AN EVALUATION OF RATIONALLY WEIGHTED BINOMIAL SUMS VIA SOME DIFFERENTIAL AND INTEGRAL OPERATORS

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Abstract. In this paper, we use differential and integral operators, as started by Scherk in his 1823 Ph.D. thesis and rediscovered by Gauthier in 1995, to find closed forms of some rationally weighted binomial sums and several generalizations. As consequences of Gauthier's approach, we prove some identities in the literature, for example, Epsteen (1904), Finkel (1909), Greenstreet (1904), or, more recent ones (like Koshy's book from 2018). In the course of the proofs, we naturally generalize the Stirling numbers of the second kind and express some of our identities in terms of these numbers.

1. Introduction

We let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ be the ring of integers, respectively, the fields of rational, real, and complex numbers.

Between 1904 and 1909, Epsteen [2], Finkel [3], and Greenstreet [9] proposed the following weighted binomial sums identities:

$$\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1}-1}{n+1},$$
$$\sum_{r=1}^{n} (-1)^{r-1}\frac{1}{r}\binom{n}{r} = \sum_{r=1}^{n} \frac{1}{r},$$
$$\sum_{r=1}^{n} (-1)^{r}\binom{n}{r} 2^{n-r} r^2 = n^2 - 2n.$$

The current authors were certain that there are other such problems proposed since the beginning of the 20th century, and a quick search revealed that [7, Identities (1.46), (1.134)] (see the similarity to our third equation in (3.3)), [8, Probs. 13, 14 on p. 63], [10, Chapter 24, Exercises 1–2, 7–9; Chapter 25, Identities 25.1–25.4, 25.9–25.14, 25.18, 25.23, 25.25–25.26, 25.32, 25.34–25.36, Exercises on pp. 520–521], [17, Chapter IV, pp. 54–56,] contain examples of such identities.

Besides proving identities, the method we use in this paper can also be used to provide alternative proofs. For example, Theorems 4.1, 4.2, and 4.3 of [1] can be shown by taking the generating function of the (even indexed) Fibonacci/Lucas, applying the differential operator as in our paper and basic algebraic manipulations. These identities have the common form $\sum_{r=1}^{n} a^r f(r)$, or $\sum_{r=1}^{n} a^r \frac{1}{f(r)}\binom{n}{r}$, where a is a constant and $f \in \mathbb{Q}[x]$ (or any other field of coefficients, for that matter). In the first case, $f(r)$ could involve the binomial coefficient, but it is not necessary for our first result.

In this paper, we will describe a differential and integral operators' method designed to find closed forms for binomial sums with polynomial or rational weights, and as a byproduct, we easily show the three identities above. The methods for addressing these identities are provided in Theorem 2.1 and 3.1. For expositional purposes, we explore some consequences of Theorem 2.1 (see Corollaries 2.3, 2.4, 2.6, 2.7, and 2.8) before stating and proving Theorem 3.1.

2. A General Result

We were made aware by Koshy's book [10, Chapter 25, pp. 514–521] that the expression $\sum_{i=1}^{n} i^2 F_i = (n+1)^2 F_{n+2} - (2n+3)F_{n+4} + 2F_{n+6} - 8$, and a similar one for the Lucas sequence, were "developed algebraically by P. Glaister [6] of the University of Reading, England, and by N. Gauthier [4] of The Royal Military College of Canada," by using a differential operator technique. The differential operator method was developed in Scherk's Ph.D. thesis [15] written in the early 19th century. Scherk got his Ph.D. under Friedrich Wilhelm Bessel and Heinrich Wilhelm Brandes and was the Ph.D. advisor for Ernst Eduard Kummer. Before we proceed, we point out that Gauthier [5] showed (we slightly change notations to match our indices in the theorem below) that

$$\sum_{k=0}^{N} k^r x^k = x^N \sum_{i=0}^{r} a_i^{(r)}(x) N^i - a_0^{(r)}(x), \text{ where } N \geq 0 \text{ is an integer and}$$

$$a_i^{(r)}(x) = \frac{x}{x-1}\left(\binom{r}{i} - \sum_{j=i+1}^{r} \binom{j}{i} a_j^{(r)}(x)\right), a_\ell^{(\ell)} = \frac{x}{x-1}, \ell \geq 0.$$

By a technique similar to the one of Scherk, or Gauthier, relying on the operator $x\dfrac{\partial}{\partial x}$, we will next slightly extend and rewrite this result to point out a connection with Stirling numbers.

