SUMS INVOLVING A FAMILY OF GIBONACCI POLYNOMIAL SQUARES: GENERALIZATIONS

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Abstract. We explore a generalization of an infinite sum involving a large family of gibonacci polynomial squares and its consequences.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2+4}$, and $2\alpha = x + \Delta$.

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

1.1. Fundamental Gibonacci Identities.

Gibonacci polynomials satisfy the following properties [2, 3, 4, 5, 6]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1.1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n; \\ (-1)^{n+k}\Delta^2 f_r f_{2k}, & \text{otherwise,} \end{cases} \tag{1.2}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas.

It follows from these two identities that

$$g_{(2pn+t)k}g_{(2pn+t-2p)k} - g_{(2pn+t-p)k}^2 = \begin{cases} (-1)^{tk+1} f_{pk}^2, & \text{if } g_n = f_n; \\ (-1)^{tk}\Delta^2 f_{pk}^2, & \text{otherwise;} \end{cases} \tag{1.3}$$

$$g_{(2pn+t)k+r}g_{(2pn+t-2p)k} - g_{(2pn+t)k}g_{(2pn+t-2p)k+r} = \begin{cases} (-1)^{tk+1} f_r f_{2pk}, & \text{if } g_n = f_n; \\ (-1)^{tk}\Delta^2 f_r f_{2pk}, & \text{otherwise,} \end{cases} \tag{1.4}$$

where k, p, r, and t are positive integers, and $t \leq 2p$.

2. A Telescoping Gibonacci Sum

With recursion, we will now explore a telescoping gibonacci sum.

Lemma 2.1. *Let k, p, r, t, and λ be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty}\left[\frac{g^{\lambda}_{(2pn+t-2p)k+r}}{g^{\lambda}_{(2pn+t-2p)k}} - \frac{g^{\lambda}_{(2pn+t)k+r}}{g^{\lambda}_{(2pn+t)k}}\right] = \frac{g^{\lambda}_{tk+r}}{g^{\lambda}_{tk}} - \alpha^{\lambda r}. \tag{2.1}$$

Proof. With recursion [2, 3], we will first establish that

$$\sum_{n=1}^{m}\left[\frac{g^{\lambda}_{(2pn+t-2p)k+r}}{g^{\lambda}_{(2pn+t-2p)k}} - \frac{g^{\lambda}_{(2pn+t)k+r}}{g^{\lambda}_{(2pn+t)k}}\right] = \frac{g^{\lambda}_{tk+r}}{g^{\lambda}_{tk}} - \frac{g^{\lambda}_{(2pm+t)k+r}}{g^{\lambda}_{(2pm+t)k}}. \tag{2.2}$$

To this end, we now let A_m denote the left side of this equation and B_m its right side. Then

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

By recursion, this implies

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

establishing the validity of equation (2.2).

Because $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$, the given result now follows, as desired. □

3. Gibonacci Sums

Using identities (1.3) and (1.4), Lemma 2.1 with $\lambda = 1$ plays a pivotal role in our explorations. To this end, in the interest of brevity, we let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu^* = \begin{cases} 1, & \text{if } g_n = f_n; \\ -1, & \text{otherwise.} \end{cases}$$

These tools serve as building blocks of our discourse, as the following theorem showcases.

Theorem 3.1. *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu\nu^* f_r f_{2pk}}{g^2_{(2pn+t-p)k} - (-1)^{tk}\mu\nu^* f^2_{pk}} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{3.1}$$

Proof. Suppose $g_n = f_n$. Coupled with identities (1.3) and (1.4), Lemma 2.1 then yields

$$\begin{aligned} \frac{(-1)^{tk} f_r f_{2pk}}{f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}} &= \frac{f_{(2pn+t)k} f_{(2pn+t-2p)k+r} - f_{(2pn+t)k+r} f_{(2pn+t-2p)k}}{f_{(2pn+t)k} f_{(2pn+t-2p)k}}, \\ \sum_{n=1}^{\infty} \frac{(-1)^{tk} f_r f_{2pk}}{f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}} &= \sum_{n=1}^{\infty}\left[\frac{f_{(2pn+t-2p)k+r}}{f_{(2pn+t-2p)k}} - \frac{f_{(2pn+t)k+r}}{f_{(2pn+t)k}}\right] \\ &= \frac{f_{tk+r}}{f_{tk}} - \alpha^r. \end{aligned}$$

