THE ACCELERATED ZECKENDORF GAME

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Abstract. The Zeckendorf decomposition of a positive integer n is the unique set of nonconsecutive Fibonacci numbers that sum to n. Baird-Smith, et al., defined a game on Fibonacci decompositions of n, called the Zeckendorf Game. This paper introduces a variant of the Zeckendorf Game, called the Accelerated Zeckendorf Game, where a player may play as many moves of the same type as possible on their turn. We prove that a sharp lower bound on the game length of the Accelerated Zeckendorf Game is $k-1$, where k is the index of the largest term in the Zeckendorf decomposition of n. We conjecture that Player 1 has a winning strategy if $n > 9$. We conjecture that the distribution of game lengths tends to a Gaussian as n goes to infinity, and that the average game length grows sublinearly in n.

1. Introduction

1.1. **Previous Work.** The Fibonacci numbers are one of the most famous sequences of integers, and have many fascinating properties. Defining them as $F_1 = 1$, $F_2 = 2$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 3$, Zeckendorf [29] proved that every natural number can be written uniquely as a sum of nonsubsequent Fibonacci numbers, called the Zeckendorf decomposition. There is extensive literature on properties and generalizations of Zeckendorf decompositions [4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 22, 26, 27, 28].

Baird-Smith, et al., [2, 3] created a two-player game on the Fibonacci numbers, called the Zeckendorf Game. Since then, several variations have been explored [1, 6, 10, 11, 24, 25]. We describe the rules of the game from [3].

Definition 1.1 (The two-player Zeckendorf Game). *Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$ for $i \geq 2$. At the beginning of the game, there is an unordered list of n copies of F_1. Let k be the index of the largest Fibonacci number in the Zeckendorf decomposition of n. A game state is represented by the tuple of integers $(a_k, a_{k-1}, a_{k-2}, \dots, a_1)$, where a_j is the current number of copies of F_j. Hence, at the beginning of the game, the game state is $(0, \dots, 0, 0, n)$. On each turn, a player can make one of the following moves.*

1. *If the list contains two consecutive Fibonacci numbers, F_{i-1}, F_i, then a player can change these to F_{i+1}. We denote this move by C_i.*
2. *If the list has two copies of a Fibonacci number F_i,*
   - (a) *if $i = 1$, a player can change F_1, F_1 to F_2. We denote this move by C_1.*
   - (b) *if $i = 2$, a player can change F_2, F_2 to F_1, F_3. We denote this move by S_2.*
   - (c) *if $i \geq 3$, a player can change F_i, F_i to F_{i-2}, F_{i+1}. We denote this move by S_i.*

The players alternate moving. The game ends when no more moves can be made. The last player to move wins.

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The moves of the game are derived from the Fibonacci recurrence, either combining terms to make the next in the sequence via a C_i move or splitting terms with multiple copies via an S_i move. We call the C_is *combining moves* and the S_is *splitting moves*. The two-player Zeckendorf Game has been generalized to positive linear recurrences [2, 6]. We define and analyze the Accelerated Zeckendorf Game using the same Fibonacci recurrence as the Zeckendorf Game, with slightly modified rules.

1.2. **Accelerated Zeckendorf Game.** Instead of each player making one combining or splitting move on their turn, each player may perform the same move as many times as they wish as long as it is a valid move. For example, if there were 7 terms on index 5, a player could perform up to three splitting moves on index 5 on their turn. We describe the game formally as follows.

Definition 1.2 (The two-player Accelerated Zeckendorf Game)**.** *Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$. At the beginning of the game, there is an unordered list of n copies of F_1. Let k be the index of the largest Fibonacci number in the Zeckendorf decomposition of n. A game state is represented by the tuple of integers $(a_k, a_{k-1}, a_{k-2}, \ldots, a_1)$, where a_j is the current number of copies of F_j. Hence, at the beginning of the game, the game state is $(0, \ldots, 0, 0, n)$. On each turn, a player can make one of the following moves.*

(1) *If the list contains at least m copies of both F_{i-1} and F_i for some $m > 0$, then a player can change m copies of F_{i-1} and m copies of F_i to m copies of F_{i+1}. We denote this move by $m \cdot C_i$.*

(2) *If the list contains at least $2m$ copies of F_i for some $m > 0$, then*
   (a) *if $i = 1$, a player can change $2m$ copies of F_1 to m copies of F_2. We denote this move by $m \cdot C_1$.*
   (b) *if $i = 2$, a player can change $2m$ copies of F_2 to m copies of F_1 and m copies of F_3. We denote this move by $m \cdot S_2$.*
   (c) *if $i \geq 3$, a player can change $2m$ copies of F_i to m copies of F_{i-2} and m copies of F_{i+1}. We denote this move by $m \cdot S_i$.*

The players alternate moving. The game ends when no moves can be made. The player to make the last move wins.