First however, we introduce some notation. Recall (see [8, pp. 257–267], or [14, Sect. 2.5.2, pp. 150–152]) that a *Stirling number of the second kind* $\left\{ {r \atop k} \right\}$ (or Stirling partition number) is the number of ways to partition a set of r objects into k nonempty subsets. They satisfy the recurrences $\left\{ {r+1 \atop k} \right\} = k \left\{ {r \atop k} \right\} + \left\{ {r \atop k-1} \right\}$, with boundary conditions $\left\{ {r \atop 0} \right\} = 0$, $\left\{ {0 \atop 0} \right\} = \left\{ {r \atop r} \right\} = \left\{ {r \atop 1} \right\} = 1$, $r \geq 1$. We introduce a sequence, denoted by $\left\{ {n \atop k} \right\}_s$, which can be regarded as a generalization of Stirling numbers of the second kind, which we call (generalized) s-Stirling numbers of the second kind, defined by the recurrences

$$\left\{ {r+1 \atop k} \right\}_s = (r(s+1) - r + k) \left\{ {r \atop k} \right\}_s + \left\{ {r \atop k-1} \right\}_s, \text{ with boundary conditions}$$
$$\left\{ {r \atop 0} \right\} = 0, \left\{ {0 \atop 0} \right\}_s = \left\{ {r+1 \atop r+1} \right\}_s = \left\{ {1 \atop 1} \right\}_s = 1, \left\{ {r+1 \atop 1} \right\}_s = (r(s+1) - r + 1) \left\{ {r \atop 1} \right\}_s, r \geq 1.$$

Pita-Ruiz, in an upcoming paper [13] announced in [12], embeds these sequences in a more general class of generalizations and connects them further with the differential operator. Given fixed integers k, s, r, we let the s-rising factorial of k be defined by $(k)^{(s,r)} = k(k+s)(k+2s)\cdots(k+(r-1)s)$. We now state and prove the main result of this section (see Scherk's thesis [15], as well as Mohamaad-Noori [11], also, for the first identity).

Theorem 2.1. *Let $N, r \geq 0$ be positive integers, $f(x) = \sum_{k=0}^{N} a_k x^k$, where a_k, $0 \leq k \leq N$ are arbitrary real numbers, and $f^{(i)}$ denotes the ith derivative of f. Then*

$$\sum_{k=0}^{N} k^r a_k x^k = \sum_{i=0}^{r} \left\{ {r \atop i} \right\} x^i f^{(i)}(x), \tag{2.1}$$

where the coefficients $\left\{ {r \atop k} \right\}$ are the Stirling numbers of the second kind. For a fixed real number s and integer r, we have

$$\sum_{k=0}^{N} (k)^{(s,r)} a_k x^{k+rs} = \sum_{i=0}^{r} \left\{ {r \atop i} \right\}_s x^{rs+i} f^{(i)}(x), \tag{2.2}$$

where the coefficients $\left\{ {r \atop k} \right\}_s$ are the generalized s-Stirling numbers of the second kind.

Proof. We let the operator $\Delta_1 = x\dfrac{\partial}{\partial x}$ (see also [10, p. 515]). By differentiating $f(x) = \sum_{k=0}^{N} a_k x^k$ side by side and multiplying by x, we obtain $\Delta_1(f) = xf'(x) = \sum_{k=0}^{N} k a_k x^k$. Differentiating again and multiplying by x, we get $\Delta_2(f) = \sum_{k=0}^{N} k^2 a_k x^k$. Continuing, for $r \geq 1$, we obtain

$$\Delta_r(f) = x\overbrace{\left(x(\cdots x(xf'(x))' \cdots)'\right)'}^{r \text{ derivatives}} = \sum_{k=0}^{N} k^r a_k x^k.$$

We now concentrate on the left side expression. We proceed to compute a few more expressions to observe the pattern

$$\begin{aligned}
\Delta_1(f) &= xf'(x),\\
\Delta_2(f) &= x^2 f''(x) + xf'(x),\\
\Delta_3(f) &= x^3 f^{(3)}(x) + 3x^2 f''(x) + xf'(x),\\
\Delta_4(f) &= x^4 f^{(4)}(x) + 6x^3 f^{(3)} + 7x^2 f''(x) + xf'(x).
\end{aligned}$$