Now, let $g_n = l_n$. With the same two identities and Lemma 2.1, we get

$$\begin{aligned}
\frac{(-1)^{tk+1}\Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}} &= \frac{l_{(2pn+t)k} l_{(2pn+t-2p)k+r} - l_{(2pn+t)k+r} l_{(2pn+t-2p)k}}{l_{(2pn+t)k} l_{(2pn+t-2p)k}}, \\
\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}\Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}} &= \sum_{n=1}^{\infty} \left[\frac{l_{(2pn+t-2p)k+r}}{l_{(2pn+t-2p)k}} - \frac{l_{(2pn+t)k+r}}{l_{(2pn+t)k}} \right] \\
&= \frac{l_{tk+r}}{l_{tk}} - \alpha^r.
\end{aligned}$$

Combining the two cases yields the desired result, as expected. □

Letting $t = p$; $t = p = k$; $t = 2p$; and $t = 2p = 2k$, Theorem 3.1 yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{pk}\mu\nu^* f_r f_{2pk}}{g^2_{2pkn} - (-1)^{pk}\mu\nu^* f^2_{pk}} = \frac{g_{pk+r}}{g_{pk}} - \alpha^r; \tag{3.2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{p^2}\mu\nu^* f_r f_{2p^2}}{g^2_{2p^2 n} - (-1)^{p^2}\mu\nu^* f^2_{p^2}} = \frac{g_{p^2+r}}{g_{p^2}} - \alpha^r; \tag{3.3}$$

$$\sum_{n=1}^{\infty} \frac{\mu\nu^* f_r f_{2pk}}{g^2_{(2n+1)pk} - \mu\nu^* f^2_{pk}} = \frac{g_{2pk+r}}{g_{2pk}} - \alpha^r; \tag{3.4}$$

$$\sum_{n=1}^{\infty} \frac{\mu\nu^* f_r f_{2p^2}}{g^2_{(2n+1)p^2} - \mu\nu^* f^2_{p^2}} = \frac{g_{2p^2+r}}{g_{2p^2}} - \alpha^r, \tag{3.5}$$

respectively.

In the interest of brevity, we use the labels below in the computational examples to follow:

A	= 20,736;	G	= 103,680;	N	= 1,043,280;
B	= 40,590;	H	= 185,472;	P	= 1,664,080;
C	= 41,216;	I	= 208,656;	Q	= 1,860,500;
D	= 46,368;	J	= 372,100;	R	= 3,328,160;
E	= 67,650;	K	= 463,680;	S	= 3,744,180;
F	= 92,736;	L	= 518,420;	T	= 6,656,320;
		M	= 832,040.		

Case 1. Suppose $p = 1$. With $r = 1$, $k \le 3$, and $t \le 2$, we get [3, 4, 5]:

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{2n} + 1} = -\frac{1}{2} + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{2n} - 5} = \frac{1}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{4n} - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{4n} + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{6n} + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{6n} - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 - 1} = \frac{3}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 + 5} = -\frac{1}{6} + \frac{\sqrt{5}}{10};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 1} = \frac{7}{18} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 5} = -\frac{1}{14} + \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(2n+1)}^2 - 4} = \frac{9}{64} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(2n+1)}^2 + 20} = -\frac{1}{36} + \frac{\sqrt{5}}{80}.$$

Case 2. Suppose $p = 2$. With $r = 1$, $k \le 3$, and $t \le 4$, the theorem implies [3, 4]:

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-1}^2 + 1} = -\frac{1}{6} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n-1}^2 - 5} = \frac{1}{6} - \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(4n-1)}^2 - 9} = \frac{1}{14} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(4n-1)}^2 + 45} = -\frac{1}{126} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(4n-1)}^2 + 64} = -\frac{1}{144} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(4n-1)}^2 - 320} = \frac{1}{576} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{8n}^2 - 9} = \frac{1}{18} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n}^2 + 45} = -\frac{1}{98} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{12} - \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(4n+1)}^2 - 9} = \frac{3}{56} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+1)}^2 + 45} = -\frac{2}{189} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(4n+1)}^2 + 64} = -\frac{19}{2,448} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(4n+1)}^2 - 320} = \frac{17}{10,944} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+2}^2 - 1} = \frac{7}{18} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+2}^2 + 5} = -\frac{1}{14} + \frac{\sqrt{5}}{30};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(4n+2)}^2 - 9} = \frac{47}{882} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(4n+2)}^2 + 45} = -\frac{1}{94} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(4n+2)}^2 - 64} = \frac{161}{A} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(4n+2)}^2 + 320} = -\frac{1}{644} + \frac{\sqrt{5}}{1,440}.$$

Case 3. Let $p = 3$. The results for $r = 1$, $k \le 3$, and $t \le 6$ are given in [6]. In the interest of brevity, we omit them here, but encourage gibonacci enthusiasts to pursue the remaining cases.

Case 4. Let $p = 4$. Then $t \le 8$. Again, for the sake of brevity, we will find the sums with $r = 1$, $k \le 3$, and $t \le 2$.

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n-3}^2 + 9} = -\frac{1}{42} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n-3}^2 - 45} = \frac{1}{42} - \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(8n-3)}^2 - 441} = \frac{1}{658} - \frac{\sqrt{5}}{1,974}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(8n-3)}^2 + 2,205} = -\frac{1}{5,922} + \frac{\sqrt{5}}{9,870};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(8n-3)}^2 + A} = -\frac{1}{D} + \frac{\sqrt{5}}{F}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(8n-3)}^2 - G} = \frac{1}{H} - \frac{\sqrt{5}}{K};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{8n-2}^2 - 9} = \frac{1}{14} - \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n-2}^2 + 45} = -\frac{1}{126} + \frac{\sqrt{5}}{210};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2(8n-2)}^2 - 441} = \frac{1}{846} - \frac{\sqrt{5}}{1,974}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(8n-2)}^2 + 2,205} = \frac{1}{4,606} - \frac{\sqrt{5}}{9,870};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{3(8n-2)}^2 - A} = \frac{1}{C} - \frac{\sqrt{5}}{F}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(8n-2)}^2 + G} = -\frac{1}{I} + \frac{\sqrt{5}}{K}.$$

Case 5. Let $p = 5$, so $t \leq 10$. When $k = 1 = r$ and $t \leq 10$, we have

$$\sum_{n=1}^{\infty} \frac{1}{F_{10n-4}^2 + 25} = -\frac{1}{110} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n-4}^2 - 125} = \frac{1}{110} - \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n-3}^2 - 25} = \frac{3}{110} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n-3}^2 + 125} = -\frac{1}{330} + \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n-2}^2 + 25} = -\frac{1}{55} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n-2}^2 - 125} = \frac{1}{220} - \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n-1}^2 - 25} = \frac{7}{330} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n-1}^2 + 125} = -\frac{3}{770} + \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n}^2 + 25} = -\frac{1}{50} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n}^2 - 125} = \frac{1}{242} - \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+1}^2 - 25} = \frac{9}{440} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+1}^2 + 125} = -\frac{2}{495} + \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+2}^2 + 25} = -\frac{29}{1,430} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+2}^2 - 125} = \frac{13}{3,190} - \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+3}^2 - 25} = \frac{47}{2,310} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+3}^2 + 125} = -\frac{21}{5,170} + \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+4}^2 + 25} = -\frac{19}{935} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+4}^2 - 125} = \frac{17}{4,180} - \frac{\sqrt{5}}{550};$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+5}^2 - 25} = \frac{123}{6,050} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+5}^2 + 125} = -\frac{1}{246} + \frac{\sqrt{5}}{550}.$$