1.3. **Results and Conjectures.** Each game of the Accelerated Zeckendorf Game can be associated to a game of the Zeckendorf Game in which each $m \cdot C_i$ and $n \cdot S_j$ is replaced by m instances of a C_i move and n instances of an S_j move, respectively. Thus, the Accelerated Zeckendorf Game terminates after a finite number of moves at the Zeckendorf decomposition follows immediately because the Zeckendorf Game does [3]. Moreover, the same reasoning shows that the maximum number of moves in the Accelerated Zeckendorf Game on n is exactly equal to the maximum number of moves in the Zeckendorf Game on n. In particular, the upper bound on game length derived in [11] also holds for the Accelerated Zeckendorf Game. We summarize these facts in the following theorem.

Theorem 1.3. *Every game terminates after a finite number of moves at the Zeckendorf decomposition of n. An upper bound for the number of moves in the Accelerated Zeckendorf Game on n is $\frac{\sqrt{5}+3}{2}n - IZ(n) - \frac{1+\sqrt{5}}{2}Z(n)$, where $Z(n)$ denotes the number of terms in the Zeckendorf decomposition of n, and $IZ(n)$ denotes the sum of the indices of the terms in the Zeckendorf decomposition of n.*

Our first result concerning the Accelerated Zeckendorf Game is the number of moves in the shortest game on n.

Theorem 1.4. *If k is the index of the greatest Fibonacci number in the Zeckendorf decomposition of n, then $k-1$ is a sharp lower bound on the number of moves in the Accelerated Zeckendorf Game on n.*

Thus, the shortest Accelerated Zeckendorf Game on n is much shorter than the shortest Zeckendorf Game on n, which takes $n - Z(n)$ moves [3].

Next, we investigate winning strategies. As with the Zeckendorf Game, one of the two players in the Accelerated Zeckendorf Game must have a winning strategy, i.e., one player must have a strategy by which they can force their victory. This is because the game terminates in a finite number of moves and one of the two players must make the last move.

Proposition 1.5. *If Player 1 has a winning strategy, Player 1 has only one correct first move. In other words, there exists only one first move that will maintain Player 1's winning strategy.*

Conjecture 1.6. *If $n > 9$, Player 1 has a winning strategy.*

Note that Conjecture 1.6 is in stark contrast with the classical situation, because Player 2 always has a winning strategy in the Zeckendorf Game on n when $n > 2$ [3]. In Section 2.2, we present a method of proving Conjecture 1.6 by reducing it to the following conjecture. A *losing state* refers to a game state in which the current player does not have a winning strategy, whereas a *winning state* refers to a game state in which the current player has a winning strategy.

Conjecture 1.7. *If Player 2 has a winning strategy, then all game states of the form $(0, \ldots, 0, k, 0, n-3k)$ are losing states.*

Theorem 1.8. *Conjecture 1.7 implies Conjecture 1.6.*

Lemma 1.9. *Conjecture 1.7 is true for $k \in \{1, 2, 3, 4, 5\}$.*

We are also interested in studying the distribution of game lengths when the Accelerated Zeckendorf Game is simulated with uniform random moves (see Appendix A).

Conjecture 1.10. *As n goes to infinity, the number of moves in a random Accelerated Zeckendorf Game on n, when all legal moves are equally likely, converges to a Gaussian.*

The same conjecture has been made for the Zeckendorf Game (see Conjecture 1.4 in [3]). However, the next conjecture is different for the Accelerated Zeckendorf Game.

Conjecture 1.11. *The average game length grows at a sublinear rate with n.*

This is in contrast to the Zeckendorf Game, where the average game length appears to grow at a linear rate with n (see Conjecture 1.6 in [3]).

2. Proofs

2.1. **Strict Lower Bound on Game Length.**

Lemma 2.1. *If k is the index of the greatest Fibonacci number in the Zeckendorf decomposition of n, then $k-1$ is a lower bound on the number of moves in the Accelerated Zeckendorf Game on n.*

Proof. Let F_k be the greatest Fibonacci number in the Zeckendorf decomposition of n. As such, we must have an F_k term when the game ends.

