FIBONACCI AND PI SQUARED

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Abstract. In this paper, we use the summation by parts method to generate an infinite number of formulas for π^2 as infinite series involving Fibonacci numbers.

1. Introduction

The constant π and the integer sequence of Fibonacci numbers are probably the two most popular objects in mathematical literature. The constant $\pi \approx 3.1415$ is defined as the ratio of a circle's circumference to its diameter. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined as

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.$$

Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

gives an explicit calculation of the Fibonacci numbers.

There are several formulas that link the constant π with the Fibonacci sequence. Here are some examples.

- π as a sum series of Fibonacci numbers [2, 3]:

$$\begin{aligned}\pi &= \sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}\, 2^{2n+3}}{(2n+1)\,(3+\sqrt{5})^{2n+1}} \\ &= 20\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}^2}{(2n+1)\,(3+\sqrt{10})^{2n+1}} \\ &= 12\sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}}{2n+1}\left(\frac{2(2-\sqrt{3})}{\sqrt{5}+\sqrt{1+16(2-\sqrt{3})}}\right)^{2n+1};\end{aligned}$$

- π as a double sum series of Fibonacci numbers [8]:

$$\pi = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{4\,(-1)^k}{(2k+1)\,F_{2n+1}^{2k+1}};$$

- A classic result of D. H. Lehmer [6, 7]:

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\operatorname{arccot}(F_{2n+1});$$

- π and the inverse trigonometric values of Fibonacci numbers [4]:

$$\begin{aligned}\frac{\pi}{4} &= \sum_{n=1}^{\infty}\arctan\left(\frac{F_n}{1+F_{n+1}F_{n+2}}\right),\\ \frac{\pi}{6} &= \sum_{n=1}^{\infty}\arctan\left(\frac{\sqrt{3}F_n}{1+3F_{n+1}F_{n+2}}\right),\\ \frac{\pi}{3} &= \sum_{n=1}^{\infty}\arctan\left(\frac{\sqrt{3}F_n}{3+F_{n+1}F_{n+2}}\right),\\ \frac{\pi}{12} &= \sum_{n=1}^{\infty}\arctan\left(\frac{(2-\sqrt{3})F_n}{7-4\sqrt{3}+F_{n+1}F_{n+2}}\right);\end{aligned}$$

- π^2 as a sum series of Fibonacci numbers [9]:

$$\pi^2 = \frac{25\sqrt{5}}{4}\sum_{k=1}^{\infty}\frac{((k-1)!)^2}{(2k)!}F_{2k} = \frac{50(25+4\sqrt{5})}{109}\sum_{k=0}^{\infty}\frac{(k!)^2}{(2k+2)!}F_{2k+1}. \tag{1.1}$$

The purpose of this paper is to generate more formulas for π^2 as infinite series involving Fibonacci numbers. We will generate two types of formulas.

- $\pi^2 = c + \sum_{k=0}^{\infty}\alpha_k F_{2k+1}$;
- $\pi^2 = c + \sum_{k=0}^{\infty}\alpha_k F_{2k+2}$.

The first type of formula involves the odd-indexed Fibonacci numbers and the second type involves the even-indexed Fibonacci numbers. A formula of the first type can be transformed into a new one of the second type and vice versa as follows.

$$\begin{aligned}\sum_{k=0}^{\infty}\alpha_k F_{2k+1} &= \alpha_0(F_2-F_0)+\alpha_1(F_4-F_2)+\alpha_2(F_6-F_4)+\cdots\\ &= (\alpha_0-\alpha_1)F_2+(\alpha_1-\alpha_2)F_4+(\alpha_2-\alpha_3)F_6+\cdots\\ &= \sum_{k=0}^{\infty}(\alpha_k-\alpha_{k+1})F_{2k+2}; \qquad (1.2)\\ \sum_{k=0}^{\infty}\alpha_k F_{2k+2} &= \alpha_0(F_3-F_1)+\alpha_1(F_5-F_3)+\alpha_2(F_7-F_5)+\cdots\\ &= -\alpha_0F_1+(\alpha_0-\alpha_1)F_3+(\alpha_1-\alpha_2)F_5+(\alpha_2-\alpha_3)F_7+\cdots\\ &= -\alpha_0F_1+\sum_{k=1}^{\infty}(\alpha_{k-1}-\alpha_k)F_{2k+1}. \qquad (1.3)\end{aligned}$$