With $r = 1$, $k = 2, 3$, and $t \leq 2$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2(10n-4)}^2 - 3,025} = \frac{1}{4,510} - \frac{\sqrt{5}}{13,530}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2(10n-4)}^2 + 15,125} = -\frac{1}{B} + \frac{\sqrt{5}}{F};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{3(10n-4)} + J} = -\frac{1}{M} + \frac{\sqrt{5}}{P}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{3(10n-4)} - Q} = \frac{1}{R} - \frac{\sqrt{5}}{10M};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{2(10n-3)} - 3,025} = \frac{7}{B} - \frac{\sqrt{5}}{13,530}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{2(10n-3)} + 15,125} = -\frac{1}{2,255} + \frac{\sqrt{5}}{E};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{3(10n-3)} - J} = \frac{9}{T} - \frac{\sqrt{5}}{P}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{3(10n-3)} + Q} = -\frac{1}{S} + \frac{\sqrt{5}}{10M}.$$

Case 6. Let $p = 6$. Then $t \leq 12$. Again, in the interest of conciseness, we will find the 12 sums with $k = 1 = r$ only, but encourage gibonacci enthusiasts to pursue other possibilities [6]:

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-5} + 64} = -\frac{1}{288} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-5} - 320} = \frac{1}{288} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-4} - 64} = \frac{1}{96} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-4} + 320} = -\frac{1}{864} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-3} + 64} = -\frac{1}{72} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-3} - 320} = \frac{11}{4,320} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-2} - 64} = \frac{7}{864} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-2} + 320} = -\frac{1}{672} + \frac{\sqrt{5}}{288};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-1} + 64} = -\frac{11}{1,440} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-1} - 320} = \frac{5}{3,168} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n} - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n} + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+1} + 64} = -\frac{29}{3,744} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+1} - 320} = \frac{13}{8,352} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+2} - 64} = \frac{47}{6,048} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+2} + 320} = -\frac{7}{4,512} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+3} + 64} = -\frac{19}{2,448} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+3} - 320} = \frac{17}{5,472} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+4} - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+4} + 320} = -\frac{55}{35,424} + \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+5} + 64} = -\frac{199}{25,632} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+5} - 320} = \frac{89}{57,312} - \frac{\sqrt{5}}{1,440};$$
$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+6} - 64} = \frac{161}{20,736} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+6} + 320} = -\frac{161}{103,680} + \frac{\sqrt{5}}{1,440}.$$

3.1. **Gibonacci Delights.** Using the above results with $p = 5$ and $p = 6$, we can extract delightful dividends.

$$\sum_{n=3}^{\infty} \frac{1}{F^2_{2n} + 25} = \sum_{n=1}^{\infty} \left(\sum_{i=-2}^{2} \frac{1}{F^2_{10n+2i} + 25} \right) = -\frac{971}{11,050} + \frac{\sqrt{5}}{22};$$

$$\sum_{n=3}^{\infty} \frac{1}{L_{2n}^2 - 125} = \sum_{n=1}^{\infty} \left(\sum_{i=-2}^{2} \frac{1}{L_{10n+2i}^2 - 125} \right) = \frac{3,455}{133,342} - \frac{\sqrt{5}}{110};$$

$$\sum_{n=3}^{\infty} \frac{1}{F_{2n+1}^2 - 25} = \sum_{n=1}^{\infty} \left[\sum_{i=-2}^{2} \frac{1}{F_{10n+(2i+1)}^2 - 25} \right] = \frac{18,569}{169,400} - \frac{\sqrt{5}}{22};$$

$$\sum_{n=3}^{\infty} \frac{1}{L_{2n+1}^2 + 125} = \sum_{n=1}^{\infty} \left[\sum_{i=-2}^{2} \frac{1}{L_{10n+(2i+1)}^2 + 125} \right] = -\frac{25,552}{1,335,411} + \frac{\sqrt{5}}{110};$$

$$\sum_{n=3}^{\infty} \frac{1}{F_{2n+1}^2 + 64} = \sum_{n=1}^{\infty} \left[\sum_{i=-2}^{3} \frac{1}{F_{12n+(2i-1)}^2 + 64} \right] = -\frac{16,257}{393,380} + \frac{\sqrt{5}}{48};$$

$$\sum_{n=3}^{\infty} \frac{1}{L_{2n+1}^2 - 320} = \sum_{n=1}^{\infty} \left[\sum_{i=-2}^{3} \frac{1}{L_{12n+(2i-1)}^2 - 320} \right] = \frac{220,953}{19,298,224} - \frac{\sqrt{5}}{240};$$