In general, assuming that

$$\Delta_r(f) = b_r^{(r)} x^r f^{(r)}(x) + b_r^{(r-1)} x^{r-1} f^{(r-1)}(x) + \cdots + b_r^{(2)} x^2 f''(x) + b_r^{(1)} xf'(x),$$

with $b_r^{(r)} = b_r^{(1)} = 1$, by differentiation and combining terms, we infer that

$$\begin{aligned}
\Delta_{r+1}(f) &= x^{r+1} f^{(r+1)}(x) + rx^r f^{(r)}(x) + b_r^{(r-1)} x^r f^{(r)}(x)\\
&\quad + (r-1)b_r^{(r-1)} x^{r-1} f^{(r-1)}(x) + +b_r^{(r-2)} x^{r-1} f^{(r-1)}\\
&\quad + b_r^{(r-2)} x^{r-2} f^{(r-2)}(x) + (r-2)b_r^{(r-2)} x^{r-2} f^{(r-2)} + \cdots\\
&\quad + b_r^{(2)} x^3 f^{(3)}(x) + 2b_r^{(2)} x^2 f''(x) + x^2 f''(x) + xf'(x)\\
&= x^{r+1} f^{(r+1)}(x) + \left(b_r^{(r-1)} + r\right) x^r f^{(r)}\\
&\quad + \left((r-1)b_r^{(r-1)} + b_r^{(r-2)}\right) x^{r-1} f^{(r-1)}\\
&\quad + \left((r-2)b_r^{(r-2)} + b_r^{(r-3)}\right) x^{r-2} f^{(r-2)} + \cdots
\end{aligned}$$

$$+ \left(3b_r^{(3)} + b_r^{(2)}\right) x^3 f^{(3)} + (2b_r^{(2)} + 1)x^2 f''(x) + xf'(x).$$

By equating coefficients, we obtain the recurrences

$$\begin{aligned} b_{r+1}^{(r)} &= rb_r^{(r)} + b_r^{(r-1)} = \binom{r+1}{2}, \\ b_{r+1}^{(r-1)} &= (r-1)b_r^{(r-1)} + b_r^{(r-2)} \\ b_{r+1}^{(r-2)} &= (r-2)b_r^{(r-2)} + b_r^{(r-3)} \\ &\cdots\cdots\cdots\cdots \\ b_{r+1}^{(3)} &= 3b_r^{(3)} + b_r^{(2)} \\ b_{r+1}^{(2)} &= 2b_r^{(2)} + b_r^{(1)} = 2^r - 1, \text{ (see Scherk [15]).} \end{aligned} \tag{2.3}$$

Observe that all these displayed identities in (2.3) are precisely the recurrences for the Stirling numbers of the second kind (with the right initial boundary conditions). Therefore, the coefficients $b_r^{(k)} = \left\{ {r+1 \atop k} \right\}$. The proof of the first claim of our theorem is shown.

We now proceed to provide the argument for the general case of equation (2.2). In lieu of the operator $\Delta_1 = x\dfrac{\partial}{\partial x}$ that we used above, for a *fixed real number* s, we can consider the operator $\Delta_s = x^{s+1}\dfrac{\partial}{\partial x}$ and apply a similar technique. Similarly, as in the first part, we obtain that the coefficients of

$$\begin{aligned} \Delta_s^{r+1}(f(x)) &= \Delta_s(\Delta_s^r)(f(x)) \\ &= a_{r+1}^{(r+1)} x^{(r+1)(s+1)} f^{(r+1)}(x) + a_{r+1}^{(r)} x^{(r+1)(s+1)-1} f^{(r)}(x) \\ &+ a_{r+1}^{(r-1)} x^{(r+1)(s+1)-2} f^{(r-1)}(x) + \cdots \\ &+ a_{r+1}^{(2)} x^{(r+1)(s+1)-(r-1)} f^{(2)}(x) + a_{r+1}^{(1)} x^{(r+1)(s+1)-r} f'(x) \end{aligned}$$

(with initial conditions $\Delta_s^1 = \Delta_s$), satisfy the recurrences

$$\begin{aligned} a_{r+1}^{(k)} &= (r(s+1) - r + k)b_r^{(k)} + b_r^{(k-1)}, \text{ with boundary conditions} \\ a_{r+1}^{(r+1)} &= a_1^{(1)} = 1,\ a_{r+1}^{(1)} = (r(s+1) - r + 1)b_r^{(1)}. \end{aligned}$$

Therefore, the above sequence $a_n^{(k)}$ satisfies the recurrences (and the boundary conditions) of our generalization of Stirling numbers of the second kind, and the proof of the theorem is shown. □

We can find the coefficients, precisely $\left\{ {r \atop k} \right\}$, in the recurrences of Theorem 2.1 via what is known about the Stirling numbers of the second kind ([8, Chapter 6, pp. 257–267], or [14, Section 2.5.2, pp. 150–152], and the references therein), or via discrete mathematics techniques of solving nonhomogeneous linear recurrences. We provide below the first few values and a general formula for Stirling numbers of the second kind, via binomial coefficients. That way, the interested readers can give their own particular identities specializing Theorem 2.2 with r larger than 2.