These transformations are special cases of the "summation by parts" formula
$$\sum u\Delta v = uv - \sum v\Delta u,$$
where Δ denotes the difference operator on the space of sequences. This is the discrete version of the familiar "integration by part" formula
$$\int u\,dv = uv - \int v\,du.$$

The operator Δ in the summation by parts formula is defined as $\Delta(X) = Y$ if and only if $Y_n = X_{n+1} - X_n$. If we apply the operator Δ on the even-indexed Fibonacci sequence, then we obtain the odd-indexed Fibonacci sequence and vice versa:
$$\Delta(\{F_{2k}\}) = \{F_{2k+1}\},$$
$$\Delta(\{F_{2k+1}\}) = \{F_{2k+2}\}.$$

The main result of this paper is Theorem 2.1 in the following section, where we generate an infinite number of formulas for π^2 as series involving Fibonacci numbers. We do this inductively via the method of summation by parts. At each step n, where n is a positive integer, two formulas for π^2 are generated. One formula is a series involving even-indexed Fibonacci terms and the other involving odd-indexed Fibonacci terms. The even-indexed formula at step $n+1$ is obtained from the previous step ns odd-indexed formula by applying the difference operator. In a similar fashion, the odd-indexed formula at step $n+1$ is derived from the even-indexed formula at the previous step n. For example, the following formulas in Theorem 2.1 are obtained at steps $n = 4$ and 5.

$$\begin{aligned}\pi^2 &= \frac{250}{50-16\sqrt{5}}\left(\frac{1}{4!} + \sum_{k=0}^{\infty} \frac{(k!)^2\ \begin{pmatrix}27k^6+486k^5+3618k^4+14214k^3+\\ 30891k^2+34872k+15756\end{pmatrix}}{(2k+8)!}F_{2k+2}\right)\\ &= \frac{250}{75-28\sqrt{5}}\left(\frac{694}{6!} - \sum_{k=0}^{\infty} \frac{(k!)^2\ \begin{pmatrix}27k^6+486k^5+3618k^4+14214k^3+\\ 30891k^2+34872k+15756\end{pmatrix}}{(2k+8)!}F_{2k+1}\right),\end{aligned} \tag{1.4}$$

$$\begin{aligned}\pi^2 &= \frac{250}{75-28\sqrt{5}}\left(\frac{694}{6!} - \sum_{k=0}^{\infty} \frac{(k!)^2\ \begin{pmatrix}81k^8+2268k^7+27594k^6+190440k^5+814185k^4+\\ 2201148k^3+3655260k^2+3381120k+1318176\end{pmatrix}}{(2k+10)!}F_{2k+2}\right)\\ &= \frac{250}{125-44\sqrt{5}}\left(\frac{24788}{8!} + \sum_{k=0}^{\infty} \frac{(k!)^2\ \begin{pmatrix}81k^8+2268k^7+27594k^6+190440k^5+814185k^4+\\ 2201148k^3+3655260k^2+3381120k+1318176\end{pmatrix}}{(2k+10)!}F_{2k+1}\right).\end{aligned} \tag{1.5}$$

2. Main Theorem

The following main theorem gives us an infinite number of formulas for π^2 as sum series involving Fibonacci numbers.

