and sophisticated alternate proof of this theorem. It still employs the gibonacci-Jacobsthal relationships, but in a slightly different way.

3.1. **An Alternate Method.** To begin with, we let

$$d = \frac{1+\nu^*}{4} = \begin{cases} 1/2, & \text{if } g_n = f_n, \\ 0, & \text{otherwise.} \end{cases}$$

We also have

$$f_n(1/\sqrt{x}) = \frac{J_n(x)}{x^{(n-1)/2}}; \qquad l_n(1/\sqrt{x}) = \frac{j_n(x)}{x^{n/2}}; \qquad g_n(1/\sqrt{x}) = \frac{c_n(x)}{x^{n/2-d}}.$$

With these new tools at our fingertips, we are ready for the alternate proof.

Proof. Replacing x with $1/\sqrt{x}$ in the rational expression on the left side of equation (2.1) and using the above substitutions, we get

$$\begin{aligned}
A &= \frac{(-1)^{tk}\mu\nu^* J_r/x^{(r-1)/2}\cdot J_{6k}/x^{(6k-1)/2}}{\left[c_{(6n+t-3)k}/x^{\frac{(6n+t-3)k}{2}-d}\right]^2-(-1)^{tk}\mu\nu^*\left[\frac{J_{3k}}{x^{(3k-1)/2}}\right]^2}\\
&= \frac{(-1)^{tk}\mu\nu^* J_r J_{6k}\cdot x^{(6n+t-3)k-2d-\frac{6k+r-2}{2}}}{c^2_{(6n+t-3)k}-(-1)^{tk}\mu\nu^* J^2_{3k}\cdot x^{[(6n+t-3)k-2d-(3k-1)}}\\
&= \frac{(-1)^{tk}\mu\nu^* J_r J_{6k}\cdot x^{(6n+t-6)k-\frac{\nu^*+r-1}{2}}}{c^2_{(6n+t-3)k}-(-1)^{tk}\mu\nu^* J^2_{3k}\cdot x^{(6n+t-6)k+\frac{1-\nu^*}{2}}};\\
\text{LHS} &= \sum_{n=1}^{\infty}\frac{(-1)^{tk}D^*\nu^* x^{(6n+t-6)k-r/2}J_rJ_{6k}}{c^2_{(6n+t-3)k}-(-1)^{tk}D^*\nu^* x^{(6n+t-6)k}J^2_{3k}},
\end{aligned}$$

where $c_n = c_n(x)$.

Turning to the right side of equation (2.1), we have

$$\begin{aligned}
B &= \frac{g_{tk+r}}{g_{tk}}-\alpha^r\\
&= \frac{c_{tk+r}/x^{\frac{tk+r}{2}-d}}{c_{tk}/x^{\frac{tk}{2}-d}}-\frac{w^r}{x^{r/2}};\\
\text{RHS} &= \frac{c_{tk+r}}{x^{r/2}c_{tk}}-\frac{w^r}{x^{r/2}},
\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the same Jacobsthal version:

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk}D^*\nu^* x^{(6n+t-6)k}J_rJ_{6k}}{c^2_{(6n+t-3)k}-(-1)^{tk}D^*\nu^* x^{(6n+t-6)k}J^2_{3k}} = \frac{c_{tk+r}}{c_{tk}}-w^r,$$

as expected, where $c_n = c_n(x)$. □

Finally, we pursue a few gibonacci and Jacobsthal consequences of Theorem 3.1.

3.2. **Gibonacci and Jacobsthal Implications.** In particular, with $J_n(1) = F_n$ and $j_n(1) = L_n$, Theorem 3.1 yields

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk}F_rF_{6k}}{F^2_{(6n+t-3)k}-(-1)^{tk}F^2_{3k}} = \frac{F_{tk+r}}{F_{tk}}-\alpha^r; \tag{3.4}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk+1}5F_rF_{6k}}{L^2_{(6n+t-3)k}+(-1)^{tk}5F^2_{3k}} = \frac{L_{tk+r}}{L_{tk}}-\alpha^r; \tag{3.5}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk}2^{(6n+t-6)k}J_rJ_{6k}}{J^2_{(6n+t-3)k}-(-1)^{tk}2^{(6n+t-6)k}J^2_{3k}} = \frac{J_{tk+r}}{J_{tk}}-2^r; \tag{3.6}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk+1}9\cdot 2^{(6n+t-6)k}J_rJ_{6k}}{j^2_{(6n+t-3)k}+(-1)^{tk}9\cdot 2^{(6n+t-6)k}J^2_{3k}} = \frac{j_{tk+r}}{j_{tk}}-2^r. \tag{3.7}$$

With $r = 1 = k$ and $t \le 4$, they yield

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n-2}^2 + 4} = -\frac{1}{16} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n-2}^2 - 20} = \frac{1}{16} - \frac{\sqrt{5}}{80};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{6n-1}^2 - 4} = \frac{3}{16} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n-1}^2 + 20} = -\frac{1}{48} + \frac{\sqrt{5}}{80};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+1}^2 - 4} = \frac{7}{48} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+1}^2 + 20} = -\frac{3}{112} + \frac{\sqrt{5}}{80};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-5}}{J_{6n-2}^2 + 9 \cdot 2^{6n-5}} = \frac{1}{21}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-5}}{j_{6n-2}^2 - 81 \cdot 2^{6n-5}} = \frac{1}{63};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-4}}{J_{6n-1}^2 - 9 \cdot 2^{6n-4}} = \frac{1}{21}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-4}}{j_{6n-1}^2 + 81 \cdot 2^{6n-4}} = \frac{1}{315};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-3}}{J_{6n}^2 + 9 \cdot 2^{6n-3}} = \frac{1}{63}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-3}}{j_{6n}^2 - 81 \cdot 2^{6n-3}} = \frac{1}{441};$$
$$\sum_{n=1}^{\infty} \frac{2^{6n-2}}{J_{6n+1}^2 - 9 \cdot 2^{6n-2}} = \frac{1}{105}; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-2}}{j_{6n+1}^2 + 81 \cdot 2^{6n-2}} = \frac{1}{1,071}.$$

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