An $m \cdot C_i$ or $m \cdot S_i$ move can create F_{i+1} terms, but cannot create terms of a higher index. Thus, we may extend backwards: to get an F_k term, we must have made an $m \cdot C_{k-1}$ or

$m \cdot S_{k-1}$ move; to get an F_{k-1} term, we must have made an $m \cdot C_{k-2}$ or $m \cdot S_{k-2}$ move; and so on. As such, we must make an $m \cdot C_i$ or $m \cdot S_i$ move for all $i < k$ to get an F_k term. Therefore, the number of moves in the Accelerated Zeckendorf Game must be at least $k - 1$. □

In the following lemma, $F_0 = 1$ and $F_{-1} = 0$

Lemma 2.2. *Let $p \geq 2$ and $i \geq 1$. Performing the move $F_{i-1} \cdot C_p$ when there are F_{i-1} terms in the index $p-1$ and F_i terms in the index p will result in no terms in the index $p-1$, F_{i-2} terms in the index p, and F_{i-1} terms in the index $p+1$.*

Proof. Note $F_i - F_{i-1} = F_{i-2}$. Performing the move $F_{i-1} \cdot C_p$ will remove F_{i-1} terms from both index $p-1$ and p, while adding F_{i-1} terms to index $p+1$. Thus, after applying the move $F_{i-1} \cdot C_p$, we are left with the state of 0 terms at index $p-1$, F_{i-2} terms at index p, and F_{i-1} terms at index $p+1$. □

Using these lemmas, we can prove that the lower bound is sharp when n is a Fibonacci number.

Lemma 2.3. *If $n = F_k$, then $k - 1$ is a sharp lower bound on the number of moves in the Accelerated Zeckendorf Game on n.*

Proof. It is easy to check the cases $k = 1, 2$. Assume that $k \geq 3$. Starting with F_k ones, we can perform the move $F_{k-2} \cdot C_1$. This removes $2 \cdot F_{k-2}$ ones, and as such leaves us with $F_k - 2 \cdot F_{k-2}$ ones. Because $F_i = F_{i-1} + F_{i-2}$ for all i, we show that $F_k - 2 \cdot F_{k-2} = F_{k-3}$.

$$F_k - 2 \cdot F_{k-2} = F_{k-1} + F_{k-2} - F_{k-2} - F_{k-2} = F_{k-3}. \tag{2.1}$$

From this position, where we have F_{k-3} ones and F_{k-2} twos, we could repeatedly perform the moves outlined in Lemma 2.2 until we run out of terms to combine, which results in the moves $F_{k-3} \cdot C_2, F_{k-4} \cdot C_3, \ldots, F_1 \cdot C_{k-2}, F_0 \cdot C_{k-1}$. This series of $k-2$ moves gives us 1 term in index k, which is the Zeckendorf decomposition of F_k. Thus, doing these moves after the initial $F_{k-2} \cdot C_1$ move results in a game of length $k-1$. From Lemma 2.1, we know that the game cannot be shorter, so this lower bound is sharp. □

We have now shown that the lower bound $k - 1$ is sharp when n is a Fibonacci number. From here, we can prove Theorem 1.4.

Proof of Theorem 1.4. Let $n = F_{k_1} + F_{k_2} + \cdots + F_{k_m}$ where each F_j is a Fibonacci number in the Zeckendorf decomposition of n, and $F_{k_1} > F_{k_2} > \cdots > F_{k_m}$. Taking each term in the Zeckendorf decomposition, we know from Lemma 2.3 that it will take $k_1 - 1$ moves to terminate the game with F_{k_1} ones, $k_2 - 1$ moves to terminate the game with F_{k_2} ones, and so on. If we do this individually for each term in the Zeckendorf decomposition of n, we get the sequence of moves

$$((F_{k_i-j-1} \cdot C_j)_{j=1}^{k_i-1})_{i=1}^{m}. \tag{2.2}$$

This gives us $(k_1 + k_2 + \cdots + k_m) - m$ moves. However, we may group all combining moves of the same index together as follows:

$$\left(\left(\sum_{i=1}^{m} F_{k_i-j-1} \right) \cdot C_j \right)_{j=1}^{k_1-1}, \tag{2.3}$$

where we interpret $F_l = 0$ when $l < 0$ and $F_0 = 1$. This results in a game of $k_1 - 1$ moves. □

An example of the shortest game for $n = 46$ can be found in Appendix B.1.

2.2. **Results and Conjectures on Winning Strategies.** Recall that a *losing state* refers to a game state in which the current player does not have a winning strategy, whereas a *winning state* refers to a game state in which the current player has a winning strategy. Here are some rules for reasoning with winning and losing states.