Theorem 2.1. *Let $\{p_n(x)\}_{n\geq 1}$ be a sequence of polynomials and $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ two number sequences defined as*

$$p_1(x) = 1,\ \ p_{n+1}(x) = (2x+2n+1)(2x+2n+2)\, p_n(x) - (x+1)^2\, p_n(x+1) \quad \text{for all } n \geq 1,$$
$$a_0 = -\frac{8\sqrt{5}}{250},\ \ a_1 = \frac{25-4\sqrt{5}}{250},\ \ a_{n+1} = a_n + a_{n-1} \quad \text{for all } n \geq 1,$$
$$b_0 = 0,\ \ b_1 = 0,\ \ b_{n+1} = b_n + b_{n-1} + (-1)^{n+1}\,\frac{p_n(0)}{(2n)!} \quad \text{for all } n \geq 1.$$

Then, for any $n \geq 1$, we have

$$\pi^2 = \frac{1}{a_{n-1}}\left(b_{n-1} + (-1)^n \sum_{k=0}^{\infty} \frac{(k!)^2\ p_n(k)}{(2k+2n)!} F_{2k+2}\right) \tag{2.1}$$
$$= \frac{1}{a_n}\left(b_n + (-1)^{n+1} \sum_{k=0}^{\infty} \frac{(k!)^2\ p_n(k)}{(2k+2n)!} F_{2k+1}\right). \tag{2.2}$$

In Theorem 2.1, the polynomial $p_n(x)$ has degree $2n-2$. The sequence $\{a_n\}$ satisfies the Fibonacci recurrence equation and the sequence $\{b_n\}$ satisfies a Fibonacci-like recurrence.

Here are a few terms in the polynomial sequence $\{p_n(x)\}_{n\geq 1}$.

$$\begin{aligned}
p_1(x) &= 1,\\
p_2(x) &= (2x+3)(2x+4)\, p_1(x) - (x+1)^2\, p_1(x+1)\\
&= 3x^2 + 12x + 11,\\
p_3(x) &= (2x+5)(2x+6)\, p_2(x) - (x+1)^2\, p_2(x+1)\\
&= 9x^4 + 90x^3 + 333x^2 + 532x + 304,\\
p_4(x) &= (2x+7)(2x+8)\, p_3(x) - (x+1)^2\, p_3(x+1)\\
&= 27x^6 + 486x^5 + 3618x^4 + 14214x^3 + 30891x^2 + 34872x + 15756,\\
p_5(x) &= (2x+9)(2x+10)\, p_4(x) - (x+1)^2\, p_4(x+1)\\
&= 81x^8 + 2268x^7 + 27594x^6 + 190440x^5 + 814185x^4 + 2201148x^3\\
&\qquad + 3655260x^2 + 3381120x + 1318176.
\end{aligned}$$

Below are some values of a_n.

$$\begin{aligned}
a_0 &= -\frac{8\sqrt{5}}{250},\\
a_1 &= \frac{25-4\sqrt{5}}{250},\\
a_2 &= \frac{25-12\sqrt{5}}{250},\\
a_3 &= \frac{50-16\sqrt{5}}{250},\\
a_4 &= \frac{75-28\sqrt{5}}{250},\\
a_5 &= \frac{125-44\sqrt{5}}{250}.
\end{aligned}$$

Below are some values of b_n.

$$\begin{aligned}
b_0 &= 0,\\
b_1 &= 0,\\
b_2 &= b_1 + b_0 + \frac{p_1(0)}{2!} = 0 + 0 + \frac{1}{2!} = \frac{1}{2!},\\
b_3 &= b_2 + b_1 - \frac{p_2(0)}{4!} = \frac{1}{2!} + 0 - \frac{11}{4!} = \frac{1}{4!},\\
b_4 &= b_3 + b_2 + \frac{p_3(0)}{6!} = \frac{1}{4!} + \frac{1}{2!} + \frac{304}{6!} = \frac{694}{6!},\\
b_5 &= b_4 + b_3 - \frac{p_4(0)}{8!} = \frac{694}{6!} + \frac{1}{4!} - \frac{15756}{8!} = \frac{24788}{8!}.
\end{aligned}$$

When $n = 1$, Theorem 2.1 gives

$$\begin{aligned}
\pi^2 &= \frac{250}{8\sqrt{5}} \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+2)!} F_{2k+2}\\
&= \frac{250}{25-4\sqrt{5}} \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+2)!} F_{2k+1}.
\end{aligned}$$