Proposition 2.2. *In addition to the boundary conditions* $\left\{ {0 \atop 0} \right\} = 1$, $\left\{ {r \atop 0} \right\} = 0$, $\left\{ {r \atop 1} \right\} = \left\{ {r \atop r} \right\} = 1$, $r \geq 1$, *we have*

$$\left\{ {r \atop 2} \right\} = \binom{r}{2} \text{ for } r \geq 2,$$

$$\left\{ {r \atop 3} \right\} = \frac{3^{r-1}+1}{2} - 2^{r-1} \text{ for } r \geq 3,$$

$$\left\{ {r \atop 4} \right\} = \frac{101 \cdot 2^{2r-9} - 3^n + 3 \cdot 2^{n-1}}{6} \text{ for } r \geq 4, \text{ and, in general}$$

$$\left\{ {r \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^r \text{ for } r \geq k.$$

As a corollary, we provide a proof for the identity of Greenstreet [9] (we changed slightly the notation to match the parameters in Theorem 2.1).

Corollary 2.3. *For all positive integers* N, *we have* $\sum_{k=1}^{N} (-1)^k \binom{n}{k} 2^{N-k} k^2 = N^2 - 2N$.

Proof. We take $r = 2$, $a_n = \binom{N}{k}$, and $x = -1/2$ in Theorem 2.1. Note that $\sum_{k=0}^{N} \binom{N}{k} x^k = (1+x)^N = f(x)$ and so, $f'(x) = N(1+x)^{N-1}$ and $f''(x) = N(N-1)(1+x)^{N-2}$. This implies

$$\begin{aligned}
2^N \sum_{k=1}^{N} \left(-\frac{1}{2}\right)^k \binom{n}{k} k^2 &= 2^N \left(\left\{ {2 \atop 1} \right\} \left(-\frac{1}{2}\right)^1 f'\left(-\frac{1}{2}\right) + \left\{ {2 \atop 2} \right\} \left(-\frac{1}{2}\right)^2 f''\left(-\frac{1}{2}\right) \right) \\
&= 2^N \left(\left(-\frac{1}{2}\right)^1 N \left(\frac{1}{2}\right)^{N-1} + \left(-\frac{1}{2}\right)^2 N(N-1) \left(\frac{1}{2}\right)^{N-2} \right) \\
&= -N + N(N-1) = N^2 - 2N,
\end{aligned}$$

using $\left\{ {2 \atop 1} \right\} = \left\{ {2 \atop 2} \right\} = 1$ (via Proposition 2.2). The claim is shown. □

In [10, Chapter 25], there are several identities such as

$$\begin{aligned}
\sum_{i=1}^{N} F_i &= F_{N+2} - 1, \\
\sum_{i=1}^{N} iF_i &= NF_{N+2} - F_{N+3} + 2, \\
\sum_{i=1}^{N} i^2 F_i &= (N+1)F_{N+2} - (2N+3)F_{N+4} + 2F_{N+6} - 8, \\
\sum_{i=1}^{N} i^3 F_i &= (N+1)^3 F_{N+2} - (3N^2+9N+7)F_{N+4} + (6N+12)F_{N+6} - 6F_{N+8} + 50,
\end{aligned} \tag{2.4}$$

and the Lucas' counterparts. We can easily provide a more general result having the above examples as particular cases, by using Theorem 2.1. Because we shall use it later, we start with a closed form for $F(x) = \sum_{i=0}^{N} F_i x^i$ (one can easily get such a formula for any other second-order recurrence sequence), which can be easily shown by multiplying both sides by the denominator (as the referee suggested),

$$F(x)=\sum_{i=0}^{N}F_ix^i=\frac{x-F_{N+1}x^{N+1}-F_Nx^{N+2}}{1-x-x^2},\ N\geq 1. \tag{2.5}$$

Corollary 2.4. *Let $t\geq 0$, $N\geq 1$ be integers, $h(x)=\sum_{r=0}^{t}A_rx^r\in\mathbb{C}[x]$, and $F(x)=\sum_{i=0}^{N}F_ix^i$. Then*

$$\sum_{k=0}^{N}h(k)F_kx^k=\sum_{r=0}^{t}A_r\sum_{i=0}^{r}\left\{ {r \atop i} \right\}x^iF^{(i)}(x). \tag{2.6}$$

In particular, taking $h(x)=1,x,x^2$, or x^3, we obtain the four identities of (2.4).