(1) The final game state is a losing state.
(2) All moves that come from a losing state lead to winning states, as if a player does not have a winning strategy, their opponent must have a winning strategy.
(3) All moves that lead to a losing state come from winning states, as if a player can place their opponent into a state where their opponent has no winning strategies, then the player has a winning strategy, namely making that move.
(4) A winning state must have at least one move that leads to a losing state.

Proof of Proposition 1.5. Assume Player 1 has a winning strategy. This means that they must have at least one correct first move. These correct moves will bring Player 2 to a losing state.

Assume for a contradiction that there are at least two correct first moves for Player 1. Because all of the moves that Player 1 can make initially are of the form $m \cdot C_1$, suppose without loss of generality that two of these correct moves are $m_1 \cdot C_1$ and $m_2 \cdot C_1$, where $m_1 > m_2$. However, if Player 1 initially made the move $m_2 \cdot C_1$, Player 2 can make the move $(m_1 - m_2) \cdot C_1$. This would bring Player 1 to a losing state, because $m_2 \cdot C_1$ followed by $(m_1 - m_2) \cdot C_1$ brings about the same state as $m_1 \cdot C_1$. Thus, the initial move $m_2 \cdot C_1$ was not a correct move, a contradiction. □

To determine which player has a winning strategy for specific values of n, we created a program (see Appendix A). We tested all games up to $n = 140$, and found that for all $n > 9$, Player 1 had a winning strategy. This supports Conjecture 1.6.

Following the rules for winning and losing states, we can represent them, and the moves between them, using a graph. We will construct these graphs in the following manner.

(1) Each node represents a game state.
(2) Valid moves are represented by arrows pointing from one node to another.
(3) Winning states are colored green.
(4) Losing states are colored yellow.

See Appendix B.2 for an example graph when $n = 8$.

Lemma 2.4. *Let k be an odd positive integer, and let $n \geq 2k$. Assume all game states of the form $(0, \dots, 0, i, 0, n-3i)$ with $i < k$ are losing states. Then, $n \geq 3k$ and $(0, \dots, 0, k, 0, n-3k)$ is the only losing state reachable from $(0, \dots, 0, k, n-2k)$ by a single accelerated move.*

Proof. See Figure 1 for a graph of the proof. By assumption, $(0, \dots, 0, n)$ is a losing state. Thus, $(0, \dots, k, n-2k)$ is a winning state because it is reachable from $(0, \dots, 0, n)$ by the move $k \cdot C_1$. Therefore, there must be at least one move from this game state that leads to a losing state. The only valid moves that can be made from this game state are of the form $m_1 \cdot C_1$, $m_2 \cdot S_2$, or $m_3 \cdot C_2$, that all lead to game states of the form $(0, \dots, 0, a, b, n - 3a - 2b)$, where $a \leq k$ and $(3a + 2b) \leq n$. When $a < k$ and $b > 0$, the move $b \cdot C_1$ leads to this game state from $(0, \dots, a, 0, n-3a)$. Because this is a losing state by our assumption, $(0, \dots, 0, a, b, n-3a-2b)$ is a winning state when $a < k$ and $b > 0$. Because k is odd, the only move that leads to a game state where $a = k$ or $b = 0$ is $k \cdot C_2$, that leads to $(0, \dots, 0, k, 0, n - 3k)$. (We cannot make the move $\frac{k}{2} \cdot S_2$ because k is odd.) Thus, $n \geq 3k$ and $(0, \dots, 0, k, 0, n - 3k)$ is the only losing state reachable from $(0, \dots, 0, k, n - 2k)$ by a single accelerated move.

□

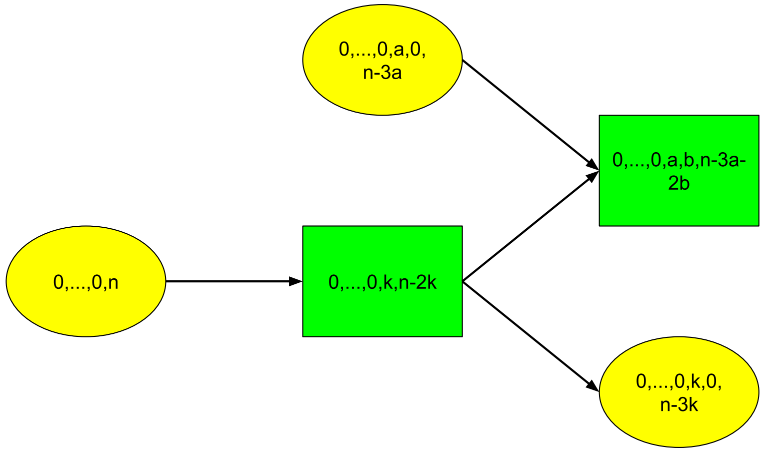

FIGURE 1. Graph for the proof of Lemma 2.4. All game states aside from $(0,\ldots,0,k,0,n-3k)$ that the winning state $(0,\ldots,0,k,n-2k)$ connects to are winning states by the initial assumption, so $(0,\ldots,0,k,0,n-3k)$ is forced losing.