When $n = 2$,

$$\begin{aligned}
\pi^2 &= \frac{250}{25-4\sqrt{5}} \sum_{k=0}^{\infty} \frac{(k!)^2\ (3k^2+12k+11)}{(2k+4)!} F_{2k+2}\\
&= \frac{250}{25-12\sqrt{5}} \left(\frac{1}{2!} - \sum_{k=0}^{\infty} \frac{(k!)^2\ (3k^2+12k+11)}{(2k+4)!} F_{2k+1} \right).
\end{aligned}$$

When $n = 3$,

$$\pi^2 = \frac{250}{25 - 12\sqrt{5}} \left(\frac{1}{2!} - \sum_{k=0}^{\infty} \frac{(k!)^2 \ (9k^4 + 90k^3 + 333k^2 + 532k + 304)}{(2k+6)!} F_{2k+2} \right)$$
$$= \frac{250}{50 - 16\sqrt{5}} \left(\frac{1}{4!} + \sum_{k=0}^{\infty} \frac{(k!)^2 \ (9k^4 + 90k^3 + 333k^2 + 532k + 304)}{(2k+6)!} F_{2k+1} \right).$$

When $n = 4$, we obtain formula (1.4); and when $n = 5$, we obtain formula (1.5).

We will prove Theorem 2.1 by induction. First, we need to establish the case $n = 1$.

Lemma 2.2.

$$\pi^2 = \frac{250}{8\sqrt{5}} \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+2)!} F_{2k+2} \tag{2.3}$$
$$= \frac{250}{25 - 4\sqrt{5}} \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+2)!} F_{2k+1}. \tag{2.4}$$

Proof. This identity is equivalent to (1.1), which is proved in Theorem 5 of [9]. However, for completeness, we provide a proof here. We will use the following arcsine function formula, which can be found in [1, 5].

$$(\arcsin x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k+1} (k!)^2 x^{2k+2}}{(2k+2)!}. \tag{2.5}$$

Applying (2.5) with $\sin(\pi/10) = (\sqrt{5} - 1)/4$, we have

$$\frac{\pi^2}{100} = \sum_{k=0}^{\infty} \frac{2^{-1} (k!)^2}{(2k+2)!} \left(\frac{1 - \sqrt{5}}{2} \right)^{2k+2}. \tag{2.6}$$

Applying (2.5) with $\sin(3\pi/10) = (\sqrt{5} + 1)/4$, we have

$$\frac{9\pi^2}{100} = \sum_{k=0}^{\infty} \frac{2^{-1} (k!)^2}{(2k+2)!} \left(\frac{1 + \sqrt{5}}{2} \right)^{2k+2}. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\frac{8\pi^2}{100} = \sum_{k=0}^{\infty} \frac{2^{-1} (k!)^2}{(2k+2)!} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{2k+2} - \left(\frac{1 - \sqrt{5}}{2} \right)^{2k+2} \right]$$
$$= \sum_{k=0}^{\infty} \frac{2^{-1} (k!)^2}{(2k+2)!} \sqrt{5}\, F_{2k+2},$$

and on solving for π^2, we get formula (2.3).

It also follows from (2.6) and (2.7) that

$$\begin{aligned}\frac{\pi^2}{100}\left(\frac{1+\sqrt{5}}{2}\right)+\frac{9\pi^2}{100}\left(\frac{\sqrt{5}-1}{2}\right)&=\frac{5\sqrt{5}-4}{100}\,\pi^2\\&=\sum_{k=0}^{\infty}\frac{2^{-1}\,(k!)^2}{(2k+2)!}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2k+1}\right]\\&=\sum_{k=0}^{\infty}\frac{2^{-1}\,(k!)^2}{(2k+2)!}\sqrt{5}\,F_{2k+1},\end{aligned}$$

and on solving for π^2, we get formula (2.4). □

Proof of Theorem 2.1. We use induction. For the case $n=1$, we have Lemma 2.2. Suppose the theorem holds for n. We will prove it for $n:=n+1$.