Proof. Replacing h by its expression in the left side of (2.6), we get

$$\begin{aligned}\sum_{k=0}^{N}\sum_{r=0}^{t}A_rk^rF_kx^k&=\sum_{r=0}^{t}A_r\sum_{k=0}^{N}k^rF_kx^k\\&=\sum_{r=0}^{t}A_r\sum_{i=0}^{r}\left\{ {r \atop i} \right\}x^iF^{(i)}(x),\ \text{via Theorem 2.1.}\end{aligned}$$

Now, taking $h(x)=1$ (thus, $A_0=1$, $A_r=0$, $r\geq 1$) in equation (2.6), we simply obtain equation (2.5) and replacing $x=1$, the first identity of (2.4) follows. Next, taking $h(x)=x$ (thus, $A_0=0$, $A_1=1$, $A_r=0$, $r\geq 2$), equation (2.6) transforms into

$$\begin{aligned}\sum_{k=0}^{N}kF_kx^k&=A_1\sum_{i=0}^{1}\left\{ {1 \atop i} \right\}x^iF^{(i)}(x)=xF'(x)\\&=\frac{x\left(F_N\left(N\left(x^2+x-1\right)+x-2\right)x^{N+1}+F_{N+1}\left(N\left(x^2+x-1\right)-x^2-1\right)x^N+x^2+1\right)}{\left(x^2+x-1\right)^2},\end{aligned}$$

and replacing $x=1$ above, we get the second identity of (2.4). The other identities follow in a similar manner. ☐

We now proceed in providing a more general consequence of Theorem 2.1.

Theorem 2.5. *For all positive integers N,r, we have (with the convention $(N)_0=1$)*

$$\sum_{k=0}^{N}k^r\binom{N}{k}F_k=\sum_{i=0}^{r}\left\{ {r \atop i} \right\}(N)_i\,F_{2N-i},$$

$$\sum_{k=0}^{N}k^r\binom{N}{k}L_k=\sum_{i=0}^{r}\left\{ {r \atop i} \right\}(N)_i\,L_{2N-i},$$

where $(N)_i=N(N-1)\cdots(N-i+1)$ is the falling factorial. In general, if $\{U_n\}_n$ is a second order recurrence sequence satisfying $U_{n+1}=aU_n+bU_{n-1}$ (for some initial conditions), $a,b\in\mathbb{R}$, and $\{V_n\}_n$, its Lucas companion sequence, then

$$\sum_{k=0}^{N}k^r\binom{N}{k}a^kb^{N-k}W_k=\sum_{i=0}^{r}\left\{ {r \atop i} \right\}(N)_i\,a^i\,W_{2n-i},$$

where $W_n=U_n$, respectively, $W_n=V_n$.

Proof. We shall apply Theorem 2.1 twice with $a_k = \frac{1}{\sqrt{5}}\binom{N}{k}$, $f(x) = \frac{1}{\sqrt{5}}(1+x)^N$ for $x = \alpha$, respectively, $x = \beta$. First, note that if $f(x) = \frac{1}{\sqrt{5}}(1+x)^N$, then $f^{(i)}(x) = \frac{1}{\sqrt{5}}(N)_i(1+x)^{N-i}$. Subtracting the two identities of Theorem 2.1, corresponding to the above two cases, we obtain

$$\sum_{k=0}^{N} k^r \binom{N}{k} F_k = \frac{1}{\sqrt{5}} \sum_{i=0}^{r} \left\{ {r \atop i} \right\} (N)_i \left(\alpha^i (1+\alpha)^{N-i} - \beta^i (1+\beta)^{N-i} \right).$$

Now, recall that α, β satisfy $y^2 - y - 1 = 0$, and so, $\alpha + 1 = \alpha^2$, $\beta + 1 = \beta^2$, which used in the relation above renders

$$\sum_{k=0}^{N} k^r \binom{N}{k} F_k = \frac{1}{\sqrt{5}} \sum_{i=0}^{r} \left\{ {r \atop i} \right\} (N)_i \left(\alpha^{2N-i} - \beta^{2N-i} \right) = \sum_{i=0}^{r} \left\{ {r \atop i} \right\} (N)_i \, F_{2N-i}.$$

A similar argument works for the Lucas sequence as well.

For the general case, we recall that $U_n = \frac{1}{\sqrt{\Delta}}(\alpha^n - \beta^n)$, $V_n = \alpha^n + \beta^n$, where $\Delta = a^2 + 4b$ is the discriminant, and $\alpha = \frac{a+\sqrt{\Delta}}{2}$, $\beta = \bar{\alpha} = \frac{a-\sqrt{\Delta}}{2}$. We now apply Theorem 2.1 for $f(x) = \frac{1}{\sqrt{\Delta}}(b+ax)^N$, $a_k = \frac{1}{\sqrt{\Delta}}\binom{N}{k} a^k b^{N-k}$, and $x = \alpha, \beta$. Subtracting the left sides of the two identities of Theorem 2.1 (for $x = \alpha, \beta$), we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\Delta}} \sum_{k=0}^{N} k^r \binom{N}{k} \left((a\alpha)^k b^{N-k} - (a\beta)^k b^{N-k} \right) \\
&\quad = \frac{1}{\sqrt{\Delta}} \sum_{k=0}^{N} k^r \binom{N}{k} a^k b^{N-k} \left(\alpha^k - \beta^k \right) \\
&\quad = \sum_{k=0}^{N} k^r \binom{N}{k} a^k b^{N-k} U_k.
\end{aligned}$$