Proof of Theorem 1.8. Assume Conjecture 1.7 and assume for contradiction that Conjecture 1.6 is false. As such, there exists an $n > 9$ such that Player 2 has a winning strategy. Because $n > 9$, there exists an odd integer k such that $2k \leq n < 3k$. By Conjecture 1.7, all game states of the form $(0,\ldots,0,i,0,n-3i)$ with $i < k$ are losing states. As such, $n \geq 3k$ by Lemma 2.4, a contradiction. □

This proof also correctly predicts Player 1's winning strategy for $n = 2$, $n = 6$, $n = 7$, and $n = 8$, because there exists odd k such that $2k \leq n < 3k$ for these n. Furthermore, it does not predict Player 1's winning strategy for $n = 1$, $n = 3$, $n = 4$, $n = 5$, and $n = 9$, and for these values of n Player 2 has a winning strategy.

Proof of Lemma 1.9. From the starting assumption that Player 1 does not have a winning strategy,

(1) $(0,\ldots,0,1,0,n-3)$ is a losing state by Lemma 2.4,
(2) $(0,\ldots,0,2,0,n-6)$ is a losing state by contradiction, see Figure 2,
(3) $(0,\ldots,0,3,0,n-9)$ is a losing state by Lemma 2.4,
(4) $(0,\ldots,0,4,0,n-12)$ is a losing state by contradiction, see Figure 3, and
(5) $(0,\ldots,0,5,0,n-15)$ is a losing state by Lemma 2.4.

□

We have not yet been able to construct a proof showing that $(0,\ldots,0,6,0,n-18)$ is a losing state. If there were a proof that $(0,\ldots,0,k,0,n-3k)$ is a losing state for even k, then we could prove Conjecture 1.7 through strong induction using that proof and Lemma 2.4.

2.3. **Conjectures on Average Game Length.** In this section, we address the game length of the Accelerated Zeckendorf Game. We support two conjectures on game length using Java code (see Appendix A). In support of Conjecture 1.10, we provide a program that runs simulations of the game with random moves for some fixed n and calculates the frequency of each game length. The data points in Figure 4 depict the results of the program after 9,999 simulations for $n = 50$.

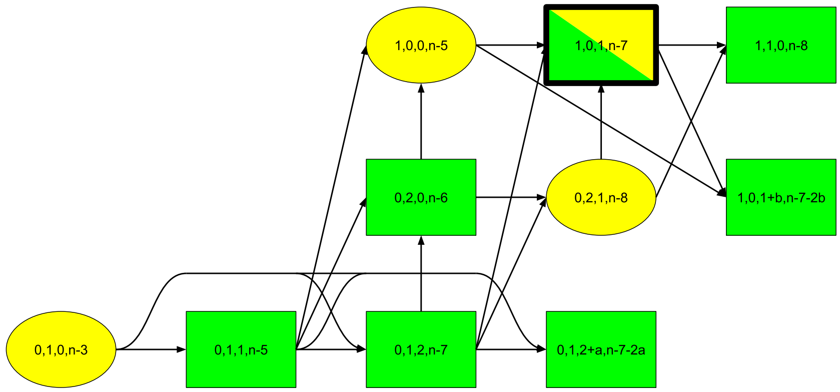

FIGURE 2. Proof by contradiction that $(0, \ldots, 0, 2, 0, n-6)$ is a losing state when Player 1 does not have a winning strategy. If it is assumed to be winning, $(0, \ldots, 0, 1, 0, 1, n-7)$ would be a winning state from which no moves lead to losing states, a contradiction.