$$\sum_{n=4}^{\infty} \frac{1}{F_{2n}^2 - 64} = \sum_{n=1}^{\infty} \left(\sum_{i=-2}^{3} \frac{1}{F_{12n-2i}^2 - 64} \right) = \frac{396,227}{7,983,360} - \frac{\sqrt{5}}{48};$$

$$\sum_{n=4}^{\infty} \frac{1}{L_{2n}^2 + 320} = \sum_{n=1}^{\infty} \left(\sum_{i=-2}^{3} \frac{1}{L_{12n-2i}^2 + 320} \right) = -\frac{130,922,537}{17,279,637,984} + \frac{\sqrt{5}}{144};$$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{F_{4n-1}^2 + 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{12n-5}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n-1}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+3}^2 + 64} \\
&= -\frac{77}{4,080} + \frac{\sqrt{5}}{96}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{12n-4}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} \\
&= \frac{183}{7,040} - \frac{\sqrt{5}}{96}; \\
\sum_{n=2}^{\infty} \frac{1}{F_{4n+1}^2 + 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{12n-3}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+1}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+5}^2 + 64} \\
&= -\frac{1,247}{55,536} + \frac{\sqrt{5}}{96}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{4n-1}^2 - 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{12n-5}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n-1}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+3}^2 - 320} \\
&= \frac{265}{40,128} - \frac{\sqrt{5}}{480}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{4n}^2 + 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{12n-4}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} \\
&= -\frac{113}{26,568} + \frac{\sqrt{5}}{480}; \\
\sum_{n=2}^{\infty} \frac{1}{L_{4n+1}^2 - 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{12n-3}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+1}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+5}^2 - 320} \\
&= \frac{5,369}{1,108,032} - \frac{\sqrt{5}}{480}.
\end{aligned}$$

3.2. **Additional Gibonacci Delectables.** With the above sums, we can compute more exotic gibonacci sums [3].

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n}^2 - 1} = \sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 1} + \sum_{n=2}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 1} = \frac{8}{9} - \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 + 5} = \sum_{n=2}^{\infty} \frac{1}{L_{4n}^2 + 5} + \sum_{n=2}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 5} = -\frac{8}{63} + \frac{\sqrt{5}}{15};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 + 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n-1}^2 + 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{2} + \frac{\sqrt{5}}{3};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 - 5} = \sum_{n=1}^{\infty} \frac{1}{L_{4n-1}^2 - 5} + \sum_{n=2}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{4} - \frac{\sqrt{5}}{15};$$

$$\sum_{n=3}^{\infty} \frac{1}{F_{2n+1}^2 + 64} = \sum_{n=1}^{\infty} \frac{1}{F_{4n-1}^2 + 64} + \sum_{n=2}^{\infty} \frac{1}{F_{4n+1}^2 + 64} = -\frac{16,257}{393,380} + \frac{\sqrt{5}}{48};$$

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{F_{4n}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{4(3n-1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{4(3n)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{4(3n+1)}^2 - 64} \\ &= \frac{183}{7,040} - \frac{\sqrt{5}}{96};\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(2n+1)}^2 + 64} = \sum_{n=1}^{\infty} \frac{1}{F_{3(4n-1)}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{3(4n+1)}^2 + 64} = -\frac{1}{68} + \frac{\sqrt{5}}{144};$$

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{F_{2(2n+1)}^2 - 64} &= \sum_{n=1}^{\infty} \frac{1}{F_{2(6n-1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{2(6n+1)}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{2(6n+3)}^2 - 64} \\ &= \frac{3,431}{145,152} - \frac{\sqrt{5}}{96};\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{3(2n+1)}^2 - 320} = \sum_{n=1}^{\infty} \frac{1}{L_{3(4n-1)}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{3(4n+1)}^2 - 320} = \frac{1}{304} - \frac{\sqrt{5}}{720};$$

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{L_{2(2n+1)}^2 + 320} &= \sum_{n=1}^{\infty} \frac{1}{L_{2(6n-1)}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+1)}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{2(6n+3)}^2 + 320} \\ &= -\frac{139}{30,268} + \frac{\sqrt{5}}{480}.\end{aligned}$$

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