Using the induction hypothesis (2.2) and applying transformation (1.2), we have

$$\begin{aligned}a_n\pi^2&=b_n+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{(k!)^2\ p_n(k)}{(2k+2n)!}F_{2k+1}\\&=b_n+(-1)^{n+1}\sum_{k=0}^{\infty}\left(\frac{(k!)^2\ p_n(k)}{(2k+2n)!}-\frac{((k+1)!)^2\ p_n(k+1)}{(2k+2n+2)!}\right)F_{2k+2}\\&=b_n+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{(k!)^2\ ((2k+2n+1)(2k+2n+2)p_n(k)-(k+1)^2p_n(k+1))}{(2k+2n+2)!}F_{2k+2}.\end{aligned}$$

Using the recursive formula for the polynomial sequence $\{p_n(x)\}$, we obtain

$$a_n\pi^2=b_n+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{(k!)^2\ p_{n+1}(k)}{(2k+2n+2)!}F_{2k+2}, \tag{2.8}$$

and this proves formula (2.1) for $n+1$.

Using the induction hypothesis (2.1) and applying transformation (1.3), we have

$$\begin{aligned}a_{n-1}\pi^2&=b_{n-1}+(-1)^{n}\sum_{k=0}^{\infty}\frac{(k!)^2\ p_n(k)}{(2k+2n)!}F_{2k+2}\\&=b_{n-1}+(-1)^{n}\left(-\frac{p_n(0)}{(2n)!}+\sum_{k=1}^{\infty}\left(\frac{((k-1)!)^2\ p_n(k-1)}{(2k+2n-2)!}-\frac{(k!)^2\ p_n(k)}{(2k+2n)!}\right)F_{2k+1}\right)\\&=b_{n-1}+(-1)^{n}\left(-\frac{p_n(0)}{(2n)!}+\sum_{k=0}^{\infty}\left(\frac{(k!)^2\ p_n(k)}{(2k+2n)!}-\frac{((k+1)!)^2\ p_n(k+1)}{(2k+2n+2)!}\right)F_{2k+3}\right)\\&=b_{n-1}+(-1)^{n}\left(-\frac{p_n(0)}{(2n)!}+\sum_{k=0}^{\infty}\frac{(k!)^2((2k+2n+1)(2k+2n+2)p_n(k)-(k+1)^2p_n(k+1))}{(2k+2n+2)!}F_{2k+3}\right).\end{aligned}$$

Using the recursive formula for the polynomial sequence $\{p_n(x)\}$, we obtain

$$a_{n-1}\pi^2 = b_{n-1} + (-1)^{n+1}\frac{p_n(0)}{(2n)!} + (-1)^n \sum_{k=0}^{\infty} \frac{(k!)^2\, p_{n+1}(k)}{(2k+2n+2)!} F_{2k+3}. \tag{2.9}$$

Adding (2.8) and (2.9), we obtain

$$(a_n + a_{n-1})\pi^2 = b_n + b_{n-1} + (-1)^{n+1}\frac{p_n(0)}{(2n)!} + (-1)^n \sum_{k=0}^{\infty} \frac{(k!)^2\, p_{n+1}(k)}{(2k+2n+2)!}(F_{2k+3} - F_{2k+2}).$$

Using the recursive formulas for the sequence $\{a_n\}$ and $\{b_n\}$, we have

$$a_{n+1}\pi^2 = b_{n+1} + (-1)^n \sum_{k=0}^{\infty} \frac{(k!)^2\, p_{n+1}(k)}{(2k+2n+2)!} F_{2k+1},$$

and this proves formula (2.2) for $n+1$. □

The following lemmas give explicit formulas for the number sequence $\{a_n\}$ and the polynomial sequence $\{p_n(x)\}$ used in Theorem 2.1.

Lemma 2.3.

$$a_n = \frac{25-4\sqrt{5}}{250}F_n - \frac{8\sqrt{5}}{250}F_{n-1} \quad \text{for all } n \geq 0.$$

Here, when $n = 0$, we use the convention that $F_{-1} = F_1 - F_0 = 1$.

Proof. We prove the lemma by induction. When $n = 0$,

$$\frac{25-4\sqrt{5}}{250}F_0 - \frac{8\sqrt{5}}{250}F_{-1} = -\frac{8\sqrt{5}}{250}F_{-1} = -\frac{8\sqrt{5}}{250} = a_0.$$

When $n = 1$,

$$\frac{25-4\sqrt{5}}{250}F_1 - \frac{8\sqrt{5}}{250}F_0 = \frac{25-4\sqrt{5}}{250}F_1 = \frac{25-4\sqrt{5}}{250} = a_1.$$

In the induction step, adding the following two equations

$$a_{n-1} = \frac{25-4\sqrt{5}}{250}F_{n-1} - \frac{8\sqrt{5}}{250}F_{n-2}$$
$$a_n = \frac{25-4\sqrt{5}}{250}F_n - \frac{8\sqrt{5}}{250}F_{n-1}$$

gives

$$a_{n+1} = \frac{25-4\sqrt{5}}{250}F_{n+1} - \frac{8\sqrt{5}}{250}F_n.$$

□

Lemma 2.4.

$$p_n(x) = \sum_{j=0}^{n-1}(-1)^j \binom{n-1}{j} \prod_{k=2j+3}^{2n}(2x+k)\prod_{t=1}^{j}(x+t)^2 \quad \text{for all } n \geq 1.$$

Proof. We prove the lemma by induction. When $n = 1$,

$$\sum_{j=0}^{0}(-1)^j\binom{0}{j}\prod_{k=2j+3}^{2}(2x+k)\prod_{t=1}^{j}(x+t)^2 = 1 = p_1(x).$$

At the induction step, we have

$$\begin{aligned}
p_{n+1}(x) &= (2x+2n+1)(2x+2n+2)\,p_n(x) - (x+1)^2\,p_n(x+1)\\
&= (2x+2n+1)(2x+2n+2)\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\prod_{k=2j+3}^{2n}(2x+k)\prod_{t=1}^{j}(x+t)^2\\
&\quad - (x+1)^2\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\prod_{k=2j+3}^{2n}(2x+k+2)\prod_{t=1}^{j}(x+t+1)^2\\
&= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\prod_{k=2j+3}^{2n+2}(2x+k)\prod_{t=1}^{j}(x+t)^2\\
&\quad - \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\prod_{k=2j+5}^{2n+2}(2x+k)\prod_{t=1}^{j+1}(x+t)^2.
\end{aligned}$$

In the second sum, letting $J = j+1$, we obtain

$$\begin{aligned}
p_{n+1}(x) &= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\prod_{k=2j+3}^{2n+2}(2x+k)\prod_{t=1}^{j}(x+t)^2\\
&\quad + \sum_{J=1}^{n}(-1)^J\binom{n-1}{J-1}\prod_{k=2J+3}^{2n+2}(2x+k)\prod_{t=1}^{J}(x+t)^2\\
&= \prod_{k=3}^{2n+2}(2x+k) + \sum_{j=1}^{n-1}(-1)^j\binom{n-1}{j}\prod_{k=2j+3}^{2n+2}(2x+k)\prod_{t=1}^{j}(x+t)^2\\
&\quad + (-1)^n\prod_{t=1}^{n}(x+t)^2 + \sum_{J=1}^{n-1}(-1)^J\binom{n-1}{J-1}\prod_{k=2J+3}^{2n+2}(2x+k)\prod_{t=1}^{J}(x+t)^2.
\end{aligned}$$

Using $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$, we obtain

$$\begin{aligned}
p_{n+1}(x) &= \prod_{k=3}^{2n+2}(2x+k) + (-1)^n\prod_{t=1}^{n}(x+t)^2 + \sum_{j=1}^{n-1}(-1)^j\binom{n}{j}\prod_{k=2j+3}^{2n+2}(2x+k)\prod_{t=1}^{j}(x+t)^2\\
&= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\prod_{k=2j+3}^{2n+2}(2x+k)\prod_{t=1}^{j}(x+t)^2,
\end{aligned}$$

and this completes the proof. □

Acknowledgement

The author thanks the anonymous reviewers for many helpful comments and suggestions that led to improvements in this paper.

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MSC2020: 11B39, 11B37, 11Y60

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