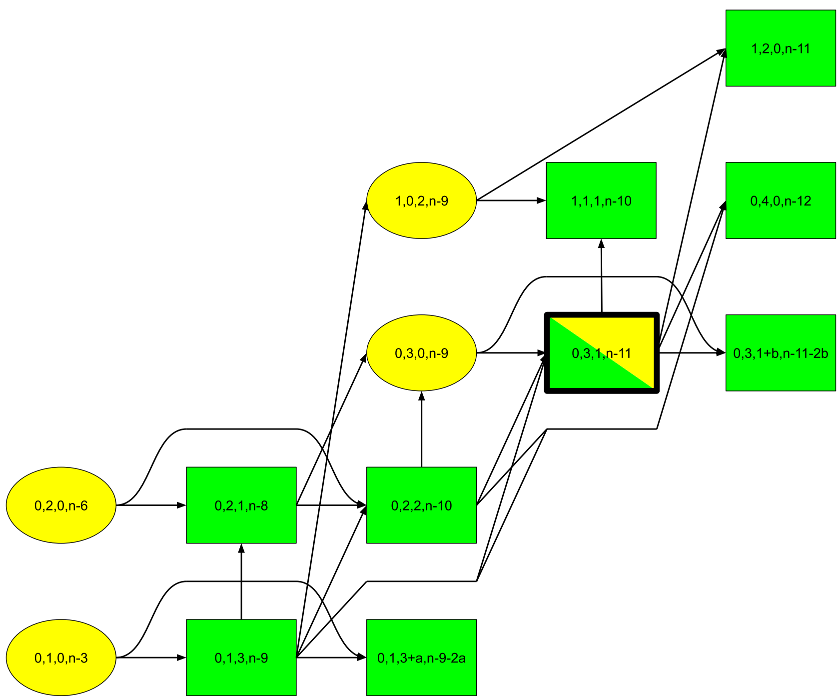

FIGURE 3. Proof by contradiction that $(0, \ldots, 0, 4, 0, n-12)$ is a losing state when Player 1 does not have a winning strategy. If it is assumed to be winning, $(0, \ldots, 0, 3, 1, n-11)$ would be a winning state from which no moves lead to losing states, a contradiction.

We have laid the best fitting Gaussian over the data, showing how tight the fit is for this experiment. Figure 5 shows data from $9,999$ simulations for $n = 100$, as well as that data set's best fitting Gaussian.

In support of Conjecture 1.11, we created a program that runs a specified number of simulations of the Accelerated Zeckendorf Game with random moves and takes the average game

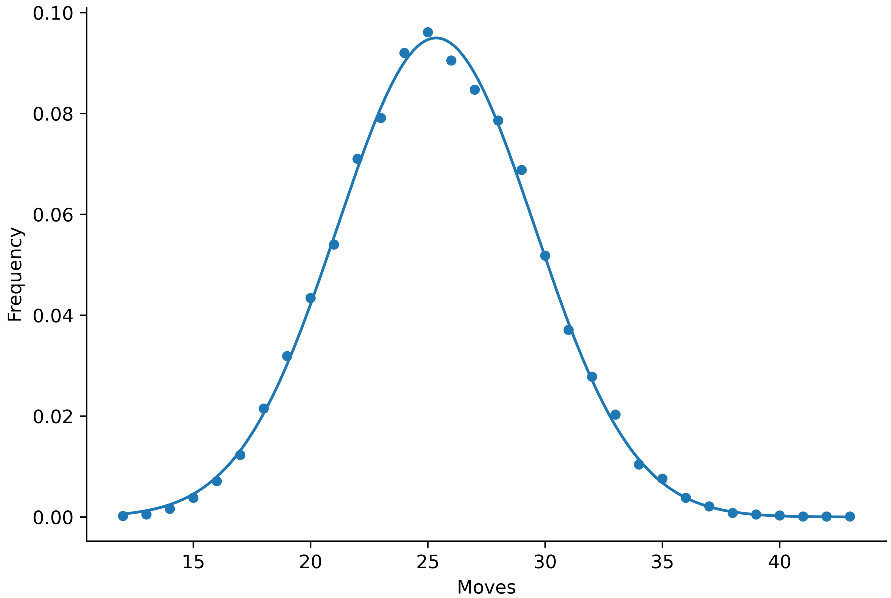

Figure 4. Graph of the frequency of the number of moves in $9,999$ simulations of the Accelerated Zeckendorf Game with random move, where each legal move has a uniform probability for $n = 50$ with the best fitting Gaussian (mean $\approx$ 25.4, standard deviation $\approx$ 4.2).