By the same operation, noting that $f^{(i)}(x) = \frac{1}{\sqrt{\Delta}}(N)_i \, a^i (b+ax)^{N-i}$ and $b + a\alpha = \alpha^2$, $b + a\beta = \beta^2$, the right side becomes

$$\begin{aligned}
&\frac{1}{\sqrt{\Delta}} \sum_{i=0}^{r} \left\{ {r \atop i} \right\} (N)_i \, \left(\alpha^i a^i (b+a\alpha)^{N-i} - \beta^i a^i (b+a\beta)^{N-i} \right) \\
&\quad = \frac{1}{\sqrt{\Delta}} \sum_{i=0}^{r} \left\{ {r \atop i} \right\} (N)_i \, a^i \left(\alpha^{2N-i} - \beta^{2N-i} \right) \\
&\quad = \sum_{i=0}^{r} \left\{ {r \atop i} \right\} (N)_i \, a^i U_{2N-i}.
\end{aligned}$$

A similar argument works for the companion Lucas sequence V_n as well. □

Taking $1 \leq r \leq 4$ in the theorem above, and using Proposition 2.2, we obtain the Fibonacci counterpart of Lucas' Theorem [10, p. 198]. Because the Lucas counterparts are similar, we omit displaying them.

Corollary 2.6. *For all integers $N > 0$, we have*

$$\sum_{k=1}^{N} k\binom{N}{k}F_k = NF_{2N-1},$$

$$\sum_{k=1}^{N} k^2\binom{N}{k}F_k = NF_{2N-1} + N(N-1)F_{2N-2},$$

$$\sum_{k=1}^{N} k^3\binom{N}{k}F_k = NF_{2N-1} + 3N(N-1)F_{2N-2} + N(N-1)(N-2)F_{2N-3},$$

$$\sum_{k=1}^{N} k^4\binom{N}{k}F_k = NF_{2N-1} + 7N(N-1)F_{2N-2} + 6N(N-1)(N-2)F_{2N-3} + N(N-1)(N-2)(N-3)F_{2N-4}.$$

A reader suggested to also look at sums of weights $\binom{n}{k}$ multiplied by a polynomial h with coefficients in $\mathbb{C}$. To this end, we can easily show the following corollary by splitting the sum for each term and applying Theorem 2.5. The second part follows from Corollary 2.6 above, with the known identity $\sum_{k=0}^{N}\binom{N}{k}F_k = F_{2N}$ [17, Formula (47) on p. 179, Appendix].

Corollary 2.7. *Let N, t be positive integers with $t \le N/2$, and $h(x) = \sum_{j=0}^{t} A_j x^j \in \mathbb{C}[x]$. Then*

$$\sum_{k=1}^{N} h(k)\binom{N}{k}F_k = \sum_{j=0}^{t} A_j \sum_{i=0}^{j} \left\{ j \atop i \right\} (N)_i\, F_{2N-i}.$$

In particular, if $h(x) = \sum_{j=0}^{4} A_j x^j \in \mathbb{C}[x]$, then

$$\sum_{k=1}^{N} h(k)\binom{N}{k}F_k = A_0F_{2N} + (A_1 + A_2 + A_3 + A_4)\binom{N}{1}F_{2N-1} + 2(A_2 + 3A_3 + 7A_4)\binom{N}{2}F_{2N-2} + 6(A_3 + 6A_4)\binom{N}{3}F_{2N-3} + 24A_4\binom{N}{4}F_{2N-4}.$$

The above identities can be further generalized, by replacing F_k by any second-order sequence U_k (with some initial conditions) raised to any power.

Corollary 2.8. *Let r, N, t be positive integers and $h(x) = \sum_{j=0}^{t} A_j x^j \in \mathbb{C}[x]$. Let $\{U_n\}_n$ be a second-order recurrent sequence satisfying $U_{n+1} = aU_n + bU_{n-1}$, where a, b, U_0, U_1 are arbitrary real numbers and the discriminant $a^2 + 4b \neq 0$. Then*

$$\sum_{k=0}^{N} h(k)\binom{N}{k}U_k^r x^k = \sum_{j=0}^{t} A_j \sum_{i=0}^{j} \left\{ j \atop i \right\} x^i \sum_{\ell=0}^{r} \binom{r}{\ell} A^j(-B)^{r-\ell}(N)_i \left(\alpha^\ell\beta^{r-\ell}\right)^i (1 + \alpha^\ell\beta^{r-\ell}x)^{N-i},$$

where $\alpha = \frac{1}{2}(a + \sqrt{a^2+4b})$, $\beta = \frac{1}{2}(a - \sqrt{a^2+4b})$, $A = \frac{U_1 - U_0\beta}{\alpha-\beta}$, $B = \frac{U_1 - U_0\alpha}{\alpha-\beta}$.

Proof. We use one of the identities of [16, Theorem 9], namely

$$\sum_{k=0}^{N}\binom{N}{k}U_k^r x^k = \sum_{j=0}^{r}\binom{r}{j}A^j(-B)^{r-j}(1+\alpha^j\beta^{r-j}x)^N =: U(x).$$

Next, we write $\sum_{k=0}^{N}h(k)\binom{N}{k}U_k^r = \sum_{k=0}^{N}\sum_{j=0}^{t}A_jk^j\binom{N}{k}U_k^r = \sum_{j=0}^{t}A_j\sum_{k=0}^{N}k^j\binom{N}{k}U_k^r$. We now use Theorem 2.1 for the inner sum, obtaining

$$\sum_{k=0}^{N}k^j\binom{N}{k}U_k^r x^k = \sum_{i=0}^{j}\left\{ {j \atop i} \right\}x^iU^{(i)}(x).$$

Observe that

$$U^{(i)}(x) = \sum_{\ell=0}^{r}\binom{r}{\ell}A^j(-B)^{r-\ell}(N)_i\left(\alpha^\ell\beta^{r-\ell}\right)^i(1+\alpha^\ell\beta^{r-\ell}x)^{N-i}.$$

By putting everything together, we get the claim. □

3. An Integration Technique

Here, we apply a technique based upon antiderivatives to obtain more identities.

Theorem 3.1. *For positive integers N, s, we have*

$$\sum_{k=0}^{N}\frac{1}{(k+1)^{(s)}}\binom{n}{k}x^{k+s} = \frac{(1+x)^{N+s}-1}{(N+1)^{(s)}} - \sum_{j=1}^{s-1}\frac{1}{j!(N+1)^{(s-j)}}x^j. \tag{3.1}$$

In general, let $f(x) = \sum_{k=0}^{N}a_kx^k \in \mathbb{R}[x]$ *and* $I_1(x) = \int f(x)dx$, $I_2(x) = \int I_1(x)dx$ *and recursively,* $I_k(x) = \int I_{k-1}(x)dx$ *be a sequence of antiderivatives starting from* f *(assuming all are defined at* 0*). Then*

$$\sum_{k=0}^{N}\frac{1}{(k+1)^{(s)}}a_kx^{k+s} = I_s(x) - \sum_{j=1}^{s}\frac{I_j(0)}{(s-j)!}x^{s-j}. \tag{3.2}$$

Proof. The claims will be shown by induction. Integrating both sides of the particular case of the binomial formula $\sum_{k=0}^{N}\binom{n}{k}x^k = (1+x)^N$, we obtain

$$\sum_{k=0}^{N}\frac{1}{k+1}\binom{n}{k}x^{k+1} = \frac{(1+x)^{N+1}}{N+1} - \frac{1}{N+1}.$$

Integrating again, we get

$$\sum_{k=0}^{N}\frac{1}{(k+1)(k+2)}\binom{n}{k}x^{k+2} = \frac{(1+x)^{N+2}}{(N+1)(N+2)} - \frac{x}{N+1} - \frac{1}{(N+1)(N+2)}.$$

Assuming that equation (3.1) is true for s (using the rising factorial $(a)^{(j)} = a(a+1)\cdots(a+j-1)$ notation), we show it for $s+1$. Integrating the identity of (3.1) and using the integration constant $\frac{1}{(N+1)^{(s+1)}}$, we obtain, after shifting the index from $j+1$ to j in the last sum,

$$\sum_{k=0}^{N} \frac{1}{(k+1)^{(s)}} \binom{n}{k} x^{k+s+1} = \frac{(1+x)^{N+s+1}}{(N+1)^{(s+1)}} - \frac{1}{(N+1)^{(s+1)}} - \sum_{j=1}^{s} \frac{1}{j!(N+1)^{(s+1-j)}} x^j,$$

and so, the first claim follows using the principle of mathematical induction.