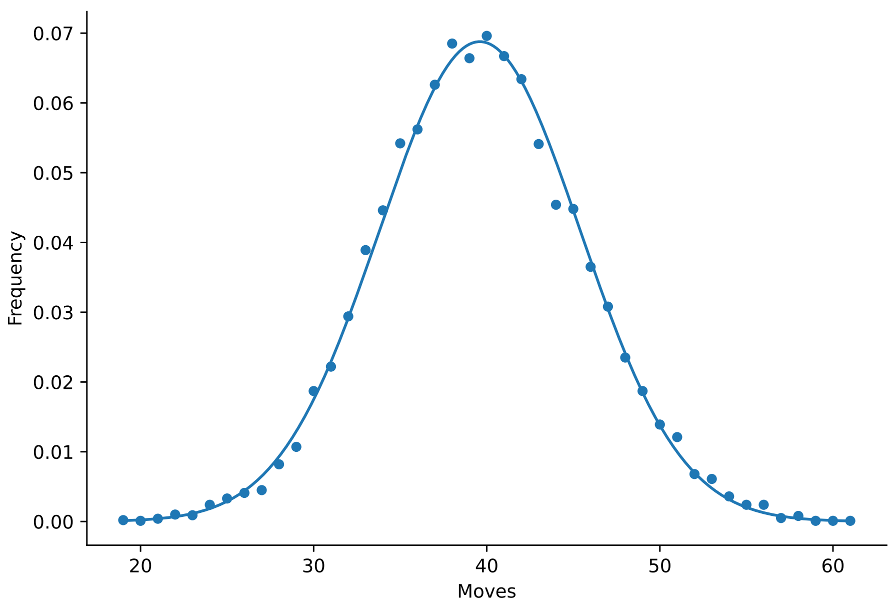

Figure 5. Graph of the frequency of the number of moves in $9,999$ simulations of the Accelerated Zeckendorf Game with random moves, where each legal move has a uniform probability for $n = 100$ with the best fitting Gaussian (mean $\approx$ 39.6, standard deviation $\approx$ 5.8).

length over these simulations for each n in a specified range. We ran the program with $9,999$ simulations and gave the starting and ending parameters of 1 and 99 for n. The results of the simulation are plotted in Figure 6.

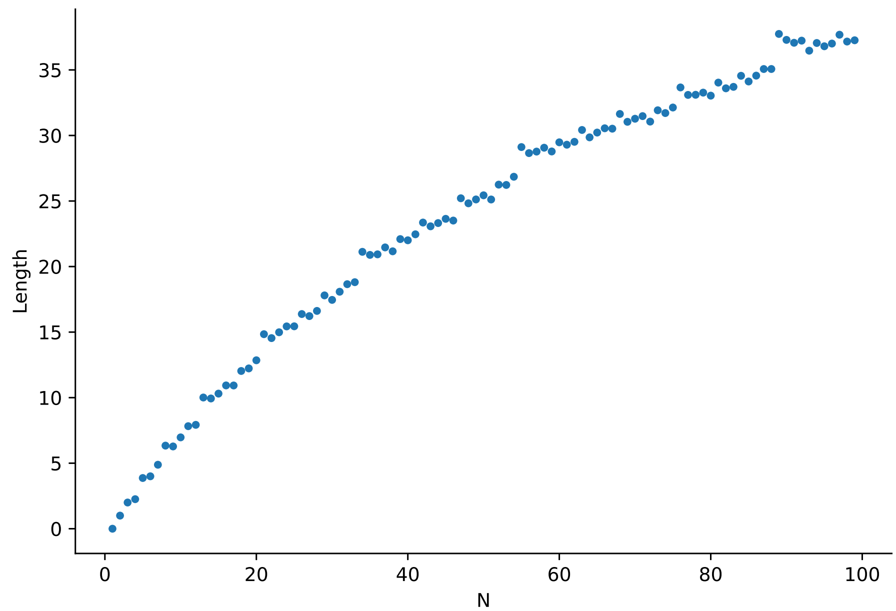

FIGURE 6. Graph of the average number of moves in the Accelerated Zeckendorf Game with random uniform moves with 9,999 simulations with n varying from 1 to 99.

One interesting observation from Figure 6 is how there seems to be a significant "jump" in the average game length when n is a Fibonacci number.

From Theorem 1.3, we know that the upper bound on the game grows linearly. Also, Theorem 1.4 shows that the shortest game for n grows linearly with k, where k is the index of the largest Fibonacci number less than or equal to n. Because the Fibonacci numbers grow exponentially, as can be seen by Binet's Formula ($F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$), the indices grow logarithmically with respect to Fibonacci numbers. Thus, the shortest Accelerated Zeckendorf Game grows logarithmically with respect to n. Because the longest game grows linearly and the shortest game grows logarithmically, the average game length must grow between a logarithmic and linear rate. The data seen in Figure 6 strongly suggests that the growth rate is sublinear.