For the general case, integrating f repeatedly (we do a few more terms than required by the induction principle, just to better see the pattern), and using our notations, we obtain

$$\sum_{k=0}^{N} \frac{1}{k+1} a_k x^{k+1} = I_1(x) - I_1(0), \tag{3.3}$$

$$\sum_{k=0}^{N} \frac{1}{(k+1)(k+2)} a_k x^{k+2} = I_2(x) - I_2(0) - I_1(0)x,$$

$$\sum_{k=0}^{N} \frac{1}{(k+1)(k+2)(k+3)} a_k x^{k+3} = I_3(x) - I_3(0) - I_2(0)x - I_1(0)\frac{1}{2}x^2.$$

We now assume the claim true for s and show it for $s+1$. Integrating (3.2), and again shifting the index from $k+1$ to k in the last sum, we obtain

$$\begin{aligned}\sum_{k=0}^{N} \frac{1}{(k+1)^{(s+1)}} a_k x^{k+s+1} &= I_{s+1}(x) - I_{s+1}(0) - \sum_{j=1}^{s} \frac{I_j(0)}{(s+1-j)!} x^{s+1-j} \\ &= I_{s+1}(x) - \sum_{j=1}^{s+1} \frac{I_j(0)}{(s+1-j)!} x^{s+1-j};\end{aligned}$$

therefore, our second claim follows via the principle of mathematical induction. The proof of our theorem is shown. □

We provide below the proofs for the identities of Epsteen and Finkel. Furthermore, the referee challenged us to find such an identity with k^2 in the denominator, and we show one such below (we prove it for any x).

Corollary 3.2. *For all integers n, we have*

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1}-1}{n+1}, \text{ respectively} \tag{3.4}$$

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}. \tag{3.5}$$

Further, the following identity holds

$$\sum_{k=1}^{n} \frac{1}{k^2} \binom{n}{k} x^k = \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} (1+x)^j - \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j}. \tag{3.6}$$

In particular, $\displaystyle\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k^2} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j}.$

Proof. Identity (3.4) follows by taking $s = 1$, $x = 1$ in the first result of Theorem 3.1 (equation (3.4) also follows from equation (3.3)).

To show (3.5), we can redo Theorem 3.1 by taking an identity involving x^{k-1}, but we prefer to avoid that by writing $\sum_{k=1}^{n}\binom{n}{k}x^k = (1+x)^n - 1 = x\left((1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x) + 1\right)$, and so, our sum becomes

$$\sum_{k=1}^{n}\binom{n}{k}x^{k-1} = (1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x) + 1.$$

We now apply once the technique of Theorem 3.1 for $f(x) = (1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x) + 1$ to obtain (recall that I_1 is any antiderivative of f),

$$\sum_{k=1}^{n}\frac{1}{k}\binom{n}{k}x^k = I_1(x) - I_1(0) = \sum_{k=1}^{n}\frac{1}{k}(1+x)^k - \sum_{k=1}^{n}\frac{1}{k}. \tag{3.7}$$

Taking $x = -1$, and multiplying the entire identity by -1, we obtain (3.5).

To show (3.6), we write (3.7) in the form

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k}\binom{n}{k}x^k &= \sum_{k=1}^{n}\frac{1}{k}\left((1+x)^k - 1\right)\\ &= \sum_{k=1}^{n}\frac{1}{k}x\left(\sum_{j=0}^{k-1}(1+x)^j\right),\end{aligned}$$

and divide by x and apply again Theorem 3.1. We infer that

$$\sum_{k=1}^{n}\frac{1}{k^2}\binom{n}{k}x^k = \sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}(1+x)^j - \sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}.$$

Taking $x = -1$, we finally obtain

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k^2}\binom{n}{k} = \sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}.$$

The corollary is shown. □

4. Further Comments

In this paper, we make use of techniques inspired by differential and integral operators to derive some identities for weighted sums of various sequences. In the process, we extend previous identities and find a connection between these and (generalized) Stirling numbers.

The involved sums do not need to be finite and the results extend easily to series (see [10, Chapter 24], where the differentiation technique is used to derive $\sum_{k=1}^{\infty}(k+1)F_k x^k = \frac{x(2-x)}{(1-x-x^2)^2}$, among others), though they have to be absolutely convergent to allow for term by term integration/differentiation.

Finally, new identities can be derived by performing a combination of differential operators. To be slightly more precise, we take a sequence of real numbers $s_1, s_2, \ldots, s_t$, and the operator $\Delta_{s_i} = x^{s_i+1}\frac{\partial}{\partial x}$ and compose these as in $\Delta_{s_1,s_2,\ldots,s_t} = \Delta_{s_1}\left(\Delta_{s_2}\left(\ldots(\Delta_{s_t})\ldots\right)\right)$. We apply this to a suitably chosen identity, like $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, to obtain new ones. We leave this to the interested reader.

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