3. Future Work

We have provided substantial evidence in support of Conjecture 1.6, verifying it computationally for all $n > 9$ up to $n = 140$. Yet, it still remains to prove Conjecture 1.6 and we would like to prove it. A more difficult problem is to find explicit winning strategies for Player 1. Figure 7 shows an explicit winning strategy for Player 1 when $n = 8$. Is there a general pattern to these winning strategies?

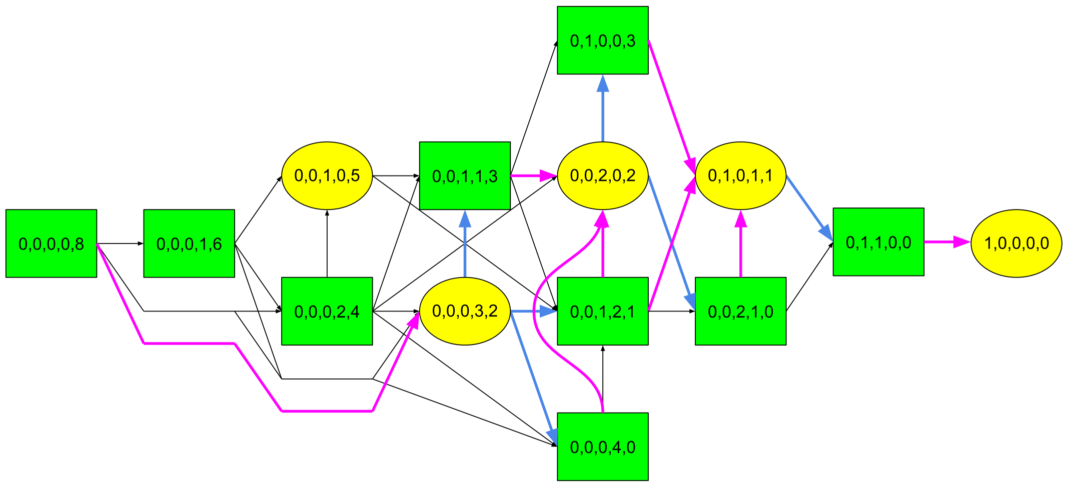

FIGURE 7. A full coloring of the graph of the Accelerated Zeckendorf Game when $n = 8$. Player 1's winning strategies are marked by the colored arrows, where Player 1's moves are colored pink and Player 2's moves are colored blue.

Assuming Conjecture 1.6, we know by Proposition 1.5 that for each $n > 9$, there is a *unique* move Player 1 can make to preserve their winning state. It is $m \cdot C_1$ for some positive integer $m = m(n)$ depending on $n > 9$. Thus, we get a sequence of positive integers $m(10)$, $m(11)$, $m(12), \ldots$. We believe it is interesting to study this sequence. Can it be determined explicitly? Does it behave randomly? What is its growth rate?

One can see in Figure 6 that the average length of the Accelerated Zeckendorf Game grows at a sublinear rate. However, we would like to investigate the precise type of sublinear growth, such as logarithmic versus n^δ for $\delta < 1$. Another avenue for future work is to explore the relationship between the conjectured "gaussianity" of random Zeckendorf Games and "gaussianity" of Random Accelerated Zeckendorf Games. Can it be shown that one implies the other?

We only looked at the Accelerated two-player Zeckendorf Game for the standard Fibonacci sequence. However, this accelerated variation can be adapted to many other versions of the Zeckendorf Game, such as the Generalized Zeckendorf Game [2], the Fibonacci Quilt Game [24], and the Multi-player Zeckendorf Game [10].

Appendix A. Code

The programs used to check which player has a winning strategy, simulate random Accelerated Zeckendorf Games, and to find the average game length are available at the repository linked below.

`https://github.com/ThomasRascon/Accelerated-Zeckendorf-Game.git`

Appendix B. Example Games and Graphs

B.1. **Shortest Game for $n = 46$.** $46 = 34+8+3+1 \to ((13+3+1)\cdot C_1, (8+2+1)\cdot C_2, (5+1)\cdot C_3, (3+1)\cdot C_4, 2\cdot C_5, 1\cdot C_6, 1\cdot C_7) \to (17\cdot C_1, 11\cdot C_2, 6\cdot C_3, 4\cdot C_4, 2\cdot C_5, 1\cdot C_6, 1\cdot C_7)$

B.2. **Example Game and Graph for n=8.** Figure 7 shows a full coloring of the graph of all of the game states for the Accelerated Zeckendorf Game when $n = 8$. One such winning game for Player 1 is $(3\cdot C_1, 1\cdot C_1, 2\cdot S_2, 1\cdot C_1, 1\cdot S_3, 1\cdot C_2, 1\cdot C_4)